Math 417 – Seventh Day – Class

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First a few elaborations on the material I sent out Tuesday.

Given that $n \equiv 1, 3, 7, 9 \mod 10$, it follows that $[n]_{10} \in (\mathbb{Z}/10\mathbb{Z})^*$, so $gcd(n, 10) = 1 = gcd(10, n)$, so that $[10]_n \in (\mathbb{Z}/n\mathbb{Z})^*$.

Rather than appealing to a previous theorem, I'll give the proof again. We know that $(\mathbb{Z}/n\mathbb{Z})^*$ is a finite set, so the sequence

$$[10]_n, \quad [10^2]_n, \quad [10^3]_n, \quad [10^4]_n, \dots$$

can't all be different.

Thus, we can find $i < j$ so that $[10^i]_n = [10^j]_n$. (This is kind of the Pigeonhole Principle.) Thus, since $gcd(n, 10) = 1$,

$$n \mid 10^j - 10^i = 10^i(10^{j-i} - 1) \implies$$
$$n \mid (10^{j-i} - 1) \implies 10^{j-i} \equiv 1 \mod n \implies [10^{j-i}]_n = [1]_n.$$

I've also gotten requests for a refresher on the geometric series. Here's one derivation. Suppose $x \neq 1$ and $n \in \mathbb{N}$. Then there is an exact identity which you can verify by cross-multiplying.

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}.$$

If $|x| < 1$, then $x^{n+1} \to 0$, and so we have the geometric series formula:

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \quad \text{if} \quad |x| < 1.$$

What about the series we saw in decimals?

$$x + x^2 + x^3 + \cdots = x(1 + x + x^2 + \dots) = \frac{x}{1 - x}.$$

In the special case that $x = \frac{1}{N}$ (our case is $N = 10^r$), we have

$$1 \quad 1 \quad 1 \qquad \frac{1}{N} \qquad 1$$

For example, if $n = 13$, then Mathematica tells me that the sequence of $10^k \mod 13$ is:

$$10^0 \equiv 1,\ 10^1 \equiv 10,\ 10^2 \equiv 9,\ 10^3 \equiv 12,\ 10^4 \equiv 3,\ 10^5 \equiv 4,\ 10^6 \equiv 1$$

Here, $i = 0,\ j = 6$, and $10^6 - 1 = 999999 = 13 \cdot 76923$.

One more question was about the order of composition, and I think it would be easiest if I used two functions which aren't permutations.

Suppose $f(x) = x + 3$ and $g(x) = x^2$ and you apply f first and then do g. Then you get $x \mapsto x + 3 \mapsto (x + 3)^2$. For comparison,

$$f(g(x)) = f(x^2) = x^2 + 3,$$
$$g(f(x)) = g(x + 3) = (x + 3)^2.$$

So in this case too: if we act in the order written, it gets written in the reverse. It's confusing but this is how it works here.

The symmetric group we will get to know best is S_3, which has $6 = 3!$ elements. I will write them down and then give them all names (the same ones in Fraleigh) and say a little bit about each one. They can also be viewed as symmetries of an equilateral triangle.

Here is the triangle:

$$\begin{matrix} 1 & & 2 \\ & 3 & \end{matrix}$$

Each element of S_3 gets its own page.

This is the first element, ρ_0. Under ρ_0,

$$1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 3$$

$$\rho_0 = \begin{pmatrix} 123 \\ 123 \end{pmatrix} = (1)(2)(3),$$

1 2

3

Thus $\rho_0 = e$ is the identity element. The triangle at the end to show the motions of these permutations and this is the starting configuration. Think of the numbers as labels that can move, and also indicate the name of the position.

This is the second element, ρ_1. Under ρ_1,

$$1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 1$$

$$\rho_1 = \begin{pmatrix} 123 \\ 231 \end{pmatrix} = (123),$$

3 1

2

The permutation ρ_1 rotates the triangle clockwise by $\frac{2\pi}{3}$. The triangle shows that the label 1 goes to the position 2, the label 2 goes to the position 3 and label 3 goes to the position 1.

This is the third element, ρ_2. Under ρ_2,

$$1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2$$

$$\rho_2 = \begin{pmatrix} 123 \\ 312 \end{pmatrix} = (132),$$

2 3

1

The permutation ρ_2 rotates clockwise by $\frac{4\pi}{3}$ or counterclockwise by $\frac{2\pi}{3}$. The triangle shows that the label 1 goes to the position 3, the label 2 goes to the position 1 and label 3 goes to the position 2.

I hope you can see that $\rho_2 = \rho_1^2$ and $\rho_1 = \rho_2^2,\ \rho_1^3 = \rho_2^3 = \rho_0 = e$. It follows that $\{\rho_0, \rho_1, \rho_2\}$ is a cyclic group of order 3, and a subgroup of S_3.

This is the fourth element, μ_1. Under μ_1,

$$1 \mapsto 1, \quad 2 \mapsto 3, \quad 3 \mapsto 2$$

$$\mu_1 = \begin{pmatrix} 123 \\ 132 \end{pmatrix} = (1)(23),$$

1 3

2

The permutation μ_1 flips 2 and 3 and fixes 1. This type of permutation is called a *transposition*. Notice that $\mu_1^2 = \rho_0 = e$. It's equivalent to flipping the triangle on a diameter through 1.

This is the fifth element, μ_2. Under μ_2,

$$1 \mapsto 3, \quad 2 \mapsto 2, \quad 3 \mapsto 1$$

$$\mu_2 = \binom{123}{321} = (13)(2),$$

3 2

1

The permutation μ_2 flips 1 and 3 and fixes 2. Also a transposition. Notice that $\mu_2^2 = \rho_0 = e$. It's equivalent to flipping the triangle on a diameter through 2.

This is the sixth element, μ_3. Under μ_3,

$$1 \mapsto 2, \quad 2 \mapsto 1, \quad 3 \mapsto 3$$

$$\mu_3 = \binom{123}{213} = (12)(3),$$

2 1

3

The permutation μ_3 flips 1 and 2 and fixes 3. Also a transposition. Again, $\mu_3^2 = \rho_0 = e$. It's equivalent to flipping the triangle on a diameter through 3.

We'll finish the multiplication table later, but I wanted to show that the operation $\circ$ is *not* always commutative. In one simple case:

$$\mu_1 \circ \mu_2 = \begin{pmatrix} 123 \\ 132 \end{pmatrix} \circ \begin{pmatrix} 123 \\ 321 \end{pmatrix} \implies$$

$$1 \mapsto 1 \mapsto 3, \quad 2 \mapsto 3 \mapsto 1, \quad 3 \mapsto 2 \mapsto 2, \implies$$

$$\mu_1 \circ \mu_2 = \begin{pmatrix} 123 \\ 312 \end{pmatrix} = \rho_2;$$

$$\mu_2 \circ \mu_1 = \begin{pmatrix} 123 \\ 321 \end{pmatrix} \circ \begin{pmatrix} 123 \\ 132 \end{pmatrix} \implies$$

$$1 \mapsto 3 \mapsto 2, \quad 2 \mapsto 2 \mapsto 3, \quad 3 \mapsto 1 \mapsto 1, \implies$$

$$\mu_1 \circ \mu_2 = \begin{pmatrix} 123 \\ 231 \end{pmatrix} = \rho_1.$$

So the product of each flip is a rotation, but they are different rotations: $\mu_1 \circ \mu_2 \neq \mu_2 \circ \mu_1$.

One more point on these before we go on. Imagine that the triangle is, say, red on the front and blue on the back. We start with it as red. Any rotation (or the identity) (that is, any ρ_i) will keep it red. Any flip (that is, any μ_i) will switch it so the front is blue.

If you think about that, then you can convince yourself, **without any calculation**, that $\rho_i \circ \rho_j$ will be some ρ_k and $\rho_i \circ \mu_j$ or $\mu_i \circ \rho_j$ will be some μ_k. Finally, two flips will turn the triangle twice, so it's in its original color position. Thus, $\mu_i \circ \mu_j$ will be some ρ_k. (We saw that twice above.) This shows up in table 8.8 on p.79 of the book. We'll talk about this on Friday.

WORKSHEET PROBLEMS

1. It is a fact from arithmetic that $77 \cdot 12987 = 999999$. Use this information to give the decimal expansion of

$$\frac{50}{77}.$$

2. Recall that

$$\rho_2 = \begin{pmatrix} 123 \\ 312 \end{pmatrix} = (132), \qquad \mu_2 = \begin{pmatrix} 123 \\ 321 \end{pmatrix} = (13)(2)$$

Calculate $\rho_2 \circ \mu_2$ and $\mu_2 \circ \rho_2$.

WORKSHEET PROBLEM SOLUTIONS

1.

$$\frac{50}{77} = \frac{50 \cdot 12987}{77 \cdot 12987} = \frac{649350}{999999} = 649350 \times \frac{1}{999999} =$$

$$649350 \times (.000001\ 000001\ 000001\ \ldots)$$

$$= .649350\ 649350\ 649350\ \ldots$$

2. Again, for reference,

$$\rho_2 = \binom{123}{312} = (132), \qquad \mu_2 = \binom{123}{321} = (13)(2)$$

So $\rho_2 \circ \mu_2$ takes $1 \mapsto 3 \mapsto 1$, $2 \mapsto 1 \mapsto 3$ and $3 \mapsto 2 \mapsto 2$ and

$$\rho_2 \circ \mu_2 = \binom{123}{132} = (1)(23)$$

Similarly, $\mu_2 \circ \rho_2$ takes $1 \mapsto 3 \mapsto 2$, $2 \mapsto 2 \mapsto 1$ and $3 \mapsto 1 \mapsto 3$ and

$$\mu_2 \circ \rho_2 = \binom{123}{213} = (12)(3)$$

That is, $\rho_2 \circ \mu_2 = \mu_1$ and $\mu_2 \circ \rho_2 = \mu_3$.