Math 417 – Sixth Day – Class

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e	e	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

$(\mathbb{Z}/4\mathbb{Z}, \oplus)$	0	1	2	3
0	0	1	2	3
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a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

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The isomorphism implicit from these tables is given by

$$\Phi(e) = [0]_4,\ \Phi(a) = [1]_4,\ \Phi(a^2) = [2]_4,\ \Phi(a^3) = [3]_4.$$

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Since the divisors of 4 are 1, 2, 4, the group $C_4 = \langle a \rangle$ has three subgroups: $\langle a \rangle$, $\langle a^2 \rangle$ and $\langle a^4 \rangle$. Of these, $\langle a \rangle = C_4$ and $\langle a^4 \rangle = \langle e \rangle = \{e\}$. The other one is $\langle a^2 \rangle = \{e, a^2\}$.

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What happens when we apply Φ to these sets? First, Φ is a bijection, so $\Phi(C_4) = \mathbb{Z}/4\mathbb{Z}$. Second, Φ maps the identity to the identity, so $\Phi(\{e\}) = [0]_4$. The interesting case is

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$$\Phi(\{e, a^2\}) = \{\Phi(e), \Phi(a^2)\} = \{[0]_4, [2]_4\}.$$

Both sides give a cyclic group of order 2, because $(a^2)^2 = e$ and $[2]_4 + [2]_4 = [0]_4$.

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What we found earlier was that $[1]_5$ is the identity, $[2]_5$ is a generator, and $[2]_5^2 = [2^2]_5 = [4]_5$ and $[2]_5^3 = [2^3]_5 = [8]_5 = [3]_5$ and $[2]_5^4 = [2^4]_5 = [16]_5 = [1]_5$.

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Again, if we define Φ_1 by

$$\Phi_1(e) = [1]_5,\ \Phi_1(a) = [2]_5,\ \Phi_1(a^2) = [4]_5,\ \Phi_1(a^3) = [3]_5,$$

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we see that it is an isomorphism, and the proper subgroup of $((\mathbb{Z}/5\mathbb{Z})^*, \odot)$ is $\{\Phi_1(e), \Phi_1(a^2)\} = \{[1]_5, [4]_5\}$.

I'll finish with a request to go over the isomorphism proofs from last night. I will not repeat them in the "Weekly Summary".

These are unchanged frames. Please stop me if you have questions!

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(ii) $\Phi(g^{-1})$ is the inverse of $\Phi(g)$ in H.

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THEOREM 1: Suppose the group $(G, *_G)$ is isomorphic to the group $(H, *_H)$ and suppose G_1 is a subgroup of G. Then

$$H_1 = \Phi(G_1) := \{\Phi(g) : g \in G_1\}$$

is a subgroup of H.

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All we have to do is prove the conditions for a subgroup. First closure. Suppose $x, y \in H_1$. We need to prove that $x *_H y \in H_1$. But $x, y \in H_1$ means that there exist $u, v \in G_1$ so that $x = \Phi(u)$ and $y = \Phi(v)$. Because Φ is an isomorphism,

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Since G_1 is a subgroup, $u *_G v \in G_1$, so this means that $x *_H y$ is the image of an element of G_1 under Φ, which means that $x *_H y \in H_1$.

PROOF: If $x \in H_1$, then there is $u \in G_1$ so that $x = \Phi(u)$.

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The other two are easier. Since G_1 is a subgroup, $e_G \in G_1$ and so $\Phi(e_G) \in H_1$. By the lemma, this means $e_H \in H_1$: it has the identity. If $x \in H_1$, then $x = \Phi(u)$ for $u \in G_1$ and then $u^{-1} \in G_1$, because G_1 is a subgroup and by the lemma, $\Phi(u^{-1}) \in H_1$ is the inverse of $x = \Phi(u)$. □

WORKSHEET PROBLEMS

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Thus, for example

$$([1]_2, [1]_3) * ([1]_2, [1]_3) = ([2]_2, [2]_3) = ([0]_2, [2]_3),$$

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because $2 \equiv 0 \mod 2$ and $2 \equiv 2 \mod 3$.

Prove that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is a cyclic group of order 6 by working out $\langle([1]_2, [1]_3)\rangle$.

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Write out the other values of $\Phi(a^k)$, and $\Phi(\langle a^2 \rangle)$ and $\Phi(\langle a^3 \rangle)$.

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These are subgroups of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ that will be "obviously" subgroups.

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$$([1]_2,[1]_3)^3 = ([1]_2,[1]_3) * ([1]_2,[1]_3)^2 = ([1]_2,[0]_3)$$
$$([1]_2,[1]_3)^4 = ([1]_2,[1]_3) * ([1]_2,[1]_3)^3 = ([0]_2,[1]_3)$$
$$([1]_2,[1]_3)^5 = ([1]_2,[1]_3) * ([1]_2,[1]_3)^4 = ([1]_2,[2]_3)$$
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([1]_2,[1]_3)^4 &= ([1]_2,[1]_3)*([1]_2,[1]_3)^3 = ([0]_2,[1]_3)\\
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([1]_2,[1]_3)^6 &= ([1]_2,[1]_3)*([1]_2,[1]_3)^5 = ([0]_2,[0]_3)
\end{aligned}
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So the first five powers are different and the sixth gives the identity and $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ is a cyclic group of order 6.

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$\Phi(a^4) = ([0]_2, [2]_3)$, $\Phi(a^5) = ([1]_2, [2]_3)$ and
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The images of $\Phi(\langle a^2 \rangle) = \Phi(\{e, a^2, a^4\})$ and $\Phi(\langle a^3 \rangle) = \Phi(\{e, a^3\}$ are then

$$\{([0]_2, [0]_3), ([0]_2, [1]_3), ([0]_2, [2]_3)\}$$

and

$$\{([0]_2, [0]_3), ([1]_2, [0]_3)\}.$$

You could write these as $\{[0]_2\} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \{[0]_3\}$