Math 417 – Fourteenth Day – Class

Bruce Reznick
University of Illinois at Urbana-Champaign

September 25, 2020

All I have for you today is one example. What happens with multiplication when you take the cosets of a subgroup that *isn't* normal?

Since every subgroup of an abelian group is normal, we don't have much choice. We only know two non-abelian groups: S_3 and D_4.

As a reminder:

$$\rho_0 = \begin{pmatrix} 123 \\ 123 \end{pmatrix} = (1)(2)(3), \quad \rho_1 = \begin{pmatrix} 123 \\ 231 \end{pmatrix} = (123),$$

$$\rho_2 = \begin{pmatrix} 123 \\ 312 \end{pmatrix} = (132), \quad \mu_1 = \begin{pmatrix} 123 \\ 132 \end{pmatrix} = (1)(23),$$

$$\mu_2 = \begin{pmatrix} 123 \\ 321 \end{pmatrix} = (13)(2), \quad \mu_3 = \begin{pmatrix} 123 \\ 213 \end{pmatrix} = (12)(3).$$

Let's take one of the subgroups of order two: $H = \{\rho_0, \mu_1\}$ and calculate the left cosets. Here they are:

$$\begin{aligned}\rho_0 H &= \{\rho_0, \mu_1\},\\ \rho_1 H &= \{\rho_1\rho_0, \rho_1\mu_1\} = \{\rho_1, \mu_3\},\\ \rho_2 H &= \{\rho_2\rho_0, \rho_2\mu_1\} = \{\rho_2, \mu_2\}.\end{aligned}$$

What happens when we take the products?

$$\begin{aligned}\rho_0 H \rho_0 H &= \{\rho_0^2, \rho_0\mu_1, \mu_1\rho_0, \mu_1^2\} = \{\rho_0, \mu_1, \mu_1, \rho_0\}\\ &= \{\rho_0, \mu_1\} = \rho_0 H.\end{aligned}$$

But that's the "easy one", because we are multiplying within a subgroup.

$$\rho_0 H \rho_1 H = \{\rho_0, \mu_1\}\{\rho_1, \mu_3\} = \{\rho_0\rho_1, \rho_0\mu_3, \mu_1\rho_1, \mu_1\mu_3\}$$
$$= \{\rho_1, \mu_3, \mu_2, \rho_2\}.$$

This is not a coset! It happens to be the union of two cosets.

I'll do one more:

$$\rho_1 H \rho_1 H = \{\rho_1, \mu_3\}\{\rho_1, \mu_3\} = \{\rho_1\rho_1, \rho_1\mu_3, \mu_3\rho_1, \mu_3\mu_3\}$$
$$= \{\rho_2, \mu_2, \mu_1, \rho_0\}.$$

Again, there are four different elements, and this is not a coset.

If you find this interesting, you could try to compute the other products.