Math 417 – Thirteenth Day – Class

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Comments on the homework, taken from my email! The main issue seems to be cosets in problem 2. I'll quote "$G = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, so the elements of G are $([i]_6, [j]_8)$, where $0 \le i \le 5,\ 0 \le j \le 7$. Let $H = \{([0]_6, [0]_8), ([0]_6, [4]_8), ([3]_6, [0]_8), ([3]_6, [4]_8)\}$ be a subgroup of G." The operation is component-wise addition.

I won't give you a left coset, but I will give you a right coset. Just pick an element from G and combine it with the group. I'll pick $([4]_6, [7]_8)$ for the element. Then

$$H + ([4]_6, [7]_8) =$$
$$\{([0]_6, [0]_8) + ([4]_6, [7]_8), ([0]_6, [4]_8) + ([4]_6, [7]_8),$$
$$([3]_6, [0]_8) + ([4]_6, [7]_8), ([3]_6, [4]_8) + ([4]_6, [7]_8)\} =$$
$$\{([4]_6, [7]_8), ([4]_6, [11]_8), ([7]_6, [7]_8), ([7]_6, [11]_8)\} =$$
$$\{([4]_6, [7]_8), ([4]_6, [3]_8), ([1]_6, [7]_8), ([1]_6, [3]_8)\}.$$

This is also $H + ([1]_6, [3]_8)$.

The last remark is based on the fact that $a \in Ha$ (or aH) for every coset. If H is a subgroup, then $e \in H$ and if $H = \{e, h_2, \ldots, h_n\}$ then

$$aH = \{ae, \ldots, ah_n\}$$

But different elements give the same coset. We have $ah_i \in aH$. What is the left coset associated to ah_i?

$$(ah_i)H = \{ah_i e, ah_i h_1, \ldots, ah_i h_n\}$$

On the one hand, we can quote a theorem that $ah_i \in aH \cap (ah_i)H$, so that $aH = (ah_i)H$. Or we can give the proof of the theorem in this case. We see that $ah_i h_j \in aH$, so $(ah_i)H \subseteq aH$, but also $ah_j = (ah_i)(h_i^{-1}h_j)$, so $aH \subseteq (ah_i)H$.

I'd like to redo that botched example, as also a review of subgroups, cosets and homomorphisms.

Suppose $K < G$ and $[G : K] = 2$, which means that, if G is a finite group, $|K| = \frac{1}{2}|G|$, and in any case, that G can be written as a disjoint union of two cosets of K.

Suppose $a \in G \setminus K$; that is, $a \in G$, $a \notin K$. Since $a \notin K$ and $a = ae \in aK$ and $a = ea \in Ka$, the left coset aK and the right coset Ka are different from K, and by an earlier result, disjoint from K. Thus we can write

$$G = K \cup aK, \quad G = K \cup Ka, \quad (\implies aK = Ka = G \setminus K).$$

One thing I didn't mention but should have is this: $a^2 \in G$, because G is closed under its operation. If $a^2 \in aK$, then $a^2 = ak$ for some $k \in K$. But this implies that $a = k \in K$, which is impossible.

Therefore, $a^2 \in K$.

I now want to look at where products of elements go. There are four cases: (i) $gh, g \in K, h \in K$, (ii) $gh, g \in K, h \in aK = Ka$, (iii) $gh, g \in aK = Ka, h \in K$, (iv) $gh, g \in aK = Ka, h \in aK = Ka$.

For (i), if $g, h \in K$, then $gh \in K$ because K is a subgroup.

For (ii), write $h = ka, k \in K$, then $gh = g(ka) = (gk)a \in Ka$, because $g, k \in K$.

For (iii), write $g = ak, k \in K$, then $gh = (ak)h = a(kh) \in aK$, because $k, h \in K$.

For (iv), write $g = k_1 a, k_1 \in K$ and $h = ak_2, k_2 \in K$. Then $gh = (k_1 a)(ak_2) = k_1 a^2 k_2$ is a product of three elements in K, so $gh \in K$.

This pattern is one we have seen in S_3, where K consists of the ρ_i's and aK consists of the μ_i's.

Now (and only now) will I define a homomorphism: Let $C_2 = \{e, u\}$ be a cyclic group of order 2, so $u^2 = e$, and define $\phi : G \to C_2$ by $\phi(x) = e$ if $x \in K$ and $\phi(x) = u$ if $x \in aK = Ka$.

I need to check that $\phi(gh) = \phi(g)\phi(h)$ for $g, h \in K$ and there are the four cases noted above.

In (i), $g, h, gh \in K$, so $\phi(g) = e, \phi(h) = e, \phi(gh) = e$.

In (ii), $g \in K, h \in Ka, gh \in Ka$, so $\phi(g) = e, \phi(h) = u, \phi(gh) = u$.

In (iii), $g \in aK, h \in K, gh \in aK$, so $\phi(g) = u, \phi(h) = e, \phi(gh) = u$.

In (iv), $g \in aK, h \in Ka, gh \in K$, so $\phi(g) = u, \phi(h) = u, \phi(gh) = e$.

Since $u^2 = e$, ϕ has been shown to be a homomorphism, and

$$Ker(\phi) = \{x \in G \mid \phi(x) = e\} = K.$$

This is what we wanted. We will have a more general version of this construction in a short while (not today).

The other topic I wanted to talk about involved the subgroups of D_4 of order four. As a reminder, here are the elements in the group.

$$\rho_0 = \binom{1234}{1234} = (1)(2)(3)(4) \quad \rho_1 = \binom{1234}{2341} = (1234),$$

$$\rho_2 = \binom{1234}{3412} = (13)(24) \quad \rho_3 = \binom{1234}{4123} = (1432),$$

$$\mu_1 = \binom{1234}{2143} = (12)(34), \quad \mu_2 = \binom{1234}{4321} = (14)(23),$$

$$\delta_1 = \binom{1234}{3214} = (13)(2)(4), \quad \delta_2 = \binom{1234}{1432} = (1)(24)(3),$$

One of the subgroups of order four is

$$\{\rho_0, \rho_1, \rho_2, \rho_3\} = \langle \rho_1 \rangle = \langle \rho_3 \rangle.$$

This group can be interpreted as the rotations of the square, or the motions which keep the front to the front.

The other subgroups are isomorphic to V: $\{\rho_0, \rho_2, \mu_1, \mu_2\}$ and $\{\rho_0, \rho_2, \delta_1, \delta_2\}$. Can these be understood in terms of motions? Let me put the squares in the first case.

$$\rho_0 = \begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} \qquad \rho_2 = \begin{matrix} 3 & 4 \\ 2 & 1 \end{matrix} \qquad \mu_1 = \begin{matrix} 2 & 1 \\ 3 & 4 \end{matrix} \qquad \mu_2 = \begin{matrix} 4 & 3 \\ 1 & 2 \end{matrix}$$

Can you see the common feature of these rotations?

What I see is that the horizontal edges (12 and 34) stay horizontal, and the vertical edges stay vertical. In the other coset,

$$\rho_1 = \begin{matrix} 4 & 1 \\ 3 & 2 \end{matrix} \qquad \rho_3 = \begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix} \qquad \delta_1 = \begin{matrix} 3 & 2 \\ 4 & 1 \end{matrix} \qquad \delta_2 = \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix}$$

the horizontal edges all go vertical, and the vertical edges go horizontal.

What do you see in the third subgroup?

$$\rho_0 = \begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} \qquad \rho_2 = \begin{matrix} 3 & 4 \\ 2 & 1 \end{matrix} \qquad \delta_1 = \begin{matrix} 3 & 2 \\ 4 & 1 \end{matrix} \qquad \delta_2 = \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix}$$

What I see is that the diagonals are preserved: (13) stay in the (13) positions and (24) in the (24) positions.

In the coset, the diagonals are flipped.

$$\rho_1 = \begin{matrix} 4 & 1 \\ 3 & 2 \end{matrix} \qquad \rho_3 = \begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix} \qquad \mu_1 = \begin{matrix} 2 & 1 \\ 3 & 4 \end{matrix} \qquad \mu_2 = \begin{matrix} 4 & 3 \\ 1 & 2 \end{matrix}.$$

WORKSHEET QUESTION

1. Suppose $\phi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is a homomorphism. We know that $\phi([0]_6) = [0]_4$. There are four possible choices for $\phi([1]_6)$:

(a) $\phi_0([1]_6) = [0]_4$,
(b) $\phi_1([1]_6) = [1]_4$,
(c) $\phi_2([1]_6) = [2]_4$,
(d) $\phi_3([1]_6) = [3]_4$,

Show that ϕ_0 and ϕ_2 are homomorphisms and, using the property of homomorphisms, write out $\phi_0([a]_6)$ and $\phi_2([a]_6)$. Determine the images and kernels.

Also, by considering the equation

$$[1]_6 + [1]_6 + [1]_6 + [1]_6 + [1]_6 + [1]_6 = [0]_6,$$

show that ϕ_1 and ϕ_3 are not homomorphisms

WORKSHEET SOLUTION

(a) We have already seen ϕ_0: $\phi_0([a]_6) = [0]_4$ for all a, so the kernel is $\mathbb{Z}/6\mathbb{Z}$ and the image is $\{[0]_4\}$.

(c) Here, we have $\phi_2([0]_6) = [0]_4$ and $\phi_2([1]_6) = [2]_4$, additivity implies that $\phi_2([2]_6) = [0]_4$, $\phi_2([3]_6) = [2]_4$, $\phi_2([4]_6) = [0]_4$, $\phi_2([5]_6) = [2]_4$. In words (though this wasn't asked), $\phi_2([a]_6) = [2a]_4$. The kernel is $\{[0]_6, [2]_6, [4]_6\}$ and the image is $\{[0]_4, [2]_4\}$

(b) and (d) The problem is that if ϕ is a homomorphism, then

$$\phi([0]_6) = \phi([1]_6 + [1]_6 + [1]_6 + [1]_6 + [1]_6 + [1]_6)$$
$$= \phi([1]_6) + \phi([1]_6) + \phi([1]_6) + \phi([1]_6) + \phi([1]_6) + \phi([1]_6)$$
$$= 6\phi([1]_6).$$

And if $\phi = \phi_1([1]_6 = [1]_4$, then $[0]_4 \neq 6[1]_4 = [6]_4$. Similarly, if $\phi = \phi_3([1]_6 = [3]_4$, then $[0]_4 \neq 6[3]_4 = [18_4$.

Yes, there's a general theorem lurking here.