Math 417 – Ninth Day – Class

Bruce Reznick
University of Illinois at Urbana-Champaign

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First a correction. The matrix form of ρ_2 is incorrect on p.1 of yesterday's material. It should have been $\rho_2 = \begin{pmatrix} 123 \\ 312 \end{pmatrix}$, with 312, not 321.

I'd like to start by saying a bit more about cosets. For cyclic groups and abelian groups, they are very easy!

Suppose H is a subgroup of G and $g \in G$ (not necessarily in H). The left coset gH and the right coset Hg are defined by

$$gH = \{gh \mid h \in H\}, \quad Hg = \{hg \mid h \in H\}.$$

For today, we'll assume that G is a finite abelian group, such as C_n, and H is a subgroup and $H = \{h_1, \ldots, h_m\}$, so

$$gH = \{gh_1, \ldots, gh_m\}, \quad Hg = \{h_1 g, \ldots, h_m g\}.$$

Since G is abelian, $gh_j = h_j g$ for every j and so $gH = Hg$ automatically. We've seen this doesn't happen necessarily for a non-abelian group.

In general (even if G is not abelian) if $H = \{e\}$, which is a subgroup of G, then for any $x \in G$,

$$xH = \{x\} = Hx.$$

So the coset is just the one-element set. If $H = G$, then

$$xH = xG = \{xg : g \in G\}$$

Suppose $y \in G$. Take $g = x^{-1}y$ to see that

$$xg = x(x^{-1}y) = (xx^{-1})y = ey = y \in xH.$$

This is true for ever $y \in G$ is in xH, so $xH = G$. The same thing is true for Hx when $H = G$. For a finite group, xG is just the x row of the multiplication table and Gx is the x column.

Let's take $G = C_6 = \langle a \rangle, a^6 = e$. The four divisors of 6 are 1, 2,3,6, and G has two proper subgroups: $H_1 = \langle a^2 \rangle = \{e, a^2, a^4\}$ and $H_2 = \langle a^3 \rangle = \{e, a^3\}$. Let's find the cosets.

$$\begin{aligned} eH_1 &= \{e, a^2, a^4\}, & aH_1 &= \{a, a^3, a^5\}, \\ a^2H_1 &= \{a^2, a^4, a^6\} = H_1, & a^3H_1 &= \{a^3, a^5, a\} = aH_1, \\ a^4H_1 &= \{a^4, e, a^2\} = H_1, & a^5H_1 &= \{a^5, a, a^3\} = aH_1 \\ eH_2 &= \{e, a^3\}, & a^3H_2 &= \{a^3, e\} = eH_2 \\ aH_2 &= \{a, a^4\}, & a^4H_2 &= \{a^4, a\} = aH_2 \\ a^2H_2 &= \{a^2, a^5\} & a^5H_2 &= \{a^5, a^2\} = a^2H_2. \end{aligned}$$

Notice for later reference, that $x \in H_1 \implies xH_1 = H_1$ and $x \in aH_1 \implies xH_1 = aH_1$ and similarly for H_2. This will hold in general.

Now, I want to talk to you about another group you will get to know well: D_4, which is the group of symmetries of a square.

There are 8 of them. You can believe this as follows: you can rotate the square in 4 ways (none, 90, 180 or 270 degrees) and then you can flip it over and rotate that 4 ways. I'll use the same names as in the book, and introduce them one at a time, as I did with S_3.

Here is the square:

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$$

This is the first element, ρ_0. Under ρ_0,

$$1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 3, \quad 4 \mapsto 4$$

$$\rho_0 = \begin{pmatrix} 1234 \\ 1234 \end{pmatrix} = (1)(2)(3)(4),$$

1	2
4	3

Thus $\rho_0 = e$ is the identity element (again). The square at the end to show the motions of these permutations and this is the starting configuration. Think of the numbers as labels that can move, and also indicate the name of the position.

This is the second element, ρ_1. Under ρ_1,

$$1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 4, \quad 4 \mapsto 1$$

$$\rho_1 = \binom{1234}{2341} = (1234),$$

4 1

3 2

Thus $\rho_1 = e$ represents clockwise rotation by $\frac{\pi}{2}$ or 90 degrees. It's very much like the ρ_1 of S_3, but it's in a different group: S_4. A small calculation shows that $\rho_1^4 = \rho_0 = e$. Or you can think about the square.

This is the third element, ρ_2. Under ρ_2,

$$1 \mapsto 3, \quad 2 \mapsto 4, \quad 3 \mapsto 1, \quad 4 \mapsto 2$$

$$\rho_2 = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = (13)(24),$$

3	4
2	1

Thus ρ_2 is rotation by π or 180 degrees, so $\rho_2 = \rho_1^2$ and $\rho_2^2 = \rho_0 = e$. This element will be of special interest when we start talking about subgroups.

This is the fourth element, ρ_3. Under ρ_3,

$$1 \mapsto 4, \quad 2 \mapsto 1, \quad 3 \mapsto 2, \quad 4 \mapsto 3$$

$$\rho_3 = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix} = (1432),$$

2 3

1 4

Here, ρ_3 represents clockwise rotation by $\frac{3\pi}{2}$ or 270 degrees, or counterclockwise by 90 degrees. We have $\rho_3 = \rho_1^3 = \rho_1^{-1}$ and $\rho_3^4 = \rho_0 = e$. This is the final rotation in D_4.

There are two kinds of flips, and I'll keep the notation of the book. This is the fifth element, μ_1. Under μ_1,

$$1 \mapsto 2, \quad 2 \mapsto 1, \quad 3 \mapsto 4, \quad 4 \mapsto 3$$

$$\mu_1 = \binom{1234}{2143} = (12)(34),$$

2	1
3	4

I hope you can see that μ_1 is equivalent to flipping on a vertical axis through the center of the square, and if there were a front and a back, the back would now be on front; also, $\mu_1^2 = \rho_0$.

This is the sixth element, μ_2. Under μ_2,

$$1 \mapsto 4 \quad 2 \mapsto 3, \quad 3 \mapsto 2, \quad 4 \mapsto 1.$$

$$\mu_2 = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = (14)(23),$$

4 3

1 2

I hope you can see that μ_2 is equivalent to flipping on a horizontal axis through the center of the square, and if there were a front and a back, the back would now be on front, and $\mu_2^2 = \rho_0$.

What is $\mu_1 \circ \mu_2$? Since it's two flips, the front is back to the front and it has to be a rotation and $\mu_1 \circ \mu_1 = \mu_2 \circ \mu_2 = \rho_0$, so it can't be that.

$$\mu_1 \circ \mu_2 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} \begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = \rho_2.$$

You should check that $\mu_2 \circ \mu_1 = \rho_2$ as well. Try to visualize.

Now the second kind of flip. This is the seventh element, δ_1. Under δ_1,

$$1 \mapsto 3, \quad 2 \mapsto 2, \quad 3 \mapsto 1, \quad 4 \mapsto 4$$

$$\delta_1 = \binom{1234}{3214} = (13)(2)(4),$$

3	2
4	1

I hope you can see that δ_1 is equivalent to flipping on a NE-SW diagonal axis, and those corners don't move; and if there were a front and a back, the back would now be on front; also, $\delta_1^2 = \rho_0$.

This is the eighth and final element, δ_2. Under δ_2,

$$1 \mapsto 1, \quad 2 \mapsto 4, \quad 3 \mapsto 3, \quad 4 \mapsto 2$$

$$\delta_2 = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix} = (24)(1)(3),$$

1 4

2 3

I hope you can see that δ_2 is equivalent to flipping on a NW-SE diagonal axis, and those corners don't move; and if there were a front and a back, the back would now be on front; also, $\delta_2^2 = \rho_0$.

What is $\delta_1 \circ \delta_2$? Since it's two flips, the front is back to the front and it has to be a rotation and $\mu_1 \circ \mu_1 = \mu_2 \circ \mu_2 = \rho_0$, so it can't be that.

$$\delta_1 \circ \delta_2 = \begin{pmatrix} 1234 \\ 3214 \end{pmatrix} \begin{pmatrix} 1234 \\ 1432 \end{pmatrix} = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = \rho_2.$$

You should check that $\delta_2 \circ \delta_1 = \rho_2$ as well.

Here are some computations.

$$\mu_1 \circ \rho_1 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} \circ \begin{pmatrix} 1234 \\ 2341 \end{pmatrix} = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix} = \delta_2$$

$$\rho_1 \circ \mu_1 = \begin{pmatrix} 1234 \\ 2341 \end{pmatrix} \circ \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} = \begin{pmatrix} 1234 \\ 3214 \end{pmatrix} = \delta_1$$

$$\mu_1 \circ \delta_1 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} \circ \begin{pmatrix} 1234 \\ 3214 \end{pmatrix} = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix} = \rho_1^3$$

$$\delta_1 \circ \mu_1 = \begin{pmatrix} 1234 \\ 3214 \end{pmatrix} \circ \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} = \begin{pmatrix} 1234 \\ 2341 \end{pmatrix} = \rho_1$$

So we have the rotations and two kinds of flips. As before rotations and rotations or flips and flips combine to be rotations and rotations and flips in either order combine to be flips. But the two kind of flips are the μ's and the δ's. It will turn out that two μ's or two δ's combine to give something in $\{e, \rho_1^2\}$ and a μ and a δ, in either order, combine to give something in $\{\rho_1, \rho_3^2\}$. See the full table on p.80.

WORKSHEET PROBLEMS

1. Let $G = C_{12} = \langle a \rangle, a^{12} = e$. Write down all 12 left cosets of the subgroup $H = \{e, a^4, a^8\}$. How many different left cosets are there? (This is the abelian case, so left cosets = right cosets, and we usually just say "cosets".)

2. Recall the following four elements of D_4:

$$\rho_0 = \begin{pmatrix} 1234 \\ 1234 \end{pmatrix} = (1)(2)(3)(4), \quad \rho_2 = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = (13)(24),$$

$$\mu_1 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} = (12)(34), \quad \mu_2 = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = (14)(23),$$

Show that $H = \{\rho_0, \rho_2, \mu_1, \mu_2\}$ is a subgroup by constructing the multiplication table to show that it is closed under $\circ$. This subgroup is isomorphic to one we've spent a lot of time with already. Which one?

WORKSHEET SOLUTIONS

1. There are four different cosets. We have

$$\begin{aligned} H = a^4H = a^8H &= \{e, a^4, a^8\}, \\ aH = a^5H = a^9H &= \{a, a^5, a^9\}, \\ a^2H = a^6H = a^{10}H &= \{a^2, a^6, a^{10}\}, \\ a^3H = a^7H = a^{11}H &= \{a^3, a^7, a^{11}\} \end{aligned}$$

Notice that in C_{12}, the element a^k is determined by $k \mod 12$, while membership in a coset is determined by $k \mod 4$. This can be generalized.

I'll repeat the entries:

$$\rho_0 = \begin{pmatrix} 1234 \\ 1234 \end{pmatrix} = (1)(2)(3)(4), \quad \rho_2 = \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = (13)(24),$$
$$\mu_1 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} = (12)(34), \quad \mu_2 = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = (14)(23),$$

Here ρ_0 is the identity, and each of the other three elements squares to ρ_0. There are a bunch of multiplications, all of which have a familiar ring; for example

$$\mu_1 \circ \rho_2 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} \circ \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix} = \mu_2.$$

What we have here is yet another incarnation of V.

H	ρ_0	ρ_2	μ_1	μ_2
ρ_0	ρ_0	ρ_2	μ_1	μ_2
ρ_2	ρ_2	ρ_0	μ_2	μ_1
μ_1	μ_1	μ_2	ρ_0	ρ_2
μ_2	μ_2	μ_1	ρ_2	ρ_0