Math 417 – Fourth Day

Bruce Reznick
University of Illinois at Urbana-Champaign

August 31, 2020
Welcome to the beginning of the second week. We’re going to start with number theory and apply it to group theory. It might not be obvious why I’m doing this. Patience will be rewarded.

\[ \mathbb{Z}/n\mathbb{Z}^\ast := \{ [a]_n | a \in \{1, \ldots, n-1\} \text{ and } \gcd(a, n) = 1 \} \]

The set consists of the classes \( x \equiv a \mod n \), where \( a \) and \( n \) are relatively prime. Let the operation be multiplication mod \( n \), then

**THEOREM** For \( n \geq 2 \), \((\mathbb{Z}/n\mathbb{Z}^\ast, \cdot)\) is a group.

Proof at the end of class.
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Our ultimate goal is to give you another family of groups based on modular arithmetic. Here’s a definition. I’ll prove that these are groups at the end of the talk.
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Proof at the end of class.
We already have an example for $n = 10$ on the first day. The possible values for $a$ are taken from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and we want their gcd with 10 to be 1. Since $D(10) = \{1, 2, 5, 10\}$, we take out the multiples of 2 ($\{2, 4, 6, 8\}$) and the multiple of 5 ($\{5\}$). This means that $(\mathbb{Z}/10\mathbb{Z})^* = \{1, 3, 7, 9\}$. 

I'll remind you of the table.

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7 7 1 9 3
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Number theory. Remember that if \( m, n \in \mathbb{N} \), then \( \gcd(m, n) \) is the largest integer \( g \) so that \( g \mid m \) and \( g \mid n \). If \( \gcd(m, n) = 1 \), then \( m \) and \( n \) are said to be relatively prime.
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\begin{align*}
x_0 &= c_0x_1 + x_2, & c_0 &\in \mathbb{N}, & x_2 &\in \{0, \ldots, x_1 - 1\}; \\
x_1 &= c_1x_2 + x_3, & c_1 &\in \mathbb{N}, & x_3 &\in \{0, \ldots, x_2 - 1\}; \\
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From Friday: \( \gcd(x_0, x_1) = \gcd(x_1, x_2) = \cdots = \gcd(x_n, x_{n+1}) = x_{n+1} \). There’s a numerical example on the next page. We’ll also have a worksheet on Monday with smallish numbers.
This class was supposed to be in 341 Altgeld. Let's find $gcd(341, 417)$. I'll emphasize the $x_j$'s by underlining them.
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$$417 = 1 \cdot 341 + 76,$$

So $gcd(341, 417) = 1$. We have

$$1 = 37 - 18 \cdot 2,$$
$$2 = 76 - 2 \cdot 37,$$
$$37 = 341 - 4 \cdot 76,$$
$$76 = 417 - 1 \cdot 341,$$

so

$$1 = 37 - 18 \cdot 2 = 37 - 18 \cdot (76 - 2 \cdot 37) = (1 + 2 \cdot 18) \cdot 37 - 18 \cdot 76 = 37 \cdot 37 - 18 \cdot 76 = 37 \cdot (341 - 4 \cdot 76) - 18 \cdot 76 = 37 \cdot 341 - 166 \cdot 76 = 37 \cdot 341 - 166 \cdot (417 - 341) = 203 \cdot 341 - 166 \cdot 417.$$
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In precisely this way, we obtain a theorem whose proof I can give in detail on request. The idea is to use the calculations of the general Euclidean Algorithm in general the way we did just now.

\[ \text{THEOREM} \quad \gcd(m, n) = rm + sn, \quad r, s \in \mathbb{Z}, \]

Computable means that there is an algorithm to find them.
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**PROOF** Rewrite the hypothesis in parametric form:

$$m = gr, \quad n = gs, \quad m/g = r, \quad n/g = s, \quad r, s \in \mathbb{N}.$$
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$$m = gr = g(hu) = (gh)u, \quad n = gs = g(hv) = (gh)v.$$ 

This means that $gh$ is a common divisor of $m$ and $n$, but $g$ was the largest one, so $gh \leq g$, so $h = 1$. In other words, the only common divisor of $r$ and $s$ is 1, so it has to be the gcd. □
LEMMA If $a, b, c \in \mathbb{N}$, $gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$. 

PROOF. Since $gcd(a, b) = 1$, there exist $r, s \in \mathbb{N}$ so that $1 = ar + bs$, and since $a \mid bc$, there exists $t \in \mathbb{N}$ so that $bc = at$. Now we get sneaky and multiply the first equation by $c$: 

$c = arc + bsc$. But now $c = arc + bsc = arc + (bc)s = arc + (at)s = a(rc + ts)$.

so $c$ is written as a multiple of $a$; that is, $a \mid c$.

LEMMA If $gcd(a, n) = 1$ and $gcd(b, n) = 1$, then $gcd(ab, n) = 1$.

PROOF This is also sneaky. There exist $r, s, t, u \in \mathbb{Z}$ so that $1 = ar + ns$ and $1 = bt + nu$. Now multiply these together:

$1 = 1 \cdot 1 = (ar + ns)(bt + nu) = abrt + naru + nsbt + n^2su = ab(rt) + n(aru + sbt + nsu)$.

Thus, if $d$ is a common divisor of $ab$ and $n$, then $d \mid ab$ and $d \mid n$, so $d \mid 1$. That is, $1$ is the only common divisor of $ab$ and $n$. 

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\[ 1 = ar + bs, \]
and since \( a \mid bc \), there exists \( t \in \mathbb{N} \) so that 
\[ bc = at. \]

Now we get sneaky and multiply the first equation by \( c \):
\[ c = arc + bsc. \]
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Bruce Reznick
University of Illinois at Urbana-Champaign
Math 417 – Fourth Day
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Now multiply these together:
\[
1 = 1 \cdot 1 = (ar + ns)(bt + nu) = abrt + naru + nsbt + n^2su
\]
\[
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Thus, if $d$ is a common divisor of $ab$ and $n$, then $d \mid ab$ and $d \mid n$, so $d \mid 1$. That is, 1 is the only common divisor of $ab$ and $n$. \hfill \square
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For (ii), since $gcd(a, n) = 1$, we can write $1 = ar + ns$ for some integers $r, s$. This means that $ar = 1 - ns \equiv 1 \mod n$ (!).

For example, $gcd(341, 417) = 1$ and $1 = 203 \cdot 341 - 166 \cdot 417$ imply that $203 \cdot 341 \equiv 1 \mod 417$, and we get as an automatic bonus that $gcd(203, 341) = 1$ as well.
Let’s shift gears and return to groups. I’ll begin with an important definition. Suppose \((G, \ast)\) is a group. and \(H \subseteq G\) is a subset of the elements of \(G\) and suppose that if you just look at \((H, \ast)\), then you have a group. In this case, we say that \(H\) is a *subgroup* of \(G\).
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If \( h_i \in H \) then \((h_1 \ast h_2) \ast h_3 = h_1 \ast (h_2 \ast h_3)\) in \( H \), because this equation holds for them as elements in \( G \), which is a group, and so is associative. To sum up:

**Theorem:** If \((G, \ast)\) is a group and \( H \subseteq G \), then \((H, \ast)\) is a group (and a subgroup of \((G, \ast)\)) if and only if 
- \( e \in H \)
- \( h \in H \Rightarrow h^{-1} \in H \)
- \( h, h' \in H \Rightarrow h \ast h' \in H \).

What are the subgroups of \( G = \mathbb{Z}_6 \)? Remember \( G = \{e, a, a^2, a^3, a^4, a^5, a^6 = e\} \).

The obvious subgroups of \( G \) are \( \{e\} \) and \( \mathbb{Z}_6 \). I claim there are two others: \( \{e, a^2, a^4\} \) and \( \{e, a^3\} \). Here are the multiplication tables:

\[
\begin{array}{c|cccccc}
  & e & a & a^2 & a^3 & a^4 & a^5 \\
\hline
  e & e & a & a^2 & a^3 & a^4 & a^5 \\
  a & a & a^2 & a^3 & a^4 & a^5 & e \\
  a^2 & a^2 & a^3 & a^4 & a^5 & e & a \\
  a^3 & a^3 & a^4 & a^5 & e & a & a^2 \\
  a^4 & a^4 & a^5 & e & a & a^2 & a^3 \\
  a^5 & a^5 & e & a & a^2 & a^3 & a^4 \\
\end{array}
\]

A cyclic group of order 3.
If \( h_i \in H \) then \((h_1 \ast h_2) \ast h_3 = h_1 \ast (h_2 \ast h_3)\) in \( H \), because this equation holds for them as elements in \( G \), which is a group, and so is associative. To sum up:

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\[
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\begin{array}{c|ccc}
\ast & e & a^2 & a^4 \\
\hline
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  a^2 & a^2 & a^4 & e \\
  a^4 & a^4 & e & a^2 \\
\end{array}
\]

A cyclic group of order 3.
We also have

\[ e \neq e \neq e \neq 3 \]

This is a cyclic group of order 2.

One more. Here is the Klein 4-group \( V \). I'll remind you of its multiplication table.

<table>
<thead>
<tr>
<th>I</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>X</td>
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\hline
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\hline
 I & I & X & Y & Z \\
X & X & I & Z & Y \\
Y & Y & Z & I & X \\
Z & Z & Y & X & I \\
\end{array}
\]
What are the proper subgroups of $V$? Well any subgroup $H$ has to have the identity, so $I \in H$. But there has to be another element. Suppose it's $X$. The set $\{I, X\}$ is a cyclic group of order 2, so it's a subgroup. Similarly, $\{I, Y\}$ and $\{I, Z\}$ are both subgroups.

Can there be more? If a subgroup $H$ has two of $\{X, Y, Z\}$, say $X, Y$, then it must have $X \ast Y = Z$, so it's all of $V$.

We've found that $V$ has five subgroups, three of which are proper. $\{I\}$, $\{I, X\}$, $\{I, Y\}$, $\{I, Z\}$, $\{I, X, Y, Z\}$. 
What are the proper subgroups of $V$? Well any subgroup $H$ has to have the identity, so $I \in H$. But there has to be another element. Suppose it’s $X$. The set $\{I, X\}$ is a cyclic group of order 2, so it’s a subgroup. Similarly, $\{I, Y\}$ and $\{I, Z\}$ are both subgroups.
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$$\{I\}, \ {I, X}, \ {I, Y}, \ {I, Z}, \ {I, X, Y, Z}.$$
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Two hints for later in the semester. We’ll show that if $H$ is a subgroup of $G$, then $|H| | |G|$, that is, the order of $H$ divides the order of $G$. Since $|V| = 4$, this will tell us automatically that any proper subgroup of $V$ has an order dividing 4, but not equal to 1 or 4, so it has to be 2, as we’ve seen.
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Suppose $(G, \ast)$ is a group and $g \in G$. Suppose $m$ is the smallest integer so that $g^m = e$. Then $m$ is called the order of $m$. We define the subgroup generated by $g$ to be
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\[
\langle g \rangle = \{e, g, g^2, \ldots, g^{m-1}\}
\]

We’ll show that for any group \( G \) and \( g \in G \), \( \langle g \rangle \) is a subgroup of \( G \). This was the case for \( C_6 \), since \( \{e, a^2, a^4\} = \langle a^2 \rangle \) and \( \{e, a^3\} = \langle a^3 \rangle \).
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Two hints for later in the semester. We’ll show that if $H$ is a subgroup of $G$, then $|H| | |G|$, that is, the order of $H$ divides the order of $G$. Since $|V| = 4$, this will tell us automatically that any proper subgroup of $V$ has an order dividing 4, but not equal to 1 or 4, so it has to be 2, as we’ve seen.

Suppose $(G, *)$ is a group and $g \in G$. Suppose $m$ is the smallest integer so that $g^m = e$. Then $m$ is called the order of $m$. We define the subgroup generated by $g$ to be

$$\langle g \rangle = \{e, g, g^2, \ldots, g^{m-1}\}$$

We’ll show that for any group $G$ and $g \in G$, $\langle g \rangle$ is a subgroup of $G$. This was the case for $C_6$, since $\{e, a^2, a^4\} = \langle a^2 \rangle$ and $\{e, a^3\} = \langle a^3 \rangle$. We will also show that if $G = \langle a \rangle$ is a cyclic group of order $n$ and $H = \langle a^k \rangle$, then $|H| = n/gcd(n, k)$. But that’s for Wednesday and for the homework.
Finally, here’s the proof that \(((\mathbb{Z}/n\mathbb{Z})^*, \circ)\) is a group.
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Finally, associativity is automatic, because multiplication in \(\mathbb{Z}\) is associative.
I showed you \((\mathbb{Z}/10\mathbb{Z})^*, \odot)\) earlier. How about \((\mathbb{Z}/8\mathbb{Z})^*, \odot)\)?

Which of \(\{1, 2, 3, 4, 5, 6, 7\}\) are relatively prime to 8? Well, \(D(8) = \{1, 2, 4, 8\}\), so we're looking at odd numbers:

\[
(\mathbb{Z}/8\mathbb{Z})^* = \{1, 3, 5, 7\}.
\]

How does the multiplication go mod 8?

\[3 \cdot 3 = 9 \equiv 1 \text{ mod 8},\]
\[3 \cdot 5 = 15 \equiv 7 \text{ mod 8},\]
\[3 \cdot 7 = 21 \equiv 5 \text{ mod 8},\]
\[5 \cdot 5 = 25 \equiv 1 \text{ mod 8},\]
\[5 \cdot 7 = 35 \equiv 3 \text{ mod 8},\]
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\]

This leads to this multiplication table. (I've written "\(a\)" for \([a]_8\)].

\[
\begin{array}{c|cccc}
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\hline
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\end{array}
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Bruce Reznick  University of Illinois at Urbana-Champaign  Math 417 – Fourth Day
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\]

We see that 1 and 4 are their own inverses and \( 2 \cdot 3 = 1 \). In fact, \( 2 \) has order 4: the powers of 2 are
\[
2^0 = 1 \mod 5,
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So the powers of 2 give \( ((\mathbb{Z}/5\mathbb{Z})^*, \cdot) \), and it's isomorphic to \( \mathbb{C}_4 \).

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