Math 417 – First Day

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August 24, 2020

Hello, and welcome to Math 417. This first lecture is intended to be very informal. No proofs and no "official" definitions, but an introduction to some of the ideas we'll be working with. And, as a warning: these lectures won't be filled with fancy imagery or video or sound effects: I'm not so good at that, and the mathematics is more important, anyway.

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The mathematical objects I'll be talking about today are motions of objects, like rotations and flips, as well as what happens to the last digits of integers when you write them out in base 10 and multiply them. These are introductions to permutations and to arithmetic "mod n". Everything I talk about will be given a more careful definition later.

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2. A slice of pizza.

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2. A slice of pizza.
3. A quesadilla.

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1. A grilled cheese sandwich.
2. A slice of pizza.
3. A quesadilla.
4. A toasted bagel with cream cheese.

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These are obviously different foods, and if you ordered one of them at a restaurant and got another, you would not be happy.

On the other hand, they have a common structure: a hot bread-like object at the bottom and melted cheese. The breads are all different, and some also have bread on top, or sauces, but these are still somehow in the same category.

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I hope the interpretation of this table is clear. We're going to be using such tables a lot. (I found the coding format at overleaf.com. If you want to do a paper in LaTeX, I highly recommend this site.)

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Look familiar?

I'll put the two tables together:

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These are very similar. In fact, if you define the function Φ so that Φ(Combine)= Plus, Φ(Nothing)=Even and Φ(Flip)=Odd, then Φ exactly maps the first table to the second. The inverse function Φ^{-1} would map the second table to the first.

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We will say (with a more formal definition later) that these two situations are "isomorphic", and Φ is the isomorphism.

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Let T define a clockwise rotation by $\frac{\pi}{2}$. Using what I hope is an obvious notation, let T^2 denote two instances of T, which amounts to a clockwise rotation by π and let T^3 denote three instances of T, which amounts to a clockwise rotation by $\frac{3\pi}{2}$, or a counterclockwise rotation by $\frac{\pi}{2}$. I don't have to define anything else. Why? Because if I do four instances of T, or T^4, then it's rotation by 2π, which is as if I did nothing at all.

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Some more notation. Usually, when we do nothing we refer to it either as "I" or the identity, as if we were multiplying matrices, or "e", which means the same thing when objects are lower case. So $T^4 = I$ (the identity), and also, by convention $I = T^0$, or doing T zero times, which is the same as doing nothing.

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Notice that T^5 would be rotation by $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$, so doing five rotations is like doing one rotation and $T^5 = T$, etc.

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Mult	I	T	T^2	T^3
I	I	T	T^2	T^3
T	T	T^2	T^3	I
T^2	T^2	T^3	I	T
T^3	T^3	I	T	T^2

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T^2	T^2	T^3	I	T
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A nice cyclic pattern. In fact, we will soon see that the set $\{I, T, T^2, T^3\}$ is an example of what is called a **cyclic group of order 4** and denoted C_4.

If you've seen arithmetic "mod 4", you might notice that, since $T^4 = I$, $T^5 = T$ and $T^6 = T^2$, *multiplication* in this table is like *addition* of the exponents of T mod 4.

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0	0	1	2	3
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2	2	3	0	1
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The wrap-around pattern of the second table has a rhythm that we'll see a lot of. I hope you can imagine what a table of size $n \times n$ would look like. In fact, if $n = 10$, what it would look like is the addition table for the usual decimal expression of the last digits of integers.

Let me give you a different instance of a C_4. One nice property of positive integers is that if you know the last digit of m and the last digit of n, then you know the last digit of mn. (This follows from the multiplication algorithm you're taught in school, but it is also true if you look at number is bases other than 10, as we'll soon see.) For example, anything ending in "3" times anything ending in "7" will end in "1".

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Now look at the powers of 3: $3^0 = 1$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$, $3^8 = 6561$, we see a repeating pattern in the last digit: 1,3,9,7,1,3,9,7,1,... .

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This suggests consideration of a multiplication table for all integers "ending" in 1, 3, 7, 9 in base 10.

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I hope you see that the second table is the same as the "$\{0, 1, 2, 3\}$" table we saw a few pages ago, except that the names are changed. Also notice that if we take the "subtable" only involving $\{1, 9\}$, we get a table like the cyclic groups of order two:

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$*$	1	9
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9	9	1

One final example of motions. Now allow two different motions of our square S: X is rotation around a vertical axis through the center of the square, so that 1 and 2 flip and 3 and 4 flip, and Y is rotation around a horizontal axis, so that 1 and 4 flip and 2 and 3 flip. (Try to guess what happens when you do X followed by Y or Y followed by X.)

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Yes, XY and YX are the same as T^2, a rotation by π. This is not too surprising: both X and Y flip front to back, so doing them twice keeps the front in front, which means we have a rotation.

Notice also that if you do any of these rotations twice, you get back to doing nothing, which I'll call I again. Here's the multiplication table, where I've written $Z = XY$:

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Also notice that this is *not* like the C_4 we saw before: every element squares to I (look at the main diagonal!) and there is no element whose powers give the whole set.

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For historical reasons, this is called V, the *Klein four-group*, after Felix Klein (1849-1925): V is *not* isomorphic to C_4.

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Consider all rotations and flips of a square. It turns out that there are eight of them, and the multiplication is *not* commutative. (That is, not *abelian*.) Check this out yourself: see what happens when you rotate a square by $\frac{\pi}{2}$ and flip on a vertical axis, or do it in the other order. We'll spend a lot of time with this situation later. It leads to what is called D_4 or the *dihedral group of order eight*.

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Now consider a regular polygon with n vertices. The rotations of the n-gon give a nice example of C_n, the cyclic group with n elements, and its rotations and flips comprise D_n, the dihedral group with $2n$ elements, which is *not* abelian.

Because $\{1, 3, 7, 9\}$ are the numbers less than 10 which have no common factor with 10, we'll give their group of multiplication the fancy name of $(\mathbb{Z}/10\mathbb{Z})^*$, and also look at the analogous set $(\mathbb{Z}/n\mathbb{Z})^*$ for any positive integer n. It takes a little work, but we'll show that this is always a group too.

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In case $n = p$ is a prime number, it turns out that there are $p - 1$ elements in $(\mathbb{Z}/p\mathbb{Z})^*$, and it is a non-trivial theorem that they form a cyclic group. If you've had Math 453, this is what the "primitive root" is all about.

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Everything I've talked about today is an example of a group. If we considered the "one's place" decimal digit addition, together with multiplication, we would have an example of a ring. Strange things can happen in a ring: for example $4 * 5 = 0$ in this ring, even though neither 4 nor 5 equals 0. Other examples of rings include polynomials.

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Thank you for lasting this far. We've actually made it through the first day.