Notes on repeating decimals. Math 417 11/23/19

1. Suppose $x = .259259259\ldots$

or if you prefer, $x = \sum_{i=1}^{\infty} \frac{a_i}{10^i}$

$a_1 = a_4 = a_7 \cdots = 2$, $a_2 = a_5 = a_8 \cdots = 5$

$a_3 = a_6 = a_9 = a_{12} \cdots = 9$.

Then

$$1000x = 259.259259\ldots$$

(subtract) $\quad x = .259259\ldots$

$$999x = 259 \Rightarrow x = \frac{259}{999} = \frac{7\cdot 37}{27\cdot 37} = \frac{7}{27}.$$

What's going on?

Another way. $x = 259 \cdot (.001001001\ldots)$

And: $.001001001\ldots = \frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \ldots = \sum_{k=1}^{\infty}\left(\frac{1}{10^3}\right)^k$.

You should remember from Calculus(!) that $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$ (we are starting at $k=1$, not $k=0$), so the sum above is

$$\frac{\frac{1}{10^3}}{1-\frac{1}{10^3}} = \frac{1}{10^3-1} = \frac{1}{999} \quad (\text{same thing!})$$

2. In general, if x is a repeating decimal, repeating after n steps, then x is a multiple of $.00\ldots \underset{n^{th}}{1} 0 \ldots \underset{2n^{th}}{1} 0 \ldots \underset{3n^{th}}{1} \ldots$

$$= \frac{1}{10^n} + \frac{1}{10^{2n}} + \cdots = \frac{\frac{1}{10^n}}{1-\frac{1}{10^n}} = \frac{1}{10^n-1}.$$

Thus $x = \frac{c}{10^n-1}$.

3. But you're really interested in the other question:

If $x = \frac{a}{b}$ (in lowest terms), when can we write $x = \frac{c}{10^n-1}$?

So, suppose $\gcd(a,b)=1$ (lowest terms)

$\frac{a}{b} = \frac{c}{10^n-1} \iff bc = a(10^n-1)$, so $\gcd(a,b)=1$ and $b \mid a(10^n-1)$

This implies that $b \mid 10^n-1$, or, that $10^n - 1 \equiv 0 \mod b$,

so $10^n \equiv 1 \mod b$.. But when can that happen?

Theorem If $\gcd(r,d)=1$, then there exist n so that $\boxed{r^n \equiv 1 \bmod d}$ (This is similar to a later group theory result)

Proof.

Consider the set $\{1, r, r^2, \ldots, r^d\}$. There are $d+1$ numbers here. Now look at $1 \bmod d$, $r \bmod d$, $r^2 \bmod d$, $\ldots$, $r^d \bmod d$. These are $d+1$ residue classes but there are only d residue classes mod d! So by the pigeonhole principle, there exists $i<j$ $0 \le i < j \le d$ so that $r^i \equiv r^j \bmod d$, or $d \mid r^j - r^i = r^i(r^{j-i}-1)$

If $i=0$ we're done: $d \mid r^j - 1$, and $r^j \equiv 1 \bmod d$.

If $i>0$, then $d \mid r(r^{i-1}(r^{j-i}-1))$. But $\gcd(r,d)=1$ so $d \mid r^{i-1}(r^{j-i}-1)$. Again, if $i=1$, we're done. We can keep peeling off factors of r until we get to $r^{i-j} \equiv 1 \bmod d$.

5 So... if $\gcd(b,10)=1$, then there exists n so that $10^n \equiv 1 \bmod b$, and so, any fraction $\frac{a}{b}$ (in $(0,1)$) will have a completely periodic decimal expansion.

6. Lots of questions remain: what is the shortest n? What if $\gcd(b,10)$ isn't equal to 1. Oh yeah,

→ When is $\gcd(b,10)=1$? If $g \mid 10$, then $g = 1, 2, 5$ or 10, so what we want is that $2 \nmid b$, $5 \nmid b$, $10 \nmid b$. But $2 \mid b$ if the last decimal digit of b is in $\{0,2,4,6,8\}$ and $5 \mid b$ if the last decimal digit is in $\{0,5\}$. We want this not to happen, so the last digit is in $\{1,3,7,9\}$.

7. Bonus fun fact. Suppose $\gcd(b,d)=1$ and $10^n \equiv 1 \bmod b$ and $10^n \equiv 1 \bmod d$. Then $10^n - 1 = b \cdot c = d \cdot e$ for some integers c, e. And, $bc = de$, $\gcd(b,d)=1$, so $b \mid de$, hence $b \mid e$, $e = b \cdot k$. Finally, $10^n - 1 = d \cdot bk \Rightarrow 10^n \equiv 1 \bmod bd$!