1. Suppose \( x = 0.259259259 \ldots \). Then
\[
1000x = 259.259259259 \ldots - 0.259259259 \ldots \quad \text{(Subtract)} \quad x = 0.259259259 \ldots
\]

\[
999x = 259 \quad \Rightarrow \quad x = \frac{259}{999} = \frac{737}{2797} = \frac{7}{2797}
\]

What's going on?

Another way: \( x = 259.001001001 \ldots \)

And:
\[
0.001001001 \ldots = \frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^{9}} + \ldots = \sum_{k=1}^{\infty} \left(\frac{1}{10^3}\right)^k
\]

You should remember from calculus(!) that \( \sum_{k=1}^{\infty} x^k = \frac{x}{1-x} \) (if we are starting at \( k=1 \), not \( k=0 \)), so the sum above is:
\[
\frac{\frac{1}{10^3}}{1 - \frac{1}{10^3}} = \frac{1}{10^3 - 1} = \frac{1}{999} \quad \text{(same thing!)}
\]

2. In general, if \( x \) is a repeating decimal, repeating after \( n \) steps, then \( x \) is a multiple of \( \frac{1}{10^n} \).

\[
\frac{1}{10^n} + \frac{1}{10^{2n}} + \ldots = \frac{\frac{1}{10^n}}{1 - \frac{1}{10^n}} = \frac{1}{10^n - 1}
\]

Thus: \( x = \frac{c}{10^n - 1} \).

3. But you're really interested in the other question:

If \( x = \frac{a}{b} \) (unabout terms), when can we write \( x = \frac{c}{10^n - 1} \)?

So, suppose \( \gcd(a, b) = 1 \) (unabout terms)

\[
\frac{a}{b} = \frac{c}{10^n - 1} \quad \Rightarrow \quad b \cdot 10^n - 1 = c \cdot \gcd(a, 10^n)
\]

This implies that \( b \div 10^n - 1 \), or, that \( 10^n - 1 \equiv 0 \mod b \),

so \( 10^n \equiv 1 \mod b \). But when can that happen?
Thereom: If \( \text{gcd}(r, d) = 1 \), then there exist \( n \) so that \( r^n \equiv 1 \mod d \).

This is similar to a Euler-type result.

Proof:
Consider the set \( \{1, r, r^2, \ldots, r^{d-1}\} \). There are \( d+1 \) numbers here. Now look at \( 1 \mod d \), \( r \mod d \), \( r^2 \mod d \), \ldots, \( r^{d-1} \mod d \). These are \( d+1 \) residue classes, but there are only \( d \) residue classes modulo \( d \). So by the pigeonhole principle, there exists \( c \) \( 0 \leq c \leq d \)

so that \( r^c \equiv r^d \mod d \) or \( d | r^c - r^d = r^c(r^{c-1} - 1) \).

If \( c = 0 \), we're done; \( d | r^0 - 1 \) and \( r^0 = 1 \mod d \).

If \( c > 0 \), then \( d | r^c - r^d = r^c(r^{c-1} - 1) \). But \( \text{gcd}(r, d) = 1 \), so \( d | r^{c-1} - 1 \). Again, if \( c = 1 \), we're done. We can keep peeling off factors of \( r \) until we get to \( r^{c_0} = 1 \mod d \).

5 So... if \( \text{gcd}(b, 10) = 1 \), then there exists \( n \) so that \( 10^n \equiv 1 \mod b \), and so any fraction \( \frac{a}{b} \) (in \( \mathbb{Q} \)) will have a completely periodic decimal expansion.

6. Lots of questions remain: what is the shortest \( n \)? What if \( \text{gcd}(b, 10) \) isn't equal to 1? Oh yeah.

→ When is \( \text{gcd}(b, 10) = 1 \)? If \( b \equiv 0 \mod 10 \), then \( q = 1, 2, 5 \), so what we want is \( b \equiv 1, 3, 7, 9 \mod 10 \). But 2 and 6 need the last decimal digit of \( b \) is in \( \{0, 2, 4, 6, 8\} \) and \( 5 | b \) if the last decimal digit is in \( \{0, 5\} \). We can tangle this not to happen, so the last digit is in \( \{1, 3, 7, 9\} \).

7. Bonus fun fact: Suppose \( \text{gcd}(b, d) = 1 \) and \( 10^n \equiv 1 \mod b \) and \( 10^n \equiv 1 \mod d \). Then \( 10^n - 1 = b \cdot c = d \cdot e \) for some integers \( c, e \). And, \( b \cdot c = d \cdot e \), so \( c = b \cdot k \). Hence \( b \cdot c = d \cdot e \). Finally, \( 10^n - 1 = b \cdot k \Rightarrow 10^n \equiv 1 \mod b \).