1. My example is \( S = (0, 1) \cup (2, 3) \)

   \[
   \begin{pmatrix}
   1 & 2 & 3 \\
   \frac{\pi}{4} & b & \theta \\
   \end{pmatrix}
   \]

   (1) \( S \) is open because it is a union of two open intervals, which are open.

   (2) \( S \) is not an interval, because \( \frac{1}{2}, \frac{5}{2} \in S, \frac{1}{2} < \frac{3}{2} \leq \frac{5}{2} \), and \( \frac{3}{2} \notin S \).

   (3) Suppose \( x, y \in S \) and \( x < y \).

   There exist \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \subseteq S \).

   Let \( \varepsilon_0 = \min (\varepsilon, y - x) \).

   Then \( \varepsilon_0 < \varepsilon \) so \( x + \varepsilon_0 \in S \) and \( x + \varepsilon_0 < y \), so \( y \in (x + \varepsilon_0, \infty) \).

   Hence \( y > x + \frac{\varepsilon}{2} \).

   And so \( x < x + \frac{\varepsilon_0}{2} < y \in S \).

   Thus, \( S \) is not an interval. This problem was inspired by an error I saw in a book. It's a fun exercise!

2. \( S \) is an open ball.

   \( 7 = 1.0 + b(5) \).

   So, for \( x \in S \), \( x \leq 7 \).

   If \( x \leq S \), then there exists \( \delta > 0 \)

   Such that \( (7 - \varepsilon, 7 + \varepsilon) \subseteq S \).

   Thus, \( \delta \) contradicts \( S \) being an open ball.

   For any \( \delta > 0 \), say \( \delta = \frac{1}{n} \).

   \( 7 - \varepsilon = 7 - \frac{1}{n} \) is not an open ball.

Thus, there exists \( x_n, 7 - \frac{1}{n} < x_n < 7 \)

with \( x_n \in S \).

This tells us

(i) \( x_n \to 7 \).

(ii) \( x_n \to 7 \).

(There are clearly many different \( x_n \).

(3) Sort of like this. Let \( A_1 = (1, 2) \cup (3, 4) \), \( A_2 = (0, 3) \cup (7, \infty) \).

\( A_1 \cap A_2 = (1, 2) \cup (3, 4) \).

\( A_2 = [0, 3] \cup [7, \infty) \).

Same thing as (1).

4. \( f(x) = 1 - \frac{x}{10} \).

   a. \( a_0 = 0, a_{n+1} = f(a_n), n \geq 0 \).

   So \( a_1 = 1 - \frac{0}{10} = 1, a_2 = 1 - \frac{1}{10} = 0.9 \)

   \( a_3 = 1 - \frac{0.9}{10} = 0.91 \).

   \( b. f(x_0) = x_0 \Rightarrow x_0 = 1 - \frac{x_0}{10} \Rightarrow x_0 \cdot 10 = 1 \Rightarrow x_0 = \frac{10}{11} \).

   \( a_{n+2} - a_{n+1} = \frac{1 - a_{n+1}}{10} - \frac{1 - a_n}{10} = \frac{a_n - a_{n+1}}{10} \).

   An upper bound of \( a_1 \) is \( a_n \) if \( a_{n+1} - a_n > 0 \).
Since \( x_n \) is an accumulation point of \( S \), there exists \( y \in S \) such that \( |y - x_n| < \varepsilon \).

1. \( |y - x| = |(y - x_n) - (x - x_n)| \)
2. \( \leq |y - x_n| + |x - x_n| < \delta + \frac{\varepsilon}{2} \)
3. \( \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \)

And we need to show \( x \neq y \), but \( |y - x| < \delta \leq |x - x| \)

So that's why we did it that way.

2. Let \( f(m) = (\sum_{i=1}^{2n} \frac{1}{3^n}) \)

\[ f(m) = \sum_{i=1}^{2n} \frac{1}{3^n} = m \frac{2}{3^n} \]
\[ \sum_{i=1}^{2n} \frac{1}{3^n} = m \frac{2}{3^n} \]

If \( 10^6 \leq 9 \),

(picture).

There are \( 10^6 + 1 \) points points in \( 10^6 \) little squares, so there exists at least one little square with two points.

Thus, assume \( x \neq x \). Then by the hypothesis, \( x \) is an accumulation point.

(i) If any \( x_n = x \), then by the hypothesis, \( x \) is an accumulation point.

(ii) Thus, assume \( x \neq x \).

Since \( x_n \to x \), there exists \( N \) such that \( |x_n - x| < \frac{\varepsilon}{2} \) for all \( n \geq N \).

Pick one such \( x_n \).

Now let \( \delta = \min\{\frac{\varepsilon}{2}, |x_n - x|\} \)

Since \( x_n \neq x \), \( |x_n - x| > 0 \) and \( 0 < \delta \leq \frac{\varepsilon}{2} \).

Choose \( \varepsilon > 0 \).