1. \( f: A \to B \) \( g: B \to C \): \( 905: A \times \)

2a. Suppose \( f \) is not injective.
Then there exist \( x, y \in A \), \( x \neq y \)
such that \( f(x) = f(y) \). Hence
\[
g(f(x)) = g(f(y))
\]

2b. Suppose \( g \) is not surjective. Then there exists \( c \in C \) so that
\( g(b) \neq c \) for all \( b \in B \).
Could there exist \( a \in A \) s.t.
\((g \circ f)(a) = c\) ? If so,
\( g(f(a)) = c \) \( \Rightarrow f(a) \) would be such a \( b \). Thus \( g \circ f \) is not surjective.

2c. \( A \)
\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
& 5 & \Rightarrow & \\
& & & 6 \\
\end{array} \]

\( f(1) = 3 \)
\( f(2) = 4 \)
\( f(3) = 6 \)
\( f(4) = 5 \)
so \( f \) is injective, not surjective.
\( f(a) = 5 \)

2d. \( g \) is surjective, but not
injective, \( g(4) = 7 \), \( g(5) = 7 \)
so \( g \circ f \) is a bijection.

3a. \( f(m) = n \), so \( f(m_1) = f(m_2) \)
\( \Rightarrow m_1 = m_2 \), so \( f \) is injective.

but \( \frac{1}{2} \in A_\uparrow \), so \( A \) is not
he \( A \) is s.t., \( f(n) = \frac{1}{2} \)

2. \( g \left( \frac{b}{q} \right) = \left( \frac{a}{q} \right)^2 \)

If \( g \left( \frac{b}{q} \right) = g \left( \frac{b'}{q'} \right) \), then
\( \frac{b}{q} = \frac{b'}{q'} \), so \( b = b' \), \( q = q' \)

But uniqueness of prime factorization \( \Rightarrow p = p' \), \( q = q' \), so \( \frac{a}{q} = \frac{a'}{q'} \).
On the other hand, \( 3 \) cannot be
written as \( 2^a q \), so \( 3 \neq g(a) \).

Let's consider integers like this:

8a. The first proof builds on what we've

8b. If \( A \subseteq B \subseteq \mathbb{N} \), then \( \#(B) \)

8c. Suppose \( |B| = k \). There are \( \binom{n}{k} \)
subsets of size \( k \) in \( A \).

For each \( B \), there are \( 2^k \)

8d. \( \binom{n}{k} \) possible subsets \( A \subseteq B \), hence the answer
is \( \sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2) = 3^n \)

8e. The second proof looks at a
particular element \( x \); it can be
in (i) not in \( A \), not in \( B \)
(ii) in \( A \), in \( B \)
(iii) in \( A \), not in \( B \)

8f. Induction. For each pair \( (A, B) \)
def \( \mathcal{A} = A \setminus \{1, \ldots, n\} \), \( \mathcal{B} = B \setminus \{1, \ldots, n\} \)
Assume there are \( 3^{n-1} \) pairs \( (A, B) \),

then each \( \mathcal{A} \) \( A = \mathcal{A} \), \( B = B \)
(\text{i}) \( A = A \), \( B = B \)
(\text{ii}) \( A = A \), \( B = B \)
(\text{iii}) \( A = A \), \( B = B \)

so \( 3 \cdot 3^{n-1} = 3^n \) choices for \( \mathcal{A}, \ldots, \mathcal{N} \).
4. Follow the hints:

If \( f : X \to Y \)
such that \( A \cap f(X) \) ad be \( f(X) \).
Then \( f(A) \) but for \( f(A) \) and \( f(B) \)
ie, \( f(A) \subseteq C \cup f(B) \).

By Inclusion-Exclusion:
The number \# of such \( f \) is \( |A\cup B| - |A| - |B| + |A \cap B| \).

\[ |A\cup B| - |A| - |B| + |A \cap B| \]

\[ = |U| - (|A| - |B| + |A \cap B|) \]

\[ = 7 \] (The shaded region here)

What is \(|U|\)? \( f(1) = f(2) \) \( f(3) \).

To find \( f(i) = f(1), \ldots, f(26) \).

A case of 26 choices each.

\[ 26^4 \]

Similarly \( |A \cup B| = 25^4 \) because \( A \) is removed as a possible \( f(i) \) ad \( B \) is \( 25^4 \) and \( |B| = 24^4 \). \( f(i) = f(1), f(2), \ldots, f(26) \)

So there are \( 26^4 - 2 \cdot 25^4 + 24^4 \) functions.

If you try \( f(i) = a, f(j) = b \)
4 ways to pick \( i, \) 3 ways to pick \( j, \) and 26 ways to pick \( i \) and \( j \) 26^2 ways to pick.

New cases, you overcount 26^2 ways to pick \( i \) and \( j \) 26^2 ways to pick.

\[ 26^2 - 2 \cdot 25^2 + 24^2 = 7502 \]

(\( 4 \cdot 3 \cdot 26^2 \cdot 8112 \))

5. \( A = B = \{0, 1, 2, 3, \ldots\} \)
\( f: A \to B \) \( f(n) = n + 2 \)
\( g: B \to A \) \( g(n) = n + 3 \)

IF \( f(n) = f(n') \), then \( n + 2 = n + 1 \)
so \( n = n' \).

If \( g(n) = g(n') \), then \( n + 3 = n' + 3 \)
so \( n = n' \).

Thus \( f \) and \( g \) are both injective.

Here is a list of the \( n \) "pullbacks":

\[ \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\epsilon \\
\zeta \\
\eta \\
\theta \\
\iota \\
\kappa \\
\lambda \\
\mu \\
\nu \\
\xi \\
\omicron \\
\pi \\
\rho \\
\sigma \\
\tau \\
\upsilon \\
\phi \\
\chi \\
\psi \\
\omega \\
\end{array} \]

\[ \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
A \\
B \\
C \\
D \\
E \\
F \\
G \\
H \\
I \\
J \\
K \\
L \\
M \\
N \\
O \\
P \\
Q \\
R \\
S \\
T \\
U \\
V \\
W \\
X \\
Y \\
Z \\
\end{array} \]

So \( n \in A \) comes.

\( A = 0, 1, 2 \) \( B = 3, 4 \) \( A = 5, 6, 7 \)

5. \( \phi (0) = f(0) = 2 \) \( \phi (5) = f(5) = 7 \)
\( \phi (1) = f(1) = 3 \) \( \phi (6) = f(6) = 8 \)
\( \phi (2) = f(2) = 4 \) \( \phi (7) = f(7) = 9 \)
\( \phi (3) = f(3) = 0 \) \( \phi (8) = f(8) = 5 \)
\( \phi (4) = f(4) = 1 \) \( \phi (9) = f(9) = 6 \)

etc...
Great! This is what we want! Let's move it to the left of the decimal point.

1000 \times \pi = 3 \, 141.59 \, 2653...

So \( L_{1000 \times \pi} = 3 \, 141 \)

100 \times \pi = 3 \, 14.1592 \, 653...

So \( L_{100 \times \pi} = 3 \, 14 

ad 1 = 3 \, 141 - 3 \, 140 = 10 \times 10 \times \pi 

Does this work for e?

\( e = 2.71828 \ldots \)

1000 \times e = 2718.28 \ldots \Rightarrow L_{1000 \times e} = 27 \, 18

100 \times e = 271.82 \ldots \Rightarrow L_{100 \times e} = 271

But = 2718 - 10 - 2710

Claim \( f(x) = L_{1000 \times x} - 10 \times L_{100 \times x} \)

How to prove? Write x in decimal.

\( x = a_k \times 10^-k + \ldots + a_1 \times 10^-1 + a_0 + \frac{b_1}{10} + \frac{b_2}{10^2} + \frac{b_3}{10^3} \ldots \)

(not all b's > 9 from the 0,1

So \( 1000x = a_k \times 10^-k + \ldots + a_1 \times 10^-1 + a_0 \times 10 + b_1 \times 10^2 + b_2 \times 10^3 + b_3 \times 10^4 + \ldots \)

Now \( 0 \leq b_k \leq 9, \) so

\( 0 \leq \frac{b_k}{10^2} \leq \frac{9}{10} + \frac{9}{10^2} = \frac{91}{100} \)

and it's never equal to 1, because that would mean all b's are 9.

Thus \( L_{1000 \times x} = a_k \times 10^-k + \ldots + a_1 \times 10^-1 + a_0 \times 10 + b_1 \times 10^2 + \frac{b_2}{10} + b_3 \)

Similarly, \( b_k = L_{10 \times x} - 10 \times L_{x} \)

If you replace 10 with 2, you get the unusual new sequence.