Usual rules. Questions on homework accepted until noon on Nov. 8. The earlier, the better.
The symbol ({\mathcal{E}}) indicates that a problem is either an old test question or could easily have
been on a test if it hadn’t been an old homework problem. No extra credit this week.

1. ({\mathcal{E}}) Suppose \((a_n)\) is a Cauchy sequence in \(\mathbb{Q}\). For \(n \in \mathbb{N}\), let \(b_n = a_n^2\). Prove that 
\((a_n) \sim (b_n)\). If you have internalized the definitions of a Cauchy sequence, this is pretty
easy.

2. This problem uses calculus. You can use anything that you already know from there. A
point \(x_0\) is called a fixpoint of a function \(f\) if \(f(x_0) = x_0\). Suppose \(I = (x_0 - c, x_0 + c)\) is an
open interval containing a fixpoint \(x_0\) of a continuously differentiable function \(f\) and suppose
there exists a rational number \(r, 0 < r < 1\), such that \(|f'(x)| \leq r < 1\) for all \(x \in I\). Let \(a\) be
a point in \(I\) and define a sequence \((a_n)\) by: \(a_0 = a\) and \(a_{n+1} = f(a_n)\) for \(n \geq 0\). Prove that
\[
\lim_{n \to \infty} a_n = x_0.
\]
Hints: I recommend that you think about the mean value theorem as you look at the ratio
\[
\frac{a_{n+1} - x_0}{a_n - x_0} = \frac{f(a_n) - f(x_0)}{a_n - x_0}.
\]
You should first try to prove that \(a_n \in I\) implies \(a_{n+1} \in I\).

3., 4. Let
\[
g(x) = \frac{2x}{x+1}.
\]
a. Determine the fixpoints of \(g\). (Hint: it’s a plural.)
b. As in #2., let the sequence \((a_n)\) be defined by \(a_0 = X \in \mathbb{R}\) and \(a_{n+1} = g(a_n)\). Compute
\(a_1, a_2, a_3\) in terms of \(X\).
c. Formulate and prove a conjecture about a “closed form” for \(a_n\) as a function of \(n\) and \(X\).
It should look simple.
d. Use your answer in c. to describe completely the behavior of the sequence \((a_n)\) as \(n \to \infty\).
This will depend on \(X\) but, in a less obvious way than you might think at first. Don’t worry
about \(\infty\), but see f. below.
e. Determine the set on which \(|g'(x)| < 1\). This example shows that the criterion in #3 does
not completely answer the question of convergence.
f. There is a small lie in this problem. For each integer \(m\) there exists a value \(X = X_m\)
which makes \(a_m = -1\), so that “\(a_{m+1} = \infty\)”.
Determine \(X_m\). (In more advanced courses, we simply say that \(g(\infty) = 2\), which is true in a limiting sense.)
5. (E) Suppose \((a_n)\) and \((b_n)\) are two sequences of real numbers (not necessarily Cauchy or convergent). Suppose \(|a_n| < 2\) for all \(n\) and \(|b_n| < 17\) for all \(n\). Prove that there is a common convergent subsequence; that is, there exists a subsequence \((m_k)\) so that \((a_{m_k})\) and \((b_{m_k})\) are both convergent. Hint: You know that there exists \(n_k\) so that \((a_{n_k})\) is convergent by Lemma 3.6.10. Can you say that the sequence \((b_{n_k})\) is bounded? What does that mean?

6. (E) Consider the sequences
\[
(a_n) = (1, 1 + \frac{1}{3}, 1 + \frac{1}{3} + \frac{1}{9}, 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}, \ldots)
\]
and
\[
(b_n) = (u, u, u, u, \ldots),
\]
where \(u\) is a fixed rational number. That is,
\[
a_n = \sum_{k=0}^{n} \frac{1}{3^k}, \quad b_n = u.
\]
Determine a value for \(u = u_0\) so that \((a_n) \sim (b_n)\). The word “limit” should not appear in your answer; instead, I want you to examine \(a_n - b_n\) carefully. Also pick a value of \(u \neq u_0\) and show that \((a_n) \not\sim (b_n)\).

Feel free to use formulas you may know about the values of the terms \(a_n\) without having to prove them.