Recall the instructions from the Course Organization and feel free to work together in trying to understand the problems. I want you to write up the solutions individually, however. The symbol (E) indicates that a problem is either an old test question or could easily have been on a test if it hadn’t been an old homework problem.

I will answer questions about this homework in class and by email, but only up to Noon on Sun. Oct. 25 due. The website comments will be finalized by 4pm on that day.

1., 2.: This problem counts for 2 points. Do this problem without using the term “limits”. Throughout, the number r is rational and 0 < r < 1.

a. Prove by induction that for a non-negative integer n,

\[ 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}. \]

b. Using a., suppose that m, n are non-negative integers, m ≤ n, and find a similar formula for

\[ r^m + r^{m+1} + \cdots + r^n. \]

c. Using your formula from b. only, and not your previous knowledge of the geometric series, prove that

\[ r^m + r^{m+1} + \cdots + r^n < \frac{r^m}{1 - r}. \]

d. Let

\[ A_r := \{ r^n : n \in \mathbb{N} \cup \{0\} \}. \]

Let L be the greatest lower bound for \( A_r \). Prove that L = 0. There are two things to do: you need to prove that 0 is a lower bound for \( A_r \), which is pretty easy. But you also have to show that \( L > 0 \) is impossible. I suggest the following. Observe that if \( L > 0 \), then \( L < \frac{L}{r} \). What does this say about \( \frac{L}{r} \) with regards to the set \( A_r \)? What does that say about \( L \)?

3. (E) Let \( \alpha = 2^{1/2} + 7^{1/2} \). Prove that \( \alpha \) is an algebraic number, by finding a non-zero polynomial \( p \in \mathbb{Z}[x] \) with the property that \( p(\alpha) = 0 \). Hint: consider \( \alpha^2 \).

4. (E) Prove that if \( m, n \in \mathbb{N} \) and

\[ \left| \frac{m}{n} - \frac{4}{7} \right| < \frac{1}{10n}, \]

then \( \frac{m}{n} = \frac{4}{7} \).

Please turn over!
5. and 6. This problem counts for 2 points. The inevitable Fibonacci number question, now with bijections! Recall that the sequence \( (F_n) \) is defined by:

\[
F_0 = 0, \quad F_1 = 1; \quad \text{for } n \geq 0, \quad F_{n+2} = F_n + F_{n+1}.
\]

Fix an integer \( m \geq 2 \). Let \( X = \mathbb{Z}_m \times \mathbb{Z}_m \) and define the map \( f : X \to X \) by: \( f(a, b) = (b, a+b) \). Everything is reduced mod \( m \). The following picture shows what \( f \) looks like when \( m = 3 \). Arrows have the usual meaning.

\[
\begin{align*}
(0, 0) & \rightarrow (1, 0) \rightarrow (1, 2) \rightarrow \cdots \\
(1, 0) & \rightarrow (0, 0) \rightarrow (2, 0) \rightarrow \cdots \\
(2, 1) & \leftarrow (2, 2) \leftarrow (0, 2)
\end{align*}
\]

a. Prove that \( f \) is a bijection on \( X \) and compute \( g = f^{-1} \). That is; show that \( f \) is one-to-one and onto; remember that \( g(a, b) = (c, d) \) just means that \( f(c, d) = (a, b) \).

b. Define a sequence \( (x_n) \) by \( x_0 \in X \) and \( x_n = f(x_{n-1}) \). That is, \( x_0, f(x_0), f(f(x_0)), \ldots \). Use the Pigeonhole Principle to explain why there must exist integers \( k \) and \( \ell \), \( 0 \leq k < \ell \leq n^2 \), so that \( x_k = x_\ell \).

c. Use parts a. and b. to show that \( x_0 = x_{\ell-k} \).

d. Take \( x_0 = (0, 1) \). Use the preceding to show that there exists an integer \( r \) (depending on \( m \) of course) with \( 1 \leq r \leq m^2 \) and the property that \( m \) divides the \( r \)-th Fibonacci number \( F_r \). Thus, for example, there exists a Fibonacci number which is a multiple of 2015; Mathematica says that

\[
F_{210} = 2015 \times 17125544943343564360858629794243622373352
\]

7. (½ point; please recall instructions not to collaborate on these!)

Backwards engineering from the last exam. Find (with proof) values of real numbers \( r, s, t \) which make the following statement true: The sequence \( (a_n) \) defined by:

\[
a_0 = 1, \quad a_1 = r; \quad \text{for } n \geq 0, \quad a_{n+2} = sa_{n+1} + ta_n.
\]
satisfies the following equation for all \( n \geq 0 \):

\[
a_0^2 + a_1^2 + \cdots + a_n^2 = 2a_na_{n+1} + 3.
\]