The first exam will officially be based on material covered on the first three homeworks, and not this one. There may of course be some overlaps of ideas, but if it wasn’t on the first 3 homeworks (e.g. #4, 5, 6), it won’t be on this test. I have to save something for the second test. No extra credit this time. Study for the test!

Recall the instructions from the Course Organization and feel free to work together in trying to understand the problems. I want you to write up the solutions individually, however.

I will answer questions about this homework in class and by email, but only up to Noon on Tue. Oct. 6 due. The website comments will be finalized by 4pm on that day.

1. (E) You’ve seen parts of this before! For each of the parts of this problem, we have \( f : A \to B \), \( g : B \to C \) and the composition map \( g \circ f : A \to C \). You should assume that \( f, g, A, B, C \) in each part are not necessarily related to each other.

   a. If \( f \) is not injective, prove that \( g \circ f \) is not injective.

   b. If \( g \) is not surjective, prove that \( g \circ f \) is not surjective.

   c. Give an explicit example of \( f, g, A, B, C \) for which \( f \) is injective (but not surjective), \( g \) is surjective (but not injective) but \( g \circ f \) is a bijection. Hint: think about finite sets. I know an example where \( A = \{1, 2\}, B = \{3, 4, 5\} \) and \( C = \{6, 7\} \).

2. (E) Let \( \mathbb{Q}_+ \) denote the set of positive rational numbers, with elements written in lowest terms as \( \frac{p}{q} \), where \( p, q \in \mathbb{N} \). Define functions

   \[
   f : \mathbb{N} \to \mathbb{Q}_+, \quad f(n) = n; \\
   g : \mathbb{Q}_+ \to \mathbb{N}, \quad g\left(\frac{p}{q}\right) = 2^p 5^q.
   \]

   a. Prove that \( f \) is an injection and not a surjection (easy).

   b. Prove that \( g \) is an injection and not a surjection. You should use the uniqueness of prime factorizations of integers. I don’t see the names Schröder or Bernstein in this problem. No bijection is requested!

3. If \( X = \{1, 2\} \), then there are exactly 9 pairs of sets \( (A, B) \) such that \( A \subseteq B \subseteq X \):

   \[
   (\emptyset, \emptyset), \quad (\emptyset, \{1\}), \quad (\emptyset, \{2\}), \quad (\emptyset, \{1, 2\}), \quad (\{1\}, \{1\}), \quad (\{1\}, \{1, 2\}), \quad (\{2\}, \{2\}), \quad (\{2\}, \{1, 2\}), \quad (\{1, 2\}, \{1, 2\}).
   \]

   Prove that if \( X = \{1, \ldots, n\} \), then there are exactly \( 3^n \) pairs of sets \( (A, B) \) such that \( A \subseteq B \subseteq X \). You can prove this by suitably modifying the proofs given earlier for the size of \( P(A) \). I know a proof by induction and I know a proof by defining a suitable function on \( X \) which takes various values at \( x \), depending on whether \( x \in A \) or \( x \in B \). Maybe you can find others.

Please turn over!
4. \((\mathcal{E})\) How many functions \(f\) are there from \(X = \{1, 2, 3, 4\}\) to the standard English alphabet \(Y = \{a, \ldots, z\}\) \((|Y| = 26)\) with the property that \(a, b \in f(X)\)? Please note that by “a” and “b”, I mean specifically the first two letters of the English alphabet. Hint: you might want to consider the sets of functions

\[
U = \{f : X \to Y\}, \quad A = \{f : X \to Y \setminus \{a\}\}, \quad B = \{f : X \to Y \setminus \{b\}\},
\]

as well as the Principle of Inclusion and Exclusion.

5. Suppose \(A = B = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}\). We define \(f : A \to B\) by \(f(n) = n + 2\) and \(g : B \to A\) by \(g(n) = n + 3\).

a. (Easy) Prove that \(f\) is an injection and prove that \(g\) is an injection.

b. The Schröder-Bernstein Theorem gives a construction for a bijection \(\Phi\) (my letter, it’s not named in the book) from \(A\) to \(B\). Using this construction, compute \(\Phi(n)\) for \(n = 0, 1, 2, 3, 4, 5, 6, 7\). It’s ok to use diagrams to explain your answer here.

6. Sally 1.7.30 (This is asking for an explicit formula for the third digit after the decimal point.) So, if \(x = \pi = 3.14159\ldots\), then \(f(x) = 1\); if \(y = e = 2.71828\ldots\), then \(f(y) = 8\). Understanding \(f\) might help you understand binary representations better.