Math 347  Homework 3  Due TBD (Class vote)

Recall the instructions from the Course Organization and feel free to work together in trying to understand the problems. I want you to write up the solutions individually, however. The symbol \((\mathcal{E})\) indicates that a problem is either an old test question or could easily have been on a test if it hadn’t been an old homework problem.

I will answer questions about this homework in class and by email, but only up to Noon on the day before it is due. The website comments will be finalized by 4pm on that day.

1. Find a bijection from \(\mathbb{N}\) to \(\mathbb{Z}\) with the following shape:

\[
f(n) = \begin{cases} 
A(n), & \text{if } n = 0 \mod 4 \\
B(n), & \text{if } n = 1 \mod 4 \\
C(n), & \text{if } n = 2 \mod 4 \\
D(n), & \text{if } n = 3 \mod 4.
\end{cases}
\]

Here, \(A(n), B(n), C(n)\) and \(D(n)\) are simple linear functions of \(n\). You are encouraged to modify Exercise 1.7.22. There are many correct solutions! Remember that \(\mathbb{N} \neq \mathbb{Z}\).

2. \((\mathcal{E})\) Suppose \(A\) and \(B\) are countably infinite sets and \(A \cap B = \emptyset\). Give a partition \(A \cup B = C_1 \cup C_2 \cup C_3\), where each of the \(C_i\)'s is a countably infinite set and \(C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \emptyset\). There are many different ways to do this. Your answer should clearly explain how the \(C_i\)'s satisfy each given condition.

3. \((\mathcal{E})\) Let \(Y = \{(m, n) : m, n \in \mathbb{Z}, 0 \leq n \leq m\}\). That is, \(Y\) is the consists of all points with integer coordinates lying in the pizza-slice-shaped portion of the first quadrant lying on or below the line \(y = x\). Give an explicit enumeration which shows that \(Y\) is a countable set. Your enumeration should be explicit enough that you can write down the 10th element. There are many possible correct solutions!

4. \((\mathcal{E})\) Continuation of HW2 #4 Suppose \(A, B, C\) are sets and \(f, g\) are functions \(f : A \to B, \ g : B \to C\), with composition \(g \circ f : A \to C\). Except for this notation, the conditions given in one part of this problem do not apply in the other parts of the problem. In the two questions, by “an example”, I mean: name the elements of \(A, B, C\) and state what the functions do.
   a. If \(f\) is injective and \(g\) is surjective (no other information given), give an example showing that \(g \circ f\) might not be surjective.
   b. If \(f\) is injective and \(g\) is surjective (no other information given), give an example showing that \(g \circ f\) might not be injective.

Please turn over!
5. (Ε) The inevitable Fibonacci number question. The sequence \((F_n)\) is defined by:
\[
F_0 = 0, \quad F_1 = 1; \quad \text{for } n \geq 0, \quad F_{n+2} = F_n + F_{n+1}.
\]
Prove by Mathematical Induction that for all \(n \geq 1\),
\[
F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.
\]

6. Sally Exercise 1.8.14. You may assume the Fundamental Theorem of Arithmetic; that is, every positive integer \(n\) can be written as a product of primes \(p_1 < \cdots < p_k\)
\[
n = p_1^{m_1} \cdots p_k^{m_k}, \quad m_k \in \mathbb{N}
\]
in a unique way. (Notice that the primes are written in increasing order.) Ask about this if it is unclear.

7. Extra-credit (\(\frac{1}{2}\) point; please recall instructions not to collaborate on these!)
This problem is actually not very hard if you think about it right.
Let \(X\) be the set \(\mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}\). We are going to try to define a version of \(\mathbb{Q}\) (plus infinity) by using the ordered pairs method and allowing \(\frac{a}{0}\) when \(a \neq 0\). This will get us pretty far, but not far enough.

Define a relation \(\mathcal{R}\) on \(X \times X\) by
\[
( (a, b) , (c, d) ) \in \mathcal{R} \iff ad = bc.
\]
This looks very much like the definition in the book and in class, except that we are allowing \(b = 0\) and/or \(d = 0\) as long as \(a \neq 0\) and/or \(c \neq 0\).

(i) Prove that \(\mathcal{R}\) is an equivalence relation. Since we have already proved this for \((a, b)\) with \(b \neq 0\), there not too much to prove.

(ii) If \(c \neq 0\), prove that \(( (a, b) , (c, 0) ) \in \mathcal{R}\) if and only if \(a \neq 0\) and \(b = 0\).

Now we can think about the class containing \((1, 0) = \frac{1}{0} = \infty\) and throw it into our definition of \(\mathbb{Q}\) using this equivalence relation.

(iii). Use the same definitions for addition and multiplication as in the definition of \(\mathbb{Q}\) in the book to prove that there are elements \((a, b)\) and \((c, d)\) in \(X\) for which \((1, 0) + (a, b)\) and \((1, 0) \cdot (c, d)\) do not live in \(X\) (think of \(\infty - \infty\) and \(\infty \cdot 0\)); that is, the closure axioms fail. Also prove that \((1, 0)\) does not have an additive inverse or a multiplicative inverse.

Moral: Even good ideas sometimes don’t work!