Let \( d > 2 \) be an integer. We've done this for \( d = 2, 3, 10 \), but any integer will do.

1. I want to show that any integer \( n > 0 \) can be written as
   \[
   n = \sum_{i=0}^{k} a_i d^i, \quad a_i \in \{0, 1, \ldots, d-1\} \text{ for some } k.
   \]

There are two approaches. The first starts with \( a_0 \).
Define \( a_0 \) to be the integer in \( \{0, 1, \ldots, d-1\} \) for which
   \[
   n \equiv a_0 \mod d.
   \]
Then \( n = a_0 + d n_1 \), for some integer \( n_1 \).
and \( n_1 = \frac{n-a_0}{d} \). Then define \( a_1 \) to be the integer for which \( n_1 \equiv a_1 \mod d \) (0 \( \leq a_1 < \)). Repeat until
   \[
   n_k = \frac{n}{d^k} < d. \text{ In the last step } n_k = a_k \text{ and } n_k = 0.
   \]
The other approach starts with \( a_k \). Let \( k+1 \) be the smallest integer so that \( d^{k+1} > n \). Then \( n > d^k \).

Let \( a_k = \left\lfloor \frac{n}{d^k} \right\rfloor \), so \( 0 \leq a_k \leq d-1 \). Then
   \[
   n = a_k d^k + n^{(1)}, \quad \text{since } \frac{d^{k+1}}{d^k} = \frac{d}{d-1} > 1, \quad \text{we have } 0 \leq n^{(1)} < 1, \text{ and we can repeat.}
   \]

Example: \( d = 8 \) \( n = 347 \)

(a) \( 347 \equiv 3 \mod 8 \), \( 347 = 3 + 8 \cdot 43 \), \( 43 = 3 + 8 \cdot 5 \), \( 5 = 5 + 80 \).
   \[
   347 = 3 + 8(3 + 8(5 + 8(0))) = 3 + 3 \cdot 8 + 5 \cdot 8^2 \leq
   \]

(b) \( 347 \leq 512 = 8^3 \), \( \frac{347}{8} = \frac{347}{64} = \frac{5 + \frac{27}{64}}{1} \), so
   \[
   347 = 5 \cdot 8^2 + 27 \quad \frac{27}{8} = 3 + \frac{3}{8}, \text{ so } 27 = 3 + 8 \cdot 3
   \]
   \[
   347 = 5 \cdot 8^2 + 3 \cdot 8 + 8.
   \]

The same thing!
(ii) Suppose $x$ is a real number. Take $x$ if necessary to get $x > 0$. Write $x = \chi_1 + 3\chi_2$, where $\chi_1 \in \mathbb{N}$ and $0 < 3\chi_2 < 1$.

Write $\chi_1 = \sum_{\sigma=0}^{k} a_{-\sigma} \chi_{-\sigma}$, $a_{-\sigma} \in \mathbb{Z}, -2 \leq \sigma \leq 0$ as on (i).

I want to show how to write

$$x = \frac{a_{-1}}{d} + \frac{a_{-2}}{d^2} + \frac{a_{-3}}{d^3} + \ldots + \frac{a_{-k}}{d^k}$$

There's only one idea here. Let $u_0 = 3\chi_2 \in [0, 1)$

$d\,u_0 \in [0, d)$, so $d\,u_0 \in \mathbb{Z}$, $d - 1 < d$

$d\,u_0 = a_{-1} + u_1(1)$, $u_1 \in (0, 1)$

This means that $\chi_1 = u_0 = \frac{d\,u_0}{d} = \frac{a_{-1}}{d} + u_1$

$3\chi_2 = \frac{a_{-1}}{d} + \frac{1}{d}\,u_1$ (where $u_1 \in (0, 1)$).

You just keep repeating

$$x = \frac{a_{-1}}{d} + d\left(\frac{a_{-2}}{d^2} + \frac{u_2}{d}\right) = \frac{a_{-1}}{d} + \frac{a_{-2}}{d^2} + \frac{u_2}{d}, \text{ etc.}$$

Example: $x = \frac{14}{21}$, $d = 8$. $0 \leq \frac{14}{21} < 1$

$8\chi = \frac{8\chi}{21} = 4 + \frac{4}{21} \Rightarrow x = \frac{4}{8} + \frac{1}{8}\left(\frac{4}{21}\right)$

$u_1 = \frac{4}{21}$, $8\,u_1 = \frac{32}{21} = 1 + \frac{14}{21}$, so...

$$\frac{1}{21} = \frac{4}{8} + \frac{1}{8}\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{21}\right) = \frac{4}{8} + \frac{1}{8^2} + \frac{1}{8^2 \cdot 21}$$

This repeats: $\frac{14}{21} = \frac{4}{8} + \frac{1}{8^2} + \frac{4}{8^3} + \frac{1}{8^4} + \ldots$

Check: $\text{RHS} = \left(\frac{4}{8} + \frac{1}{8^2} + \ldots\right) \left(1 + \frac{1}{8^2} + \frac{1}{8^4} + \ldots\right) = \frac{33}{64} - \frac{1}{82} = \frac{33}{64} - \frac{1}{82} = \frac{14}{21}$. 


On algebraic numbers

A number \( \alpha \) is algebraic if there exists a non-zero polynomial \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \) \((a_i \in \mathbb{Z})\) such that \( p(\alpha) = 0 \); i.e., \( \sum_{k=0}^{n} a_k \alpha^k = 0 \).

Example (1) \( a_1 \alpha + a_0 = 0 \Rightarrow \alpha = -\frac{a_0}{a_1} \); \( \alpha = \frac{m}{n} \Rightarrow p(x) = nx - m \).

Given \( a_2 x^2 + a_1 x + a_0 \Rightarrow \alpha = -\frac{a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \), so after we write \( a_2^2 - 4a_0 a_2 = d^2 \cdot m \) where \( d, m \in \mathbb{Z} \) and \( m = 1 \) product of primes, \( \alpha = c_1 \pm c_2 \sqrt{m} \) for integers \( c_1, c_2, c_3 \).

Conversely, such \( \alpha \) satisfies \((c_3 \alpha - c_1)^2 = m \cdot \alpha^2 \), and this gives a quadratic:

A minimal polynomial of \( \alpha \) is a polynomial \( p \) such that \( p(\alpha) = 0 \) and the degree of \( p \) is as small as possible.

Thus if \( p(x) = 9 x^2 - 4 \), then \( p(\frac{2}{3}) = 0 \), but \( \frac{2}{3} \) isn't the minimal polynomial, because \( q(\frac{2}{3}) = 0 \) for \( q(x) = 3x - 2 \).

Note: If \( p \) is a minimal polynomial for \( \alpha \), then \( p'(\alpha) \neq 0 \).

(Why? If \( p'(\alpha) = 0 \), then \( \deg p' = \deg p - 1 \), so \( p \) isn't minimal.)

Theorem: Suppose \( \alpha \in \mathbb{C} \) is an algebraic number and the degree of a minimal polynomial of \( \alpha \) is \( n \). Then there exists \( \varepsilon > 0 \) and \( C \) so that if \( \frac{a}{b} \) is any rational number, then

\[
\left| \frac{a}{b} - \alpha \right| < \varepsilon \Rightarrow \left| \frac{a}{b} - \alpha \right| > \frac{C}{b^n}.
\]

In other words, if \( \frac{a}{b} \) is close to \( \alpha \), then it can't be too close.

(This was \( \frac{a}{b} = \frac{1}{6} \), and then I remembered the minimal polynomial... )
Fine Point: It can be shown in an algebra class, hence 425
in 4.7. That is \( \frac{a}{b} < c \) and \( P\left(\frac{a}{b}\right) = 0 \), then \( P(x) = (bx-a)q(x) \)
where \( q(x) \) is also a polynomial with integer coefficients.
Thus, if \( P \) is a minimal polynomial for \( c \), then \( P\left(\frac{a}{b}\right) = 0 \).
Because \( P\left(\frac{a}{b}\right) = 0 \), then \( P(x) = (bx-a)q(x) \) and \( q(x) = 0 \) as
well, but \( q \) has a smaller degree.

\[ P\left(\frac{a}{b}\right) = \sum_{k=0}^{n} a_k \cdot \left(\frac{a}{b}\right)^k = \frac{a_0 b^n + a_1 b^{n-1} b + \cdots + a_n b^0}{b^n} \]

So the numerator is \( \neq 0 \), hence \( \left| P\left(\frac{a}{b}\right) \right| > \frac{1}{b^n} \).

Now let's use the mean value theorem. Suppose \( \left|\frac{a}{b} - d\right| < \varepsilon \)

\[ P\left(\frac{a}{b}\right) - P(d) = P'(\xi) \quad \text{where } \frac{a}{b} \text{ and } d, \quad \xi \in \left(\frac{a}{b}, d\right) \]

\[ |\frac{a}{b} - d| \leq |\frac{a}{b} - \xi| + |\xi - d| < \varepsilon. \]

\[ |P\left(\frac{a}{b}\right) - P(d)| = \left| P'(\xi) \right| |\frac{a}{b} - d| \quad \text{(From above)} \]

But \( P(d) = 0 \) and \( |P\left(\frac{a}{b}\right)| \geq \frac{1}{b^n} \), so

\[ |\frac{a}{b} - d| = \frac{|P\left(\frac{a}{b}\right) - P(d)|}{|P'(\xi)|} \geq \frac{1}{b^n} \cdot \frac{1}{|P'(d)|} \geq \frac{1}{b^n} \]

we conclude

because \( P'(d) \to 0 \)

Take \( C = \frac{1}{b^n} \) to complete the proof.