Pythagorean Theorem Algebra Proof

What is the Pythagorean Theorem?
You can learn all about the Pythagorean Theorem, but here is a quick summary:

\[ a^2 + b^2 = c^2 \]

Proof of the Pythagorean Theorem using Algebra
We can show that \( a^2 + b^2 = c^2 \) using Algebra.

Take a look at this diagram ... it has that "abc" triangle in it (four of them actually):

Area of Whole Square
It is a big square, with each side having a length of \( a+b \), so the total area is:

\[ A = (a+b)(a+b) \]
Area of The Pieces

Now let's add up the areas of all the smaller pieces:

First, the smaller (tilted) square has an area of \( A = c^2 \)

And there are four triangles, each one has an area of \( A = \frac{1}{2}ab \)
So all four of them combined is \( A = 4(\frac{1}{2}ab) = 2ab \)

So, adding up the tilted square and the 4 triangles gives: \( A = c^2 + 2ab \)

Both Areas Must Be Equal

The area of the large square is equal to the area of the tilted square and the 4 triangles. This can be written as:

\[
(a+b)(a+b) = c^2 + 2ab
\]

NOW, let us rearrange this to see if we can get the pythagoras theorem:

Start with:

\[
(a+b)(a+b) = c^2 + 2ab
\]

Expand \((a+b)(a+b): \)

\[
a^2 + 2ab + b^2 = c^2 + 2ab
\]

Subtract "2ab" from both sides:

\[
a^2 + b^2 = c^2
\]

DONE!

Now we can see why the Pythagorean Theorem works ... and it is actually a proof of the Pythagorean Theorem.

This proof came from China over 2000 years ago!

There are many more proofs of the Pythagorean theorem, but this one works nicely.
\[
\frac{y - 0}{x - (-1)} = \frac{y_0 - 0}{x_0 - (-1)} = t
\]
\[
y = t(x_0 + 1).
\]
\[
x_0^2 + y_0^2 = 1
\]
\[
\Rightarrow x_0^2 + (t(x_0 + 1))^2 = 1
\]
\[
\Rightarrow x_0^2 - 1 + t^2(x_0 + 1)^2 = 0
\]
\[
\Rightarrow (x_0 + 1) \left[ x_0 - 1 + t^2(x_0 + 1) \right] = 0
\]
\[
\Rightarrow (x_0 + 1) \left( (1 + t^2)x_0 + t^2 - 1 \right) = 0
\]
\[
\Rightarrow x_0 = -1, \quad x_0 = \frac{1 - t^2}{1 + t^2} \Rightarrow y_0 = t(x_0 + 1) = \frac{2t}{1 + t^2}
\]
Worksheet for the Pythagorean Circle.

\[ x_0 = \frac{1-t^2}{1+t^2} \quad y_0 = \frac{2t}{1+t^2} \]

\[ t = \frac{r}{s} \Rightarrow x_0 = \frac{1-(\frac{5}{s})^2}{1+(\frac{5}{s})^2} \quad y_0 = \frac{2(\frac{5}{s})}{1+(\frac{5}{s})^2} \cdot \frac{s^2}{s^2} \]

\[ 0 < \frac{r}{s} < 1 \Rightarrow x_0 = \frac{s^2-r^2}{s^2} \quad y_0 = \frac{2rs}{s^2+r^2} \]

\[(s^2-r^2)^2 + (2rs)^2 = (s^2+r^2)^2 \]

\[
\begin{array}{cccccc}
\frac{r}{s} & s & s^2-r^2 & 2rs & s^2+r^2 & r < s \\
1 & 2 & \frac{s^2-r^2}{3} & \frac{2rs}{4} & \frac{s^2+r^2}{5} & \\
1 & 3 & \_ & \_ & \_ & \\
2 & 3 & \_ & \_ & \_ & \\
1 & 4 & \_ & \_ & \_ & \\
3 & 4 & \_ & \_ & \_ & \\
1 & 5 & \_ & \_ & \_ & \\
2 & 5 & \_ & \_ & \_ & \\
3 & 5 & \_ & \_ & \_ & \\
4 & 5 & \_ & \_ & \_ & \\
\end{array}
\]

\text{Your Picks} \quad \_ \_ \_ \_ \_ 
\_ \_
\begin{verbatim}
In[12]:= TableForm@Table[i^2 + j^2, {i, 0, 10}, {j, 0, 10}]
TableForm[Table[Mod[i^2 + j^2, 4], {i, 0, 10}, {j, 0, 10}]]
\end{verbatim}

\begin{verbatim}
\begin{tabular}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{tabular}
\end{verbatim}
```math
TableForm[Table[
{Prime[n], Mod[Prime[n], 4]}, {n, 1, 25}]]
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Which primes \( p \) can be written as \( \alpha^2 + 6^2 \) where \( \alpha \), \( \beta \) are integers?