1. \( \delta 3.3 \quad -4a, 4c \)
I'll do these the 'direct' way

4a. \( T(z) = \frac{az + b}{cz + d} \)

\( T(1) = -1, \ T(i) = i \quad T(-1) = 1 \)

\( \frac{a + b}{c + d} = -1 \)

\( \frac{a + b}{c + d} = -1 \)

\( \frac{-a + b}{c + d} = 1 \)

\( a + b = c + d \quad a + b = c + d \)

\( -a + b = c + d \quad -a + b = c + d \)

\( a + b + c + d = 0 \)

\( -a + b + c + d = 0 \)

\( b + c = 0, \ a + d = 0 \Rightarrow c = -b, \ d = -a \)

b) \( a + b = i(c + i + d) \)

\( a + b = c + i \cdot d \)

\( a + b = b - ai \Rightarrow a = 0 \)

\( \Rightarrow A = 0, \ c = -b \)

\( T(z) = \frac{b}{b \sqrt{2}} = \frac{-1}{\sqrt{2}} \quad \text{(It checks!)} \)

4c. \( T(1) = 1 \quad T(0) = 0 \quad T(i) = 1 + i \)

\( \frac{a + b}{c + d} = 1, \ \frac{b}{d} = 0, \ \frac{a + b}{c + d} = 1 + i \)

\( a + b = c + d \quad b = 0 \quad a + b = c + i \cdot d \)

\( \text{Since } d = 0, \ a = c + d \quad \text{and} \)

\( c + i \cdot d = (1 + i)(c + i \cdot d) \)

\( c + i \cdot d = (1 + i)c + (1 + i)d \)

\( \Rightarrow -c + d = 0 \Rightarrow c = d \)

\( a = c + d = x \)

\( T(z) = \frac{2z^2}{2z^2 + 2z + 1} \)

\( (T(i) = \frac{2}{2}, \ T(0) = 0), \ T(1) = \frac{2i}{1 + 1} = i + i \quad \text{It checks!} \)

There are other valid ways
to do this

2. \( \text{Let } T(z) = \frac{az + b}{cz + d} \)

If \( T(i) = 1 \quad \text{and} \quad T(-1) = -1 \)

Then \( \frac{a + b}{c + d} = 1 \quad \text{and} \quad \frac{-a + b}{c + d} = -1 \)

So \( a + b = c + d \quad \Rightarrow \quad a + b = c + d \)

\( b + c = 0, \ a + d = 0 \Rightarrow c = -b, \ d = -a \)

So \( c = b, \ d = a \)

\( T(z) = \frac{az + b}{bz + a} \)

Two cases: If \( a = 0 \), then \( T(z) = \frac{b}{bz + a} \)

and \( T(0) = \infty \)

If \( a \neq 0 \), divide both numerator and denominator by \( a \) \( T(z) = \frac{\frac{2}{b} + \frac{a}{a}}{z + \frac{1}{b}} \)

We have \( T(0) = \frac{b}{a} = 0 \quad \text{and} \quad \frac{2}{b} + \frac{a}{a} = 1 \)

\( T(0) = \frac{2}{b} + \frac{a}{a} = 1 \)

12. Suppose

\( T(p) = p, \ T(q) = q \quad \text{p,q} \)

Let \( S \) be any nondiagonal matrix

\( S(1) = p \quad S(-1) = q \)

(there are a lot of these)

Then \( S \) is an S transform.

Then \( S^{-1}(p) = 1, \ S^{-1}(q) = -1 \)

If \( U = S^{-1} T \) to \( S \)

Then \( U(1) = S^{-1}(T(S(1))) \)

\( = S^{-1}(T(p)) \)

\( = S^{-1}(p) = 1 \)

\( \text{and} \quad U(-1) = S^{-1}(T(S(-1))) \)

\( = S^{-1}(T(q)) = S^{-1}(q) = -1 \)

Due 11/4/10
\[
\begin{align*}
\theta(t) &= e^{(t^2 + 1)} \left[ \frac{1}{z^2} \left( \frac{z^2 - 1}{z^2 - 1} \right) + \frac{1}{2} \left( \frac{z^2 + 1}{z^2 - 1} \right) \right] \\
&\quad + \frac{i}{2} (\cos t + i) + \frac{1}{2} (\cos t - i) + \frac{1}{2} (\sin t + i) + \frac{1}{2} (\sin t - i)
\end{align*}
\]

4. Here we can use the

\[ f(t) = 2 \Rightarrow f(2) = 2 = e^{(1+1)} \]

\[ f'(2) = f(1+1) = e^{(1+1)} \cdot \ln 2 \]

\[ f''(2) = f'(1+1) = e^{(1+1)} \cdot (1+1) \ln 2 = e^{(1+1)} \cdot 2 \cdot \ln 2 \]

\[ f'(1) = f(1+i) = e^{(1+i)} \cdot \ln 2 = e^{(1+i)} \cdot (1+i) \ln 2 = e^{(1+i)} \cdot 2 \cdot \ln 2 \]

so

\[ f'(1) = f(1+i) = e^{(1+i)} \cdot \ln 2 \]

\[ f''(1) = f'(1+i) = e^{(1+i)} \cdot (1+i) \ln 2 = e^{(1+i)} \cdot 2 \cdot \ln 2 \]

\[ f'(1) = f(1+i) = e^{(1+i)} \cdot \ln 2 \]

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so

\[ f'(1) = f(1+i) = e^{(1+i)} \cdot \ln 2 \]

\[ f''(1) = f'(1+i) = e^{(1+i)} \cdot (1+i) \ln 2 = e^{(1+i)} \cdot 2 \cdot \ln 2 \]

so

\[ f'(1) = f(1+i) = e^{(1+i)} \cdot \ln 2 \]

\[ f''(1) = f'(1+i) = e^{(1+i)} \cdot (1+i) \ln 2 = e^{(1+i)} \cdot 2 \cdot \ln 2 \]

so

\[ f'(1) = f(1+i) = e^{(1+i)} \cdot \ln 2 \]

\[ f''(1) = f'(1+i) = e^{(1+i)} \cdot (1+i) \ln 2 = e^{(1+i)} \cdot 2 \cdot \ln 2 \]
\[ T(z) = \frac{z + i}{2 + z} \]

\( a \)

\[ T(0) = \frac{0 + i}{1} = i, \quad T(1) = \frac{1 + i}{2} = \frac{1}{2} + \frac{i}{2} \]

\[ T(-1) = \infty, \quad T(i) = \frac{i + 1}{1 + i} = \frac{1}{2} + \frac{i}{2}, \quad T(-i) = \frac{-i + 1}{1 - i} = 0 \]

\[ T(\infty) = 1, \quad T(1+i) = \frac{4 + 3i}{5} \]

\( B, C, D \) determine the line \( u = v \), which contains infinity.

The first quadrant is the intersection of the UHP and the RHP.

\( 4. \)

\[ T(1) = C, \quad T(0) = B, \quad T(\infty) = D \]

The line \( u = v \) is in the UHP, and \( i \to B \).

Thus, the UHP goes to the upper half-plane above + to the right of the line.

\( 12i \) goes to the circle through \( \text{ABDE} \).

\( i \) is in the right half-plane.

\( 1 \) goes to \( C \), which is inside the circle, and the RHP goes to the interior of the circle, and the mapped region is the intersection of the two.

As a check, \( 1+i \) is in the interior of the 1st quadrant, \( T(1+i) = \frac{4 + 1}{2} + \frac{3}{5} \frac{5}{5} = \frac{7}{5} > 1 \) (in right half).

\[ (\frac{4}{5} - \frac{1}{2})^2 + (\frac{3}{5} - \frac{1}{2})^2 = \frac{1}{10} < \frac{1}{2} \] (inside circle)

It works!
8. On |z| = 1
   \[ 1 \leq |f(z)| \leq 2 \]

   Let \( g(z) = (z + i)z^2 \)

   On |z| = 1, \( |g(z)| = |(z + i)z^2| = |z|^2 = 1 \)

   So by Rouché,
   \[ |g(z)| > |f(z)| \text{ on } |z| = 1 \]

   So \( g(z) \) and \( f(z) \) have the same number of zeros inside \( |z| = 1 \), counting multiplicity.

   \( g(z) \) has a double-zero at \( z = 0 \), hence

   \( h(z) = g(z) - f(z) \) has two zeros, counting multiplicity.

   But...
   \( h(z) = g(z) - f(z) = (2 + i)z^2 - f(z) \)

   and we know that \( h(z) \) has only one zero in all of \( \mathbb{C} \).

   Therefore, \( h \) has a double zero at \( z = 0 \).

   \( h(0) = 0 \), \( h'(0) = 0 \)

   \( h''(0) = 2(2 + i) \cdot 0 - f''(0) \)

   So \( f''(0) = 2(2 + i) \cdot 0 \)

9. The boxed one

   \[ f(z) = \frac{1}{z^2 - 1} \]

   Let \( g(z) = \frac{1}{(z - 1)(z + 1)} = \frac{1}{z^2 - 1} \)

   Then, as before, \( g \) has friendly removable singularities at \( z = \pm 1 \).

   \( g(0) = \frac{1}{1} = -4 \)

   Since \( g \) is analytic in \( |z| < 3 \),

   The maximum principle says that

   There exists \( z_0 \) with \( |z_0| = 2 \)

   and \( |g(z_0)| > 4 \). For that \( z_0 \)

   \[ |g(z)| = |(z^2 - 1)|g(z_0)| = |z^2 - 1| \cdot 1 \cdot |g(z)| \]

   Since \( |z^2 - 1| > 1 \), \( |z| = 3 \) and \( |g(z)| > 4 \)

   We have \( |f(z)| > 3 \cdot 4 = 12 \)

   If \( f(z) = 12i \), then \( g(z) = 4i \)

   So any analytic function \( g(z) \) s.t. \( g(0) = 4 \) and \( g(z) = 4i \)

   will do. You don't know what \( g \) \( z \) will be. E.g., \( g(z) = 4z + z^2 \)

10. First approach.

   \[ T(z) = \frac{az + b}{cz + d} \]

   \[ T(z_1) = \frac{a z_1 + b}{c z_1 + d} \]

   \[ T(z_2) = \frac{a z_2 + b}{c z_2 + d} \]

   \[ \Rightarrow \frac{a z_1 + b}{c z_1 + d} = z_1 \]

   \[ \Rightarrow \frac{a z_2 + b}{c z_2 + d} = z_2 \]

   \[ \Rightarrow a z_1 + b = c z_1 z_2 + d z_2 \]

   \[ \Rightarrow a z_2 + b = c z_1 z_2 + d z_1 \]

   Subtract to get

   \[ a (z_1 - z_2) = d (z_2 - z_1) \]

   \[ \Rightarrow d = -a \Rightarrow \]

   \[ a z_1 + b = c z_1 z_2 - a z_2 \]

   \[ \Rightarrow b = c z_1 z_2 - a (z_1 + z_2) \]
Third approach

As noted in class, any
Fractal linear transforation preserves the cross ratios
of 4 points so

\[
(\frac{z_1}{z_2} : \frac{z_3}{z_4}) = (T(z_1) : T(z_2) : T(z_3) : T(z_4)) = (\frac{z_2}{z_1} : \frac{z_4}{z_3} : \frac{w}{w_1})
\]

\[
\frac{z_1 - z_2}{z_1 - z_3} \cdot \frac{z_4 - z_3}{z_4 - z_2} = \frac{z_2 - z_1}{z_2 - z_4} \cdot \frac{w - w_1}{w - w_2}
\]

So

\[
\frac{w - w_2}{w - w_1} = \frac{\frac{z_1 - z_2}{z_1 - z_3} \cdot \frac{z_4 - z_3}{z_4 - z_2}}{\frac{z_2 - z_1}{z_2 - z_4} \cdot \frac{w - w_1}{w - w_2}} = \frac{\frac{z_1 - z_2}{z_1 - z_3}}{\frac{z_2 - z_1}{z_2 - z_4}} = \frac{z_1 - z_2}{z_1 - z_3} \cdot \frac{z_4 - z_3}{z_4 - z_2} \cdot \frac{w - w_1}{w - w_2}
\]

Thus \(T(z_2) = \frac{z_4}{z_3} \Rightarrow T(z_4) = z_3\)

Second approach:

Let \(S\) be any Möbius transformation, so that \(S(1) = \frac{z_1}{z_2}\) and \(S(1) = \frac{z_1}{z_2}\)

and let \(U = S^{-1} \circ T \circ S\)

Then

\[
U(1) = S^{-1}(T(S(1))) = S^{-1}(T(\frac{z_1}{z_2})) = S^{-1}(\frac{z_1}{z_2}) = \frac{1}{z_2}
\]

And similarly \(U(-1) = 1\)

So \(w\) is a point where \(f_g = f_H\) for \(g = f_H \circ f\). Let \(f_1 = f_H(\frac{z_1}{z_2})\).

Using the previous calculation,

\[
b = C \cdot 1(-1) - \alpha \cdot (1-1) = -C
\]

\[
T(2) = \frac{a \cdot z - c}{c \cdot z - a}
\]

\[
r_4 = \frac{a \cdot z - c}{c \cdot z - a}, \quad T(r_4) = \frac{a \cdot r_4 - c}{c \cdot r_4 - a} = 13
\]

Thus is a contradiction.