\[
\int_0^\infty \frac{x^\alpha}{x^2 + 3x + 2} \, dx, \quad 0 < \alpha < 1
\]

Recall \( C_1: z = e^{i\theta}, \quad 0 < \theta < 2\pi \),
\( C_2: z = Re^{i\theta}, \quad 0 \leq \theta \leq \pi, \quad R > 1 \),
\( C_3: z = e^{i(2\pi - \theta)}, \quad 0 \leq \theta \leq \pi \),
\( C_4: z = e^{i\theta}, \quad 0 < \theta < 2\pi \).

Note the signs on \( C_3, C_4 \) to signify counterclockwise direction.

\[
z^\alpha = e^{\alpha \log z}, \quad \text{where} \quad \log z = \ln|z| + i\operatorname{arg}(z)
\]

so if \( z = Re^{i\theta}, \quad z^\alpha = R^\alpha e^{i\alpha\theta} \).

On \( C_1: \quad \text{Length} = (2\pi - 2\delta) \cdot R \)
\[
|z^\alpha| = R^\alpha, \quad |z^2 + 3z + 2| \geq R^2 - 3R - 2.
\]

So
\[
\left| \int \frac{z^\alpha}{z^2 + 3z + 2} \, dz \right| \leq \frac{2\pi \cdot R \cdot R^\alpha}{R^2 - 3R - 2}
\]

\( \to 0 \) as \( R \to \infty \) because \( < 1 \).

On \( C_4: \quad \text{Length} = (\pi - 2\delta) \cdot R \)
\[
12^\alpha 1 = 3^\alpha, \quad 12^2 + 32 + 21 \geq 2 - 3R - 2^\alpha.
\]

So \( \frac{3^\alpha}{2 - 3R - 2^\alpha} \to 0 \) as \( \alpha \to -1 \).

So as \( \alpha \to 0 \), \( R \to \infty \), \( \delta \to 0 \).

We get
\[
\left( \int_0^\infty \frac{x^\alpha \, dx}{x^2 + 3x + 2} \right) \left( 1 - e^{i2\pi} \right)
\]

\( = 2\pi i \sum \text{Residues} \).

\( f(z) = \frac{2}{z+2} \) has poles at
\( z = -2, -1 \).

Since these are odd powers
\[
\text{Res}(f(z), -1) = \frac{e^{i\pi x}}{2(-1) + 3} = \frac{1}{1}.
\]

\( \text{Res}(f(z), -2) = \frac{-2^{x}}{2(-2) + 3} = \frac{e^{i\pi x} \cdot 2^x}{-1} \)

Putting it all together,
\[
\left( \int_0^\infty \frac{x^\alpha}{x^2 + 3x + 2} \right) = \frac{e^{i\pi (1 - 2^\alpha)} \cdot 2\pi i}{1 - e^{i2\pi}}
\]

Once again, since shows up multiplying top & bottom by \( e^{i\pi x} = -1 \)
\[
= \frac{2^{x-1} \cdot 2\pi i}{e^{i\pi x} - e^{-i\pi x}} = \frac{\pi (2^{x-1})}{2^{x} \sin \pi x} = \frac{\pi (2^{x-1})}{2^{x} \sin \pi x}.
\]

Way too long for a test!

and also valid for \( -1 < \alpha < 1 \).
Let \( u = -t \) so \( du = -dt \).

The integral over \( C_3 \) becomes

\[
\int_0^\infty \frac{4}{1+x^2} \left( \log x \right)^4 \, dx
\]

\[
= 2 \int_0^\infty \frac{(\log x)^4}{1+x^2} \, dx
\]

There is a simple pole

at \( z = i \), with

residue \( = \frac{(\log i)^4}{2i} = \frac{\pi^4}{2i} \).

Since \( |\log(ne^{i\theta})| = |\ln R + i\theta| \leq 1 \ln R + 10|\theta| \),

we may move the contour to \( C_1 \).

On \( C_2 \), \( 0 \leq \theta \leq \pi/2 \).

Since \( \ln R + i\pi/2 < c \cdot R \) for large \( R \), Thus \( \int_{C_2} \rightarrow 0 \) as \( R \rightarrow \infty \).

Similarly,

\[
\int_{C_2} \frac{(\log z)^4}{1+z^2} \, dz \leq \pi \cdot \frac{\pi^4}{2 - 3} = \frac{\pi^5}{16}
\]

and \( |\log(z)| = 1 \ln z \)

goes to \( 0 \) slower than any power of \( \varepsilon \), so \( |\varepsilon_0 \rightarrow 0 \).

Now the key part.

\( C_3 \) is in the wrong direction:

\[
\int_{C_3} \frac{4}{1+u^2} \, du
\]

If \( u < 0 \) is real, \( \log u = \ln(-u) + i\pi \).
#3 \[6 \times 3.1-2\]

\[e + f(z) = 2^4 - 3z^2 + 3\]

Cheat sheet:

\[f(2) = 0 \implies z^2 = \frac{3 \pm \sqrt{3}}{2}\]

\[= \frac{3 \pm \sqrt{3}}{2} i\]

\[= \sqrt{3} e^{\frac{\pi}{3}}\]

where \(e^{i\theta} = \frac{\sqrt{3}}{2} + \frac{i}{2}\) so \(\theta = \frac{\pi}{6}\)

and the roots are

\[3^{\frac{1}{4}} e^{\frac{i\pi}{12}}, 3^{\frac{1}{4}} e^{\frac{i\pi}{4}}, 3^{\frac{1}{4}} e^{\frac{3i\pi}{12}}\]

so there is one zero in each quadrant.

In particular.

\[p(x) = x^4 - 3x^2 + 3 = (x^2 - \frac{3}{2})^2 + \frac{3}{4} > 0\]

\[f(1) = 1^4 + 3 \cdot 1^2 + 3 = 1(3 + \frac{3}{2})^2 > 0\]

so \(p(0) = 3 > 0\)

as \(z\) goes from \(0\) to \(R\)

\(f(z)\) increases on the real line

from \(3\) to \(2^4 - 3R^2 + 3\).

If \(z = Re^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2}\)

\[f(z) = R^4 e^{4i\theta} - 3R^2 e^{2i\theta} + 3\]

\[= R e^{4i\theta} \left( 1 - \frac{3}{R^2} e^{-2i\theta} + \frac{3}{R^4} e^{-4i\theta}\right)\]

This factor will be close to 1 and any \(f(z)\) increases by \(2\pi\) as \(\theta\) goes from \(0\) to \(\frac{\pi}{2}\), midway up at \(f(1) = 2^4 + 3R^2 + 3\).

Then \(f(\theta) = \theta^4 + 3\theta^2 + 3\) monotonically drops from \(2^4 + 3R^2 + 3\) back to 3. Here's Mathematica for \(R = 3\)

![Graph of f(x) = x^4 + 3x^2 + 3](image)

The charge on a disk is \(2\pi\)

and \(\frac{\theta}{\pi} = 2\pi = 1\), so one zero.

This is way more than I'd expect on a homework paper.
4. Now they get shorter:
\[ f(z) = 1 \]
\[ g(z) = (z^2 + 1)(z^2 + 4)(z^2 + 9) \]
\[ \text{deg } f = 0 \quad \text{deg } g = 6 \geq 2 \quad \checkmark \]

The poles of \( g \) are all simple: at \( \pm i, \pm 2i, \pm 3i \), so by the residue theorem:
\[
\int_{C} \frac{f(z)}{g(z)} \, dz = 2\pi i \sum_{k=1}^{\infty} \text{Res}(f(z); k) \]
(The roots of \( g \) are \( i, 2i, 3i \)).

The residues are:
\[
\frac{1}{(z^2 + 1)(z^2 + 9)} \quad \text{at } z = i: \quad \frac{1}{2i} = \frac{1}{4i} \\
\frac{1}{(z^2 + 1)(z^2 + 4)} \quad \text{at } z = 2i: \quad \frac{-3i}{4i} = -\frac{3}{4i} = \frac{1}{6i} \\
\frac{1}{(z^2 + 1)(z^2 + 4)} \quad \text{at } z = 3i: \quad \frac{-8i}{6i} = -\frac{4}{3} = \frac{1}{2i} \\
\]

Answer: \( 2\pi i \left( \frac{1}{48i} - \frac{1}{60i} + \frac{1}{2i} \right) \)
\[ = 2\pi i \cdot \frac{5 - 4 + 1}{240i} = \frac{\pi}{60}. \]

5. If \( z \) is outside \( C \),
\( f(z) = 0 \) because
\[ \frac{e^{5z}}{(z - 2)} \quad \text{does not include any singularities}. \]

If \( z \) is inside \( C \), then
\[ \text{there is a pole of order 3 at } z \text{, and the residue is} \]
\[ \left( e^{5z} \right)^{3} \left|_{z = 2} \right. = \frac{25 \times e^{52}}{2!} \]

\[ \text{So } f(z) = \begin{cases} 
\frac{25}{2} e^{5z}, & \text{if } z \text{ is inside } C \\
0, & \text{if } z \text{ is outside } C 
\end{cases} \]

Note that \( f \) is analytic on \( \mathbb{C} \),
which is not connected, and has two components.

6. \[ \sum_{n=0}^{\infty} \frac{1}{3} \left( \frac{2}{3} \right)^{n} = 1 - \frac{2}{3} = \frac{1}{2} \]

Provided \( \left| \frac{2}{3} \right| < 1 \) or \( \left| 1 - \frac{2}{3} \right| < 1 \)
\[ \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{2}{3} \right)^{n} = \frac{1}{2} \cdot \frac{1}{1 - \frac{2}{3}} = \frac{2}{3} \]

Provided \( \left| \frac{2}{3} \right| < 1 \) or \( \left| 1 - \frac{2}{3} \right| < 1 \).

Thus, \( f(z) = \frac{3}{4} - \frac{2}{z} + \frac{2}{z - 2} \)

has the given Laurent series provided \( 2 < 1 - \frac{1}{1} < 1 \).

7. \( \log z = \ln |z| + i \theta \)

And so, if \( |z| = r \) and \( 0 \leq \theta \leq \pi \),
then
\[ |\log z| = \sqrt{(\ln r)^2 + \pi^2} \leq |\ln r| + \pi \]

Then \( \theta \) is okay.

If \( r \) is big,
\[ 1 + z^2 + 4 |z| \geq r^2 - 4 \]

If \( r \) is small,
\[ 1 + z^2 + 4 |z| \geq 4 z^2 - r^2 \]

are the relevant inequalities.

Putting these together,
on the next page.
\[ r = 10^3, \text{ you can take} \]
\[ M_1 = \pi \cdot 10^{-3} \frac{(\ln 10^3 + \pi)}{(10^{-3})^2 - 4 \cdot 8} \]
\[ M_2 = \pi \cdot 10^{-3} \frac{(\ln 10^3 + \pi)}{448 - (10^{-3})^2} \]
Note: \( M_2, \ln 10^{-3} = \ln 10^{-3} \).
\[ M_1 \approx 0.217, \quad M_2 \approx 0.00048 \]

8. \( f(z) = \frac{z}{z^4 + 4} \) has singularities when \( z^4 + 4 = 0 \Rightarrow z^4 = -4 \Rightarrow z = \pm \sqrt[4]{-4} \), of which only \( \pm 1 + i \) is inside \( C \).
These are simple poles.
\[ \text{Res} \left( f(z); 1+i \right) = \frac{1}{4(i+1)^3} = \frac{1}{8i} \]

b) \[ \int \frac{dx}{z^4 + 4} \]
\[ \text{For } C_{3R}, \text{ we note the direction and say } z = ti \]
\[ \int f(z)dz = \int (ti)^4i \, dt \]
\[ C_{3R} = \int_0^{R} \frac{e^{it}}{(2^4 + 4)} \, dt \]
\[ \text{So } d = 1, \text{ hey } 1 \in C! \]

C) \[ \int \frac{z}{z^4 + 4} \, dz \]
\[ \text{max } \left\{ \frac{12 \pi}{16} \right\} \]
\[ \text{and } \Phi(R) \to 0 \text{ because } 4 < 2 \]

\[ \text{so let } R \to \infty, \text{ we get} \]
\[ (1 + i) \int_0^\infty \frac{x \, dx}{x^4 + 4} = 2\pi i \cdot \frac{1}{8} \]

\[ \Rightarrow \int_0^\infty \frac{x \, dx}{x^4 + 4} = \frac{\pi}{8} \]

Do it directly: \( u = \frac{x^2}{2}, \quad x = j2u \).
Then \( 2u = x^2 \Leftrightarrow 2du = 2x \, dx \), so it becomes \( \int_0^\infty \frac{du}{u^2 + 1} = \int_0^\infty \frac{1}{u^2 + 1} \)