A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra

Victoria Powers
Department of Mathematics and Computer Science,
Emory University,
Atlanta, GA 30322

Bruce Reznick
Department of Mathematics,
University of Illinois at Urbana-Champaign,
Urbana, IL 61801

January 12, 2000

1 Introduction

Fix a positive integer n and let $\mathbf{R}[X] := \mathbf{R}[x_1, \ldots, x_n]$. We write Δ_n for the simplex $\{(x_1, \ldots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$.

Pólya's Theorem ([6], [4, pp.57-59]) says that if $f \in \mathbf{R}[X]$ is homogeneous and positive on Δ_n, then for sufficiently large N all the coefficients of

$$(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$$

are positive. In this note, we give an explicit bound for N and give an application to some special representations of polynomials positive on polyhedra. In particular, we give a bound for the degree of a representation of a polynomial positive on a convex polyhedron as a positive linear combination of products of the linear polynomials defining the polyhedron.

We use the following multinomial notation: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, let X^α denote $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Let us fix homogeneous $f \in \mathbf{R}[X]$ of degree d,

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$.

Our main theorem is:

Theorem 1. *Suppose that $f \in \mathbf{R}[X]$ is a form as above. If*

$$N > \frac{d(d-1)}{2} \frac{L}{\lambda} - d,$$

then $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ has positive coefficients.

Note in particular that the bound does not depend on n, the number of variables. This bound improves (by a factor of roughly $4n$) the bound in the paper [1], which in any case contains an error in the proof, see [2]. In [7, Ex. 3.5], we considered a special case equivalent to $f(x,y) = x^2 - (2-\delta)xy + y^2$, for which $L = \min\{1, 1 - \frac{2-\delta}{2}\} = 1$ and

$$\lambda = \min_{0\le t\le 1} (t - (1-t))^2 + \delta t(1-t) = \min_{0\le t\le 1} 1 - (4-\delta)t(1-t) = \frac{\delta}{4};$$

thus Theorem 1 gives $N > \frac{4}{\delta} - 2$. In fact, $(x+y)^N f(x,y)$ has positive coefficients precisely when $N \ge 2\lceil \frac{2}{\delta} \rceil - 3$, so Theorem 1 is sharp.

The authors thank Markus Schweighofer for bringing his unpublished work to our attention, and for useful comments and suggestions. The second author happily acknowledges useful conversations with Matthias Aschenbrenner, John D'Angelo, Zoltán Füredi and Doron Zeilberger.

2 The bound

In this section, we prove Theorem 1. We begin with some notation. For a positive number t, a non-negative integer m, and a single real variable x, define

$$(x)_t^m := x(x-t)\cdots(x-(m-1)t) = \prod_{i=0}^{m-1} (x - it)\,.$$

Note for later reference that

$$(ty)_t^d = \prod_{i=0}^{d-1} \bigl(ty - (i-1)t\bigr) = t^d (y)_1^d, \tag{1}$$

and if $m > n$ are both integers, then $(n)_1^m = 0$, since one of the factors in the definition is zero. It follows immediately that in the special case where $x = k/M$ and $t = 1/M$, where M is a positive integer, we have

$$\left(\frac{k}{M}\right)_{1/M}^m = \frac{1}{M^m} \prod_{i=0}^{m-1} (k-i) = \begin{cases} \frac{1}{M^m}\frac{k!}{(k-m)!} = \frac{m!}{M^m}\binom{k}{m}, & \text{if } m \le k; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We fix $f = \sum a_\alpha X^\alpha$ and suppose that $f > 0$ on Δ_n. We assume throughout that $d = \deg f > 1$; the $d = 1$ case is trivial. Following Pólya, we make the explicit computation:

$$(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n) =$$
$$\sum_{|\beta|=N} \frac{N!}{\beta_1! \cdots \beta_n!} x_1^{\beta_1} \cdots x_n^{\beta_n} \times \sum_{|\alpha|=d} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Write $\alpha \preceq \beta$ if $\alpha_i \le \beta_i$ for $1 \le i \le n$. For $|\beta| = N + d$, denote the coefficient of $x_1^{\beta_1} \ldots x_n^{\beta_n}$ in $(x_1 + \cdots + x_n)^N f(x_1, \ldots, x_n)$ by A_β. Then

$$A_\beta = \sum_{|\alpha|=d,\ \alpha\preceq\beta} \frac{N!}{(\beta_1-\alpha_1)!\cdots(\beta_n-\alpha_n)!} \cdot a_\alpha$$
$$= \frac{N!(N+d)^d}{\beta_1!\cdots\beta_n!} \sum_{|\alpha|=d,\ \alpha\preceq\beta} a_\alpha \prod_{\ell=1}^{n} \frac{\beta_\ell!}{(\beta_\ell-\alpha_\ell)!(N+d)^{\alpha_\ell}}.$$

We now express A_β using the $(x)_t^m$ notation and (2):

$$A_\beta = \frac{N!(N+d)^d}{\beta_1!\cdots\beta_n!}\sum_{|\alpha|=d} a_\alpha(\tfrac{\beta_1}{N+d})^{\alpha_1}_{(N+d)^{-1}}\cdots(\tfrac{\beta_n}{N+d})^{\alpha_n}_{(N+d)^{-1}}. \tag{3}$$

(If $\alpha \not\leq \beta$, then the extra terms added in (3) are just 0.) Still following Pólya, define

$$f_t(x_1,\ldots,x_n) := \sum_{|\alpha|=d} a_\alpha (x_1)_t^{\alpha_1}\cdots(x_n)_t^{\alpha_n}.$$

Clearly, $f_t \to f$ uniformly on Δ_n, so that for t sufficiently small, f_t is also positive on Δ_n. In view of the foregoing, this means that for N sufficiently large, and all β with $|\beta| = N+d$ (so that $f_{(N+d)^{-1}}$ is evaluated on Δ_n),

$$A_\beta = \frac{N!(N+d)^d}{\beta_1!\cdots\beta_n!} f_{(N+d)^{-1}}(\tfrac{\beta_1}{N+d},\ldots,\tfrac{\beta_n}{N+d}) > 0. \tag{4}$$

It follows that all the coefficients of $(x_1+\cdots+x_n)^N f(x_1,\ldots,x_n)$ are positive.

We now extend Pólya's work. Let us drop the constant factor in (4) and set $t = \frac{1}{N+d}$, $y_k = \frac{\beta_k}{N+d}$, and keep in mind that $\sum_k y_k = 1$. We have

$$f_t(y_1,\ldots,y_n) = f(y_1,\ldots,y_n) - \sum_{|\alpha|=d} a_\alpha \left(y_1^{\alpha_1}\cdots y_n^{\alpha_n} - (y_1)_t^{\alpha_1}\cdots(y_n)_t^{\alpha_n}\right).$$

Using the information about f, we see that

$$f_t(y_1,\ldots,y_n) \geq \lambda - L\sum_{|\alpha|=d}\frac{d!}{\alpha_1!\cdots\alpha_n!}\left|y_1^{\alpha_1}\cdots y_n^{\alpha_n} - (y_1)_t^{\alpha_1}\cdots(y_n)_t^{\alpha_n}\right| \tag{5}$$

If $\alpha_k > \beta_k$, then $(y_k)_t^{\alpha_k} = 0$, so $y_k^{\alpha_k} \geq (y_k)_t^{\alpha_k} \geq 0$ for all k; hence we may drop the absolute value in (5).

By the Multinomial Theorem,

$$\sum_{|\alpha|=d}\frac{d!}{\alpha_1!\cdots\alpha_n!} y_1^{\alpha_1}\cdots y_n^{\alpha_n} = (y_1+\cdots+y_n)^d = 1,$$

and by the iterated Vandermonde-Chu identity (see below for a proof),

$$\sum_{|\alpha|=d}\frac{d!}{\alpha_1!\cdots\alpha_n!}(y_1)_t^{\alpha_1}\cdots(y_n)_t^{\alpha_n} = (y_1+\cdots+y_n)_t^d = \prod_{k=0}^{d-1}(1-kt). \tag{6}$$

Thus by (5), we are done if we can show that

$$\lambda - L\big(1-(1-t)\cdots(1-(d-1)t)\big) \geq 0. \tag{7}$$

Suppose now that

$$t = \frac{1}{N+d} < \frac{2}{d(d-1)}\frac{\lambda}{L}.$$

It is easy to prove by induction that if $0 \leq w_j \leq 1$, then $\prod(1-w_j) \geq 1 - \sum w_j$. Since $\lambda \leq f(1,0,\ldots,0) \leq L$ and $d \geq 2$, we have $t < \frac{1}{d-1}$, hence

$$(1-(1-t)\cdots(1-(d-1)t)) < t(1+2+\cdots+(d-1)) = t\frac{(d-1)d}{2} < \frac{\lambda}{L},$$

and we are done.

What remains is to prove the iterated Vandermonde-Chu identity (reference thanks to Doron Zeilberger):

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (X_1)_t^{\alpha_1} \cdots (X_n)_t^{\alpha_n} = (X_1 + \cdots + X_n)_t^d. \tag{8}$$

We first prove (8) combinatorially in the special case that $t = 1$ and $X_k = y_k$ is a non-negative integer:

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)_1^{\alpha_1} \cdots (y_n)_1^{\alpha_n} = (y_1 + \cdots + y_n)_1^d. \tag{9}$$

Consider n sets $S_1, \ldots, S_n$ of distinct elements, where $|S_k| = y_k$, and let $S = \cup S_k$. Then $|S| = \sum y_k := y$, and the number of d-tuples of distinct elements from S is plainly $y(y - 1) \cdots (y - (d-1))$, which is the right-hand side of (9). We now count the number of d-tuples in a different way. For each n-tuple α with $|\alpha| = d$, consider the number of such d-tuples in which there are α_k distinct elements from S_k, $1 \leq k \leq n$. There are $\binom{y_k}{\alpha_k} = (y_k)_1^{\alpha_k}/\alpha_k!$ ways to choose these elements, and $d!$ ways to arrange them, and so, altogether,

$$d! \prod_{k=1}^{n} \frac{(y_k)_1^{\alpha_k}}{\alpha_k!} = \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)_1^{\alpha_1} \cdots (y_n)_1^{\alpha_n}$$

d-tuples. We sum over all possible choices of α to get the left-hand side of (9), completing the proof in the special case that the y_k's are non-negative integers. But both sides of (9) are polynomials in the y_k's, and their difference is a polynomial which vanishes on $\mathbf{N}^n$. It's easy to see that such a polynomial must vanish identically, and so (9) is in fact an identity for all real y_k. Finally, let $y_k = X_k/t$ in (9) and multiply through by t^d, keeping (1) in mind. Then we have proved (8).

3 Polynomials positive on polyhedra

Suppose $P \subseteq \mathbf{R}^n$ is a convex polyhedron with non-empty interior, bounded by linear polynomials $\lambda_1, \ldots, \lambda_k \in \mathbf{R}[X]$. We always choose the sign of the λ_i's so that $P = \{X \mid \lambda_i(X) \geq 0 \text{ for all } i\}$. Then it is an remarkable fact that any polynomial which is strictly positive on P can be written as a positive linear combination of powers of the λ_i's:

Theorem 2. *Given P as above and suppose $f \in \mathbf{R}[X]$ is strictly positive on P. Then for some $m \in \mathbf{N}$, f has a representation*

$$f = \sum_{|\alpha| \leq m} b_\alpha {\lambda_1}^{\alpha_1} \ldots {\lambda_n}^{\alpha_n}, \tag{10}$$

where $b_\alpha \geq 0$ for all α.

Theorem 2 was first proven by Handelman [3]. His proof is non-constructive; it uses a representation theorem similar to the Kadison-Dubois Theorem. In this section, using our bound for Pólya's Theorem, we give an upper bound for the degree m of a representation (10).

We begin with the case where P is a simplex. In this case, a degree bound for the representation follows almost immediately from the bound in Pólya's Theorem. Let S be an n-simplex in $\mathbf{R}^n$, with vertices $\{v_0, \ldots, v_n\}$ and let $\{\lambda_0, \ldots, \lambda_n\}$ be the set of barycentric coordinates of S, i.e., each $\lambda_i \in \mathbf{R}[X]$ is linear and

$$X = \sum_{i=0}^{n} v_i \lambda_i(X), \quad 1 = \sum_{i=0}^{n} \lambda_i(X), \quad \lambda_i(v_j) = \delta_{ij}.$$

Given $f \in \mathbf{R}[X]$ of degree d, then for any $m \geq d$, there exists a homogeneous polynomial $\tilde{f}_m$ in $n+1$ variables of degree m such that $\tilde{f}_m(\lambda_0, \ldots, \lambda_n) = f(X)$. We can construct $\tilde{f}_m$ as follows: Suppose $f(X) = \sum_{|\alpha| \leq d} a_\alpha X^\alpha$, then

$$f(X) = \sum_{|\alpha| \leq d} a_\alpha \left(\sum_{i=0}^{n} v_i \lambda_i(X) \right)^\alpha \left(\sum_{i=0}^{n} \lambda_i(X) \right)^{m-|\alpha|},$$

thus we set

$$\tilde{f}_m(y_0, \ldots, y_n) = \sum_{|\alpha| \leq d} a_\alpha \left(\sum_{i=0}^{n} v_i y_i \right)^\alpha \left(\sum_{i=0}^{n} y_i \right)^{m-|\alpha|}. \tag{11}$$

Note that for $d = \deg f$, $\tilde{f}_d$ is the Bernstein-Bézier form of f with respect to S.

The following theorem is a generalization of [7, Thm. 6]. Without the concrete bound and with a different proof, it was proven by Miccheli and Pinkus [5, 2.6].

Theorem 3. *Suppose $f \in \mathbf{R}[X]$ of degree d is strictly positive on S and $\tilde{f}_d(y_0, \ldots, y_n)$ is as defined as in* (11)*. Let λ be the minimum of f on S and $L = L(\tilde{f}_d)$. Then for*

$$N \geq \frac{d(d-1)}{2} \frac{L}{\lambda} - d,$$

f has a representation of the form (10) *of degree N.*

Proof. Since $f > 0$ on S, it follows easily that $\tilde{f}_d > 0$ on Δ_{n+1}. Thus we can apply Pólya's Theorem to $\tilde{f}_d$ to find N such that $(\sum y_i)^N \tilde{f}_d(Y)$ has positive coefficients. Then we have

$$(\sum y_i)^N \tilde{f}_d(Y) = \sum_{|\beta|=N} b_\beta Y^\beta,$$

where $b_\beta \geq 0$ for all β. Substituting λ_i for y_i yields $f(X)$ on the left, and a representation of degree N on the right. The bound on N comes from Theorem 1 if we first note that the minimum of $\tilde{f}_d$ on Δ_{n+1} is the same as the minimum of f on S. □

Now we turn to the more general case of the polyhedron P described in the beginning of the section. Given f strictly positive on P, we want to use the same technique as for simplices, i.e., find a homogeneous polynomial g which is positive on Δ_k such that when we "plug in" the λ_i's we obtain f. In this case, however, finding g is not quite so straightforward.

We fix P as above and $\{\lambda_1, \ldots, \lambda_k\}$ such that $P = \{\lambda_i \geq 0\}$. We first note that by [3], there must exist positive real c_i such that $\sum_i c_i \lambda_i = 1$. The c_i's are found by solving a linear system. We replace each λ_i by $c_i \lambda_i$ so that we have

$$\sum_i \lambda_i = 1 \tag{12}$$

Furthermore, there exist constants $b_{i,j} \in \mathbf{R}$ so that, for $j = 1, \dots, n$,

$$x_j = \sum_{j=1}^{k} b_{i,j} \lambda_i.$$

Again, explicitly finding the $b_{i,j}$'s is an easy linear algebra problem. Thus we are almost in the situation for simplices, although the $b_{i,j}$'s need not be positive. Let B be the real $n \times k$ matrix $(b_{i,j})$, then

$$B \cdot (\lambda_1, \dots, \lambda_k)^T = (x_1, \dots, x_n) \tag{13}$$

As in [9], we formalize the notion of "plugging in" the λ_i's. Let $\mathbf{R}[Y] := \mathbf{R}[y_1, \dots, y_k]$ and define $\phi : \mathbf{R}[Y] \to \mathbf{R}[X]$ by $y_i \mapsto \lambda_i$. By (12) and (13), ϕ is onto. More explicitly, given a polynomial $f = \sum_{|\alpha| \le d} a_\alpha X^\alpha$, define homogeneous $\tilde{f} \in \mathbf{R}[Y]$ by

$$\tilde{f} := \sum_{|\alpha| \le d} a_\alpha (B \cdot Y^T)^\alpha (\sum_{j=1}^{k} y_j)^{d-|\alpha|}. \tag{14}$$

Then $\phi(\tilde{f}) = f$.

Suppose now that $f > 0$ on P and we have a point $\gamma \in \Delta_k$. Then $\tilde{f}(\gamma) = f(B \cdot \gamma)$. Since the point $B \cdot \gamma$ need not be in P, we do not necessarily have that $\tilde{f}(\gamma)$ is positive. Thus we cannot apply Polya's Theorem directly to $\tilde{f}$. However, by a theorem of Schweighofer [9], it turns out that there is a polynomial positive on Δ_k of the form $\tilde{f} + c(\sum_j r_j^2)$, where $\{r_1, \dots, r_t\}$ is any basis for the kernel of ϕ. Note that any g of this form has the property $\phi(g) = f$. The following result is (essentially) [9, Lemma 3.1]:

Lemma 4. *Suppose P and ϕ are as above and $f > 0$ on P. Let $\{r_1, \dots, r_t\}$ be a basis for the kernel of ϕ, set $r := \sum_{j=1}^{t} r_j^2$, and define $\tilde{f}$ as in (14). Then for sufficiently large c, $\tilde{f} + cr$ is strictly positive on Δ_k. More explicitly, if $\tilde{f}$ is already strictly positive on Δ_k, then we take $c = 0$ and otherwise, this holds for $c > \frac{-m_1}{m_2}$, where m_1 is the minimum of $\tilde{f}$ on Δ_k and m_2 is the minimum of r on $\Delta_k \cap \{\beta \in \mathbf{R}^k \mid \tilde{f}(\beta) \le 0\}$.*

Proof. Let U be the compact set $\Delta_k \cap \{\beta \in \mathbf{R}^k \mid \tilde{f}(\beta) \le 0\}$ and assume that $U \ne \emptyset$. By [9, §3], $r > 0$ on U and hence, since U is compact, the minimum m_2 of r on U exists and is positive. Then on U, $\tilde{f} + cr \ge m_1 + cm_2 > 0$. On $\Delta_k \setminus U$, $\tilde{f} + cr \ge \tilde{f} > 0$. □

Theorem 5. *Given P, ϕ, r, f, and $\tilde{f}$ as above. Fix c such that $F := \tilde{f} + cr > 0$ on Δ_k. Let d be the degree of f and let λ be the minimum of $\tilde{F}$ on Δ_k. For*

$$N \ge \frac{d(d-1)}{2} \frac{L(\tilde{F})}{\lambda} - d,$$

f has a representation

$$f = \sum_{|\alpha|=N} b_\alpha \lambda_1^{\alpha_1} \dots \lambda_k^{\alpha_k},$$

where $b_\alpha \ge 0$ for all α.

Proof. Since $\phi(\tilde{F}) = f$, this follows from Theorem 1 applied to $\tilde{F}$, exactly as in the proof of Theorem 3. □

Remark. Note that for a specific P and f, we can calculate all elements needed for the bound in the theorem, and then can easily find a representation for f. Thus the theorem yields an algorithm for finding a representation for f of the form (10).

Algorithm. Given a compact, convex polyhedron $P \subseteq \mathbf{R}^n$ defined by $\{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0\}$, where $\sum_i \lambda_i = 1$, and $f > 0$ on P. We will describe a procedure for constructing a representation of f of the form (10). We proceed as follows:

1. Using (14), construct homogeneous $\tilde{f} \in \mathbf{R}[Y]$ with the same degree as f such that $\phi(\tilde{f}) = f$.

2. Construct a basis $\{r_1, \ldots, r_t\}$ for the kernel of ϕ. We can use the following well-known procedure for this: Construct a Groebner Basis G for the ideal generated by $\{y_1 - \lambda_1, \ldots, y_k - \lambda_k\}$, using lex order with $x_1 > \cdots > x_n > y_1 > \cdots > y_k$. Then $G \cap \mathbf{R}[y_1, \ldots, y_k]$ is the desired basis.

3. Calculate the minima m_1 and m_2 needed for the c of Lemma 4, e.g., by using Lagrange multipliers. Set $F := \tilde{f} + cr$ and find the homogeneous $\tilde{F}$.

4. Calculate $L(\tilde{F})$ and the minimum of $\tilde{F}$ on Δ_k and then find N as in Theorem 5. Use the coefficients of $Y^N \tilde{F}$ to obtain the desired representation.

Example. Let P be the square unit square centered at the origin in $\mathbf{R}^2$, and let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{1/4 + 1/4\,x, 1/4 - 1/4\,x, 1/4 + 1/4\,y, 1/4 - 1/4\,y\}$. With ϕ the map defined above, we have that $\{r_1, r_2\} := \{y_1 + y_2 - 1/2, y_3 + y_4 - 1/2\}$ is a Groebner Basis for the kernel of ϕ. Consider $f := 3/2 - x^2 + y^2 > 0$ on P, then

$$\tilde{f} = -\frac{5}{2}y_1^2 + 11y_1y_2 - \frac{5}{2}y_2^2 + 3y_1y_3 + 3y_2y_3 + \frac{11}{2}y_3^2 + 3y_1y_4 + 3y_2y_4 - 5y_3y_4 + \frac{11}{2}y_4^2$$

The minimum of $r := r_1^2 + r_2^2$ on $\{\tilde{f} \leq 0\} \cap \Delta_4$ is 1 and the minimum of $\tilde{f}$ on Δ_4 is $-5/2$. Thus we need $c > 5/2$. We choose $c = 3$, and set $F := \tilde{f} + 3r$. Then

$$\tilde{F} = \frac{7}{2}y_1^2 + 23y_1y_2 + \frac{7}{2}y_2^2 - 9y_1y_3 - 9y_2y_3 + \frac{23}{2}y_3^2 - 9y_1y_4 - 9y_2y_4 + 7y_3y_4 + \frac{23}{2}y_4^2$$

which is positive on Δ_4. The minimum of $\tilde{F}$ on Δ_4 is $3/10$ and $L(\tilde{F}) = 23/2$. Hence the bound in Theorem 1 is 75. This means that that $(y_1 + y_2 + y_3 + y_4)^{38}\tilde{F}$ must have positive coefficients. Expanding, and plugging in the λ_i's, we could then obtain an explicit representation for f. In point of fact, f has an explicit representation of degree 3.

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