Seminar announcement for November 1, 2011

Linearly dependent powers of quadratic forms, Bruce Reznick

Number Theory (1pm, 241 AH), Geometry/AGC (2pm, 243 AH)

These *independent* talks will cover the same general topic for two consecutive hours, although there will be no more than 15 minutes overlap in the material presented. Let $\Phi(d)$ be the smallest r so that there exist r pairwise non-proportional complex quadratic forms $\{q_i\}$ with the property that $\{q_i^d\}$ is linearly dependent. **Problem:** compute $\Phi(d)$ and characterize the minimal sets. Any set of $2r+2$ q_i^d's is dependent, so $\Phi(d) \le 2d+2$, but a "general" set of $2r+1$ q_i^d's is linearly independent.

The Pythagorean parameterization gives the unique minimal set up to change of variable: $\Phi(2) = 3$ and $(x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2$. Liouville's proof of Fermat's Last Theorem for non-constant polynomials implies that for $d \ge 3$, $\Phi(d) \ge 4$. A deep theorem of Mark Green implies that $\Phi(d) \ge 1 + \sqrt{d+1}$. In the other direction, many 19th century examples show that $\Phi(3) = \Phi(4) = \Phi(5) = 4$. New results: if $d \ge 6$, then $\Phi(d) \ge 5$, $\Phi(6) = \Phi(7) = 5$, $\Phi(14) \le 6$; $\Phi(d) \le 2 + \lfloor (d+1)/2 \rfloor$.

My work on this was motivated by a 1999 seminar of Bruce Berndt about an question of Ramanujan, who wanted a generalization of the identity of the form $f_1^3 + f_2^3 = f_3^3 + f_4^3$ for four specific quadratic forms in $\mathbb{Z}[x,y]$. Neither Ramanujan, nor Narayanan, who solved his question, noted that there existed other quadratic forms f_j so that $f_1^3 + f_2^3 = f_3^3 + f_4^3 = f_5^3 + f_6^3$ and $f_1^3 - f_4^3 = f_3^3 - f_2^3 = f_7^3 + f_8^3$, but nothing further for $f_1^3 - f_3^3 = f_4^3 - f_2^3$. This is typical. For $\alpha \in \mathbb{C}$,

$$(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 = (\alpha^2 - 1)(\alpha x^3 + y^3)(x^3 + \alpha y^3),$$

and if $y \mapsto \omega y$, where $\omega^3 = 1$, then the right-hand side is unchanged, hence there are two other pairs of quadratic forms whose cubes which have the same sum. Up to change of variable, these are *all* the minimal solutions of degree 3. In some cases, solutions coalesce: $x^6 + y^6$ is a sum of two cubes in four different ways and $xy(x^4+y^4)$ in six ways. There are three different minimal solutions of degree 4 and one of degree 5, but no families of solutions, as there are in degree 3.

Felix Klein promoted the idea of associating each linear form $x - \alpha y$, $\alpha \in \mathbb{C}$ with the image of α on the Riemann map from $\mathbb{C}$ to the unit sphere (and y to the north pole.) We associate quadratic forms to the *pairs* of points of their factors. In this way, the Pythagorean parameterization corresponds to antipodal pairs of the vertices of an octahedron, the unique solution for $d = 5$ corresponds to antipodal pairs of the vertices of a cube and the example for $d = 14$ corresponds to antipodal pairs of the vertices of a regular icosahedron. This cannot be an accident.

It's also useful to consider sums of the form $\sum_{k=0}^{m-1} (\zeta_m^k x^2 + \beta xy + \zeta_m^{-k} y^2)^d$ where $\zeta_m = e^{2\pi i/m}$ and $m > 2d$; the sum on roots of unity kills the coefficient of all terms but $x^{d \pm m} y^{d \mp m}$ and $x^d y^d$, and β is chosen to leave a multiple of $(xy)^d$. In this way, one can show that $\Phi(d) \le 2 + \lfloor (d+1)/2 \rfloor$, although this is not best possible for $d = 3, 5, 7, 14$. These sets of quadratic forms have a Klein correspondence with a polyhedron whose vertices are the two poles and two antipodal horizontal m-gons.