Ternary forms with lots of zeros
(Slightly corrected version)

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Structures algébriques ordonnées et leurs interactions
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This talk is dedicated to the memory of Murray Marshall.

The new work in this presentation is joint with Greg Blekherman. My apologies to those who have seen earlier versions. We have not made progress since my presentations in San Diego and Daejeon. I have supplemented the talk with results about sums of cubes, as suggested by Eberhard Becker's talk on Monday.

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For such forms, we are particularly interested in the zero set of p, written $\mathcal{Z}(p)$, and the *projective* number of zeros, $|\mathcal{Z}(p)|$, counted this way because forms vanish on lines through the origin. We will describe $\mathcal{Z}(p)$ by picking a representative from each such line.

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If e.g. $p(x,y,z) = (x-z)^2 q(x,y,z)$ for some psd q, then the entire line $\{x = z\}$ is contained in $\mathcal{Z}(p)$, so $|\mathcal{Z}(p)| = \infty$. We will only be interested in those cases where $|\mathcal{Z}(p)|$ is finite, so we assume no indefinite square factors.

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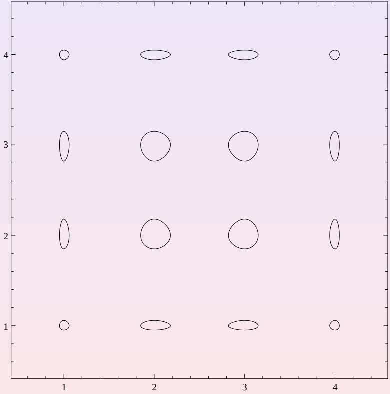
If $p(a,b,0) = 0$, then p has a "zero at infinity". In the absence of these, it makes sense to dehomogenize to $p(x,y,1)$. If $\epsilon > 0$ is sufficiently small, then the real solutions to $p(x,y) = \epsilon$ will consist of $|\mathcal{Z}(p)|$ disjoint ovals in the plane, one around each of the zeros.

Here is one example of 16 ovals for the octic q_4:

In[6]:=
```
ContourPlot[Product[(x - i)^2, {i, 1, 4}] +
   Product[(y - j)^2, {j, 1, 4}] == .1, {x, .5, 4.5},
  {y, .5, 4.5}, ContourStyle → Black, PlotPoints → 100]
```

Out[6]=



M. D. Choi, T. Y. Lam and I studied this topic systematically in 1980. Here are some of our results:

- There is an integer $\alpha(2k)$ with the property that if $p \in P_{3,2k}$ and $|\mathcal{Z}(p)| > \alpha(2k)$, then there exists an indefinite form h so that $p = h^2 q$. (If p is irreducible over $\mathbb{C}$ and $p(\pi) = 0$, then p is singular at π, and p has at most $(k-1)(2k-1)$ singular points.) (Failure in 4 variables: $x^2y^2 + z^2w^2$!)

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- $\alpha(2rk) \ge r^2\alpha(2k)$. (Argument to follow.)

Examples. If p is a real ternary form of degree $2k = 2, 4$, then psd implies sos, so the upper bounds are $1^2, 2^2$. These are achieved by:

$$\mathcal{Z}(x^2 + y^2 + z^2 - xy - xz - yz) = \{(1, 1, 1)\};$$
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Let $F = x(x^2 - z^2)$ and $G = y(y^2 - z^2)$. Then F and G have 9 common real zeros, at $\{(a, b, 1) : a, b \in \{-1, 0, 1\}\}$; that is, on a 3×3 grid. We pick the 8 zeros minus the center and note that $K(x, y, z) = (x^2 - z^2)(y^2 - z^2)(z^2 - x^2 - y^2)$ is singular at the first 8.

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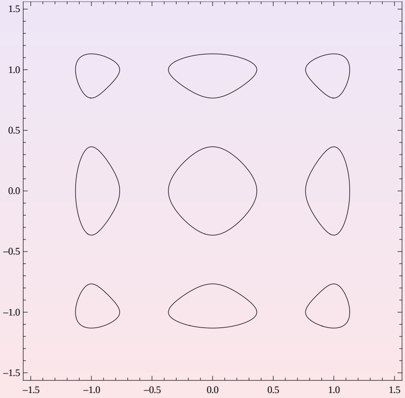
It turns out that $R := F^2 + G^2 + K$ is psd and has the original 8 zeros plus 2 at infinity. Miraculously, R is symmetric in x, y, z, even though z was treated differently from x and y.

Here are some dehomogenized ($z = 1$) pictures. This shows the set $F^2 + G^2 = .1$.

```
In[7]:= ContourPlot[x^2 (x^2 - 1) ^2 + y^2 (y^2 - 1) ^2 == .1,
  {x, -1.5, 1.5}, {y, -1.5, 1.5},
  ContourStyle → Black, PlotPoints → 100]
```

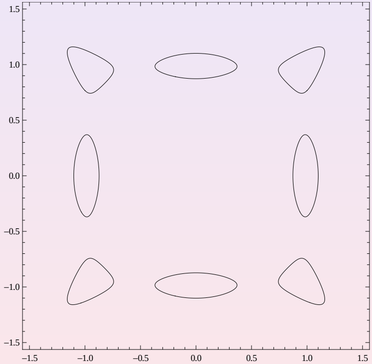
Out[7]=



This shows the set $R = F^2 + G^2 + K = .1$. You can't see the zeros at infinity.

```
In[8]:= ContourPlot[x^2 (x^2 - 1)^2 + y^2 (y^2 - 1)^2 +
           (x^2 - 1) (y^2 - 1) (1 - x^2 - y^2) == .1, {x, -1.5, 1.5},
         {y, -1.5, 1.5}, ContourStyle → Black, PlotPoints → 100]
```

Out[8]=



After algebraic simplification,

$$R(x,y,z) = x^6 + y^6 + z^6$$
$$-(x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4)$$
$$+3x^2y^2z^2.$$

We have

$$\mathcal{Z}(R) = \{(\pm 1, \pm 1, 1), (\pm 1, 0, 1), (0, \pm 1, 1), (1, \pm 1, 0)\}.$$

The last two zeros are at infinity; note that $|\mathcal{Z}(R)| = 10$ as promised. Both the singularity upper bound and the oval upper bound for sextics give 10, so $\alpha(6) = 10$.

Let $T_r(t) := \cos(r \arccos(t))$ be the r-th Chebyshev polynomial ($deg(T_r) = r$); e.g. $T_3(t) = 4t^3 - 3t$. Chebyshev polynomials have the property that $T_r : [-1, 1] \mapsto [-1, 1]$ in such a way that for $u \in (-1, 1)$, $|\{T_r^{-1}(u)\}| = r$.

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If $p \in P_{3,2k}$ and $|\mathcal{Z}(p)| = m$, then after an invertible linear change of variables, we may assume that

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We construct a new polynomial of degree $2kr$:

$$p_r(x, y, z) := z^{2kr} p(T_r(x/z), T_r(y/z), 1) \implies$$
$$\mathcal{Z}(p_r) = \{(T_r^{-1}(a_i), T_r^{-1}(b_i), 1) : 1 \le i \le m\},$$

so we see that $|\mathcal{Z}(p_r)| = r^2 m$. This guarantees quadratic growth.

Now some "new" results; this is joint work with Greg.

- $\alpha(8) \geq 17$. (In fact, "morally", we have $\alpha(8) \geq 18$; the oval upper bound is 19.) Added: Claus Scheiderer gave a beautiful octic in his talk showing $\alpha(8) \geq 18$ unconditionally.

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The octic examples come from emulating Robinson's construction, but starting with a 4×4 grid. First ignore two zeros. It turns out that the set of quartics which vanish on these 14 points is a pencil with generators, say, F and G. We then look at octic forms which are singular at these 14 points. When we are lucky, they form a subspace of ternary octics with basis $\{F^2, FG, G^2, K\}$ for some K. We then play with taking $\phi(F, G) + \lambda K$ where ϕ is a pd quadratic form, and, when things work out just right, we find the examples.

The example with 17 zeros comes from a variation. We start with a 3×4 grid and a symmetric pair above and below.) The resulting $F_1(x, y, z)$ is unfortunately, quite ugly: $F_1 \in \mathbb{Q}(\sqrt{345})[x, y, z]$, and the three new zeros are at infinity; at $(0, 1, 0)$ and $(a, b, 0)$, where $3\sqrt{345}a^2 = 23b^2$. We have varied the starting points and found many similar examples, but none with rational coefficients.

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$$
\begin{aligned}
F_1(x, y, z) := {} & -y^2(5x^2 + 9y^2 - 81z^2)(5x^2 + y^2 - 9z^2)(y^2 - 4z^2) \\
& + \frac{2}{27}(675 + 23\sqrt{345})x^2y^2(y^2 - 4z^2)^2 \\
& + 9(5x^4 - y^4 - 50x^2z^2 + 4y^2z^2 + 45z^4)^2
\end{aligned}
$$

In 1893, Hilbert proved that if $p \in P_{3,2k}$ and $2k \geq 4$, then there exists $q \in P_{3,2k-4}$ so that $pq \in \Sigma_{3,4k-4}$ is a sum of *three* squares of forms of degree $2k - 2$.

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This example F_1 has the property that the only quadratic q (up to multiple) so that qF_1 is a sum of squares is $q = q_1$:

$$q_1(x,y,z) = 90x^2 + \sqrt{345}\ y^2 + 14\sqrt{345}\ z^2.$$

It turns out that $q_1 F_1$ is a sum of four squares, not three, so this example shows that, for at least one octic in Hilbert's Theorem, you really need a multiplier of degree $8-4$, not $8-6$.

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Now we turn to the "morally 18 zero" example. It has 16 zeros, but two of them are "deep", with the polynomial vanishing to fourth order in a certain direction. In a geometric sense, this happens when two zeros coalesce at a point, and $16 + 2 = 18$.

The 14 zeros we start with are

$$\{(a, b, 1) : a, b \in \{\pm 1, \pm 3\}, (a, b) \neq (3, 3), (-3, -3)\};$$

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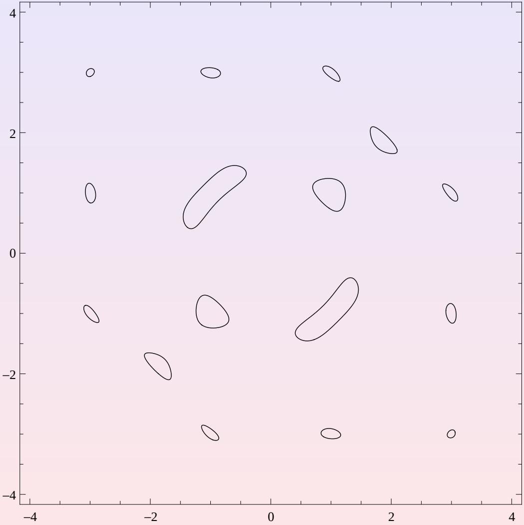
the two new zeros turn out to be at $(\pm s, \pm s, 1)$, where $s = \sqrt{\frac{45}{13}}$.

$$F_2(x,y,z) =$$
$$25x^8 + 72x^6y^2 + 144x^5y^3 + 194x^4y^4 + 144x^3y^5 + 72x^2y^6$$
$$+25y^8 - 572x^6z^2 - 144x^5yz^2 - 1436x^4y^2z^2 - 1728x^3y^3z^2$$
$$-1436x^2y^4z^2 - 144xy^5z^2 - 572y^6z^2 + 4192x^4z^4$$
$$+1584x^3yz^4 + 6584x^2y^2z^4 + 1584xy^3z^4$$
$$+4192y^4z^4 - 9720x^2z^6 - 1440xyz^6 - 9720y^2z^6 + 8100z^8$$

The next page shows $F_2(x,y,1) = 400$; 400 is small!

You can count 16 zeros and you can see the squeezed shape of the zeros at $(\pm 1, \mp 1)$, which is consistent with their 4th order.

Out[28]=



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As Charu said the other night, we have constructed symmetric and even symmetric forms of degree d in n variables for every (n, d) for which it is possible. Unfortunately for these purposes, the basic examples are mainly reducible, and are usually multiplied by squares of indefinite factors, so they are not very useful in the quest for a large finite number of zeros.

I had forgotten about my student William Harris's example in degree 10 until I was reminded of it in 2014 by my collaborators Charu Goel and Salma Kuhlmann at the University of Konstanz.

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Harris actually constructed a one-parameter family of psd ternary decics with 30 zeros. I have made a choice of parameter to make it as simple as possible.

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$$W(x,y,z) = 16\sum x^{10} - 36\sum x^8y^2 + 20\sum x^6y^4$$
$$+57\sum x^6y^2z^2 - 38\sum x^4y^4z^2.$$

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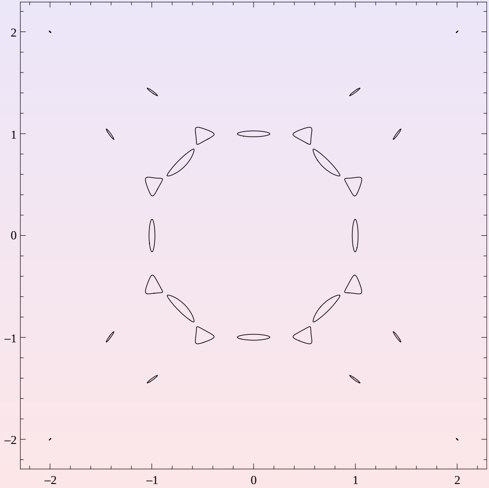
(The sums above should be taken so as to make W symmetric.)

Harris showed that W is psd and $\mathcal{Z}(W)$ consists of $(1,1,\sqrt{2})$, $(1,1,\frac{1}{2})$, and $(1,1,0)$ with all choices of sign and permutation. This gives $12+12+6=30$ zeros, of which 28 zeros are not at infinity. (It seems likely that the future examples in higher degree will be symmetric.) The next page shows $W(x,y,1)=.08$.

The zeros are at $(\pm 1, \pm\frac{1}{2})$, $(\pm\frac{1}{2}, \pm 1)$, $(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}})$, $(\pm 1, \pm\sqrt{2})$, $(\pm\sqrt{2}, \pm 1)$, $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 2, \pm 2)$. The last 4 are barely visible, but choosing a larger ϵ makes the ovals coalesce.

Out[38]=



On the conjecture, Choi, Lam and I remarked in 1980 that because of the Chebyshev-fueled quadratic growth, we have

$$\alpha(6s) \ge 10s^2,$$
$$\alpha(6s+2) \ge 10s^2 + 1,$$
$$\alpha(6s+4) \ge 10s^2 + 4.$$

This is already enough to prove that $\alpha(2k) \ge k^2 + 1$ for all but 18 cases: $6s+2$ for $1 \le s \le 6$ and $6s+4$ for $1 \le s \le 12$. The new information about $\alpha(8)$ and $\alpha(10)$ reduces the number of open cases to eight: $2k \in \{14, 22, 26, 28, 34, 38, 46, 58\}$.

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We think the conjecture is true, because it's hard to believe that there's something interesting about ternary forms which only occurs in these degrees.

We mention two applications. The first is taken from my 1992 *Memoir*. Let $Q_{3,2k}$ be the closed cone of sums of $2k$-th powers of real linear forms; this is the dual cone to $P_{3,2k}$.

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Suppose $p \in P_{3,2k}$, $\mathcal{Z}(p) = \{(a_i, b_i, c_i)\}$ and the $|\mathcal{Z}(p)|$ forms $\{(a_i x + b_i y + c_i z)^{2k}\}$ are linearly independent. Then any expression of the form

$$\sum_{i=1}^{|\mathcal{Z}(p)|} \lambda_i (a_i x + b_i y + c_i z)^{2k}, \quad (\lambda_i > 0)$$

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The *a priori* lower bound on "maximal width" is $\frac{(k+1)(k+2)}{2}$, which e.g. for $2k = 10$ is 21. It is easy to find sums of 10th powers of ternary linear forms which need 21 summands. The Harris example demonstrates the existence of forms needing 30 summands.

The second application is stolen from an overheard conversation between Greg and Cordian (and possibly involving others here.) Any misinterpretation is my responsibility. A *quadrature formula of strength d on* $(S^{n-1}, d\mu)$ is an equation:

$$\int_{u \in S^{n-1}} \cdots \int p(u)\ d\mu = \sum_{k-1}^{N} \lambda_k p(u_k)$$

which is valid for all forms p of degree $\le d$ in n variables, where $d\mu$ is a positive measure on S^{n-1}, $\lambda_k \in \mathbb{R}$, $u_k \in S^{n-1}$.

Observe that if $0 \ne p$ is psd, then the left-hand side is positive, hence it cannot be the case that $p(u_k) = 0$ for $1 \le k \le N$. The Harris example shows that for $d = 10$, $n = 3$, the minimum number of nodes in a strength 10 quadrature formula must be ≥ 31.

Sums of two squares are an important theme in mathematics. What can one say about a sum of two cubes? (Warning: this is over $\mathbb{C}$, not $\mathbb{R}$!) There is a simple result which I have not seen in the literature. If you know it, please give me the proper source.

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Theorem

Suppose $F \in \mathbb{C}[x_1, \ldots, x_n]$. Then F is a sum of two cubes in $\mathbb{C}[x_1, \ldots, x_n]$ if and only if it is itself a cube, or has a factorization $F = G_1 G_2 G_3$, into non-proportional, but linearly dependent factors.

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Let ω denote a primitive cube root of unity. This theorem is actually valid for any field K with $\mathbb{Q}(\omega) \subseteq K \subseteq \mathbb{C}$.

Proof.

First observe that

$$F = G^3 + H^3 = (G+H)(G+\omega H)(G+\omega^2 H),$$
$$(G+H) + \omega(G+\omega H) + \omega^2(g+\omega^2 H) = 0.$$

If two of the factors $G+\omega^j H$ are proportional, then so are G and H, and hence F is a cube.

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If two of the factors $G+\omega^j H$ are proportional, then so are G and H, and hence F is a cube.
Conversely, if F has such a factorization, there exist $0 \neq \alpha, \beta \in \mathbb{C}$ so that $F = G_1 G_2(\alpha G_1 + \beta G_2)$. It is easily checked that

$$F = \frac{(\omega^2 a G_1 - \omega b G_2)^3 - (\omega a G_1 - \omega^2 b G_2)^3}{3ab(\omega - \omega^2)}.$$

$\square$

An immediate corollary is a special case of an old theorem.

Corollary (Sylvester, 1851)

If a binary cubic p is not a cube, then it is a sum of two cubes of linear forms unless it has a repeated factor.

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Write $p(x,y) = \prod_{j=1}^{3}(\alpha_j x + \beta_j y)^3$. The three factors $\{\alpha_j x + \beta_j y\}$ are always linearly dependent, so p is a sum of two cubes iff they are not pairwise proportional. $\square$

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Sylvester's proof also gives an algorithm for the coefficients of the cubes in the representation of p as a sum of two cubes. That's for another talk.

In case p is a binary sextic form, there are 7 coefficients in p and 6 coefficients in two quadratic forms.

Theorem

The binary sextic $f(x, y)$ is a sum of two cubes of quadratic forms if and only if either
(i) $p = \ell^3 g$, where ℓ is a linear form and g is a cubic which is a sum of two cubes, or
(ii) After an invertible linear change of variables, p is even; i.e., $p(ax + by, cx + dy) = g(x^2, y^2)$, where g is a cubic which is a sum of two cubes.

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Proof.

Write $p = F^3 + G^3$. If F and G have a common linear factor ℓ, we are in case (i). If F and G are quadratic forms which are relatively prime, there is an invertible linear change of variables under which they are simultaneously diagonalized. This is case (ii) and implies that p is even, after that change of variables. $\square$

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This is a sketch of the proof. Write a binary sextic as

$$p(x, y) = \sum_{k=0}^{6} \binom{6}{k} a_k x^{6-k} y^k.$$

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Given $p \neq 0$, we may make an invertible change of variable to assume that $p(0, 1)p(1, 0) \neq 0$; that is, $a_0 a_6 \neq 0$. By an observation of *ad hoc*, the coefficients of x^6, x^5y, x^4y^2 in p and $a_0 q^3$ are the same, where

$$q(x, y) = x^2 + 2\frac{a_1}{a_0}\ x\ y + \frac{5a_0a_2 - 4a_1^2}{a_0^2}\ y^2.$$

Hence there is a cubic c such that.

$$p(x, y) - a_0 q(x, y)^3 = y^3 c(x, y).$$

Usually, $(p - a_0q^3)/y^3 = c$ is a sum of 2 cubes of linear forms, from which it follows that p is a sum of 3 cubes. This algorithm can only fail if c has a square factor. The discriminant of c is a polynomial in the a_i's of degree 18, divided by a_0^{14}.
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where c is a cubic which has a square linear factor.

We believe that in these cases there is always (or nearly always) a value of the parameter t such that the algorithm works for $p_t(x, y) = p(x, tx + y)$. Work is ongoing in this case, though the argument I outlined on 10/16/15 had errors. In any case we can (could?, did?) have lunch now (then?).

I thank the organizers for the invitation to speak and the audience for its patience and attention.