# 1  
\[ 1 - \frac{3}{2} = -1 \] 
Where is \( e^z - 1 = 0 \)  
\[ e^z = 1 \implies z = 2n \pi i \] 
If \( f(z) = (e^z - 1)^2 = e^{2z} - 2e^z + 1 \)  
Then \( f'(z) = 2e^{2z} - 2e^z \)  
\[ f''(z) = 4e^{2z} - 2e^z \]  
So if \( e^{2z} = 1 \), then \( f''(z) = 0 \)  
and \( f'(z) = 0 \) and \( f \) has a zero of order 2 at each \( 2n \pi i \)  
4.  
\[ (z - z_0)^3 = (e^{z-1})^3 \]  
This has a zero of order 6 at \( z = z_0 \).  

\[ i \rightarrow \text{ } 1 \rightarrow 10, 12 \] 
10. Expand \( e^z \) at \( z = 10i \)  
\[ e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(10i)}{n!} (z-10i)^n \]  
If \( f(z) = e^z \rightarrow f^{(n)}(z) = f(z) = e^z \)  
all \( z \) and \( f^{(n)}(10i) = e^{10i} \)  
So \( e^z = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{e^{10i}}{n!} (z-10i)^n \)  
Q.E.D.  
12. \( \frac{z^2}{1-z} \) about \( z = 0 \) is easiest  
to do by manipulating  
\[ z^2 + \frac{1}{1-z} = z^2 - \sum_{n=0}^{\infty} 2^n \]  
\[ \frac{z^2}{1-z} = \frac{(-1-z)(1-z)}{1-z} \]  
\[ = -1-z + 1 - z = -2 + \sum_{n=0}^{\infty} z^n \]  
The first two terms cancel  
\[ \sum_{n=2}^{\infty} z^n \]  

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# 3  
\[ e^{2z-x} \]  
If \( f \) is entire and \( f(z) = f(x+iy) = u(x,y) + iv(x,y) \)  
and \( g(z) = e^{f(z)} \), then \( g \) is also entire and \( g(z) = e^{u(x,y)} \)  
If \( Re(z) \leq C \), i.e., \( u(x,y) \leq C \),  
then we have  
\[ |g(z)| \leq e^C \]  
Since \( g \) is entire, this means  
that \( g \) is constant by Liouville’s Theorem. In particular, \( u(x,y) \)  
is constant and, as we know,  
this implies that \( f(x+iy) = f(z) \)  
is constant.  

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# 4, 5  
\[ \frac{1}{2\pi i} \int \frac{z^3}{(z+1)(z+2)(z+3)} \, dz = \frac{1}{2\pi i} \int \frac{z^3}{(z^3+3z^2+2+1)} \, dz \]  
\[ \text{Res}_{z=0} = \text{Res}_{z=1} = \]  
all residues are zero except  
the third:  
\[ \frac{1}{2\pi i} \int \frac{z^3}{z^2+3z} \, dz \]  
\[ \frac{1}{2\pi i} \int \frac{\frac{1}{z^2}}{z+3} \, dz \]  
has poles at \( z = 0, 3 \)  
and 3 is outside.  
\[ = \frac{1}{2\pi i} \int \frac{1}{z^2} \, dz = \frac{1}{0-3} = \frac{1}{3} \]  
\[ \frac{1}{2\pi i} \int \frac{e^{2z}}{z^2} \, dz = \text{coeff of } z^4 \text{ in } e^{2z} \]  
by Thm 1? (p. 723)  
\[ \text{(This is pre 7.2.5, but you can also use) Cauchy: } e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} \frac{z^n}{n!} = \frac{z^n}{n!} \]  
\[ = \text{so the integral is } 4^4 \text{ at } 2^4 = \frac{2^5}{3} = \frac{32}{3} \]  
\[ e^{-z^2} \]  
is analytic in \( z = 2 \), so  
\[ \frac{1}{2\pi i} \int_{|z|=2} e^{-z^2} \, dz = 0 \]
8. \( z^2 + 2iz + 1 = 0 = (z - p)(z - q) \)
\( \Rightarrow z = -2i \pm \sqrt{6i^2 - 4} \)
\( = -2i \pm \sqrt{6}i \)
\( = -i \pm \sqrt{2}i \)

Here \( p = (\sqrt{2} - 1)i \) \quad \|p\| = \sqrt{2 - 1} \approx 1.4 \)
\( q = (-1 - \sqrt{2})i \) \quad \|q\| = \sqrt{1 + 2i} \approx 1.4 \)

b. Thus
\( \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z^2 + 2iz + 1} = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z - p} \)
\( = \frac{1}{p-q} \) by Cauchy's Theorem.

\( p-q = 2\sqrt{2}i \), so the value of the integral is \( \frac{1}{2\sqrt{2}i} = \frac{1}{4} \).

c. Let \( z = e^{it} \), \( dz = ie^{it} dt \), so
\( \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z^2 + 2iz + 1} = \frac{1}{2\pi i} \int_{t=0}^{2\pi} \frac{ie^{it} dt}{e^{2it} + 2ie^{it} + 1} \)
\( = \frac{1}{2\pi} \int_{t=0}^{2\pi} \frac{dt}{e^{2it} + 2i + e^{-it}} \)

Now \( e^{2it} + 2i + e^{-it} = 2 \cos 2t + 2i \cos t \)
\( = \frac{2 \cos 2t - 2i}{4 \cos^2 t + 4} \) so we get
\( -\frac{\sqrt{2}i}{4} = \frac{1}{2\pi} \int_{t=0}^{2\pi} \frac{2 \cos 2t dt}{4 \cos^2 t + 4} + \frac{1}{2\pi} \int_{t=0}^{2\pi} \frac{i}{4 \cos^2 t + 4} \)

Taking the real part,
\( 0 = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{2 \cos 2t dt}{4 \cos^2 t + 4} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{2 \cos 2t dt}{4 \cos^2 t + 4} \)
\( -\frac{\sqrt{2}i}{4} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{-2 dt}{4 \cos^2 t + 4} \Rightarrow \frac{2 \pi dt}{4 \cos^2 t + 4} = \int_{0}^{2\pi} \frac{dt}{4 \cos^2 t + 4} \)
\( -\frac{\sqrt{2}i}{4} = \sqrt{2} \pi, \) or sculped van't hatus
Suppose $z = \text{re}^{i\theta}$, $r < 1$.

(a) Multiplying by the conjugate of the denominator, we get

$$\frac{e^{it} \text{re}^{i\theta}}{1 - \text{re}^{i\theta}} \times \frac{1 + \text{re}^{-i\theta}}{1 + \text{re}^{-i\theta}} = \frac{e^{it} + e^{-i\theta}}{1 - \text{re}^{-i\theta}}$$

The denominator is

$$1 - r(e^{i\theta} + e^{-i\theta}) + r^2 = 1 - 2r \cos(\theta) + r^2.$$  

The numerator is

$$1 + r(e^{i\theta} - e^{-i\theta}) - r^2 = 1 + 2i r \sin(\theta) - r^2.$$  

So the real part of the quotient is

$$\frac{1 - r^2}{1 - 2r \cos(\theta) + r^2}$$  as desired.

(b) Doing the algebra,

$$\frac{e^{it} + \frac{2}{1 - e^{-i\theta}}}{e^{it} + \frac{2}{1 - e^{-i\theta}} + 1} = \frac{e^{it} - 1}{e^{it} - 1} = 1.$$

$2 \pi i \int_0^{2\pi} f(e^{it}) \frac{e^{it}}{e^{it} - z} dt = f(z)$

by Cauchy's Theorem, and by taking $\xi = e^{it}$,

$$\frac{1}{2\pi i} \int_{|\xi| = 1} f(e^{it}) \frac{e^{it}}{e^{it} - z} \frac{d\xi}{\xi} = f(z).$$

(c) Adding (a) and (b)

$$\frac{1}{2\pi i} \int_0^{2\pi} f(e^{it}) \left[ \frac{e^{it} - 1}{1 - 2e^{it} + e^{it}} \right] dt = f(z)$$

by (b), this is

$$\frac{1}{2\pi i} \int_0^{2\pi} f(e^{it}) \frac{e^{it} + 2}{e^{it} - 1} dt = f(z)$$

and by (a), this is

$$\frac{1}{2\pi i} \int_0^{2\pi} f(e^{it}) \frac{1 - r^2}{1 - 2r \cos(\theta) + r^2} dt = f(z).$$

Who cares?

Well, this formula gives the value of $f$ inside the unit circle as a weighted integral of $f$ on the unit circle.

The derivation is ugly.

The result is beautiful.
10. 22.4 – 27. 28.

Suppose \( f(z) \) and \( A(z) \) are analytic in a simply connected domain and

\[
f'(z) = A(z)f(z)
\]

for \( z \in \mathbb{C} \).

Let \( g(z) = e^\int_{z_0}^{z} A(w)dw \)

By the chain rule,

\[
g'(z) = g(z) \cdot \left( -\int_{z_0}^{z} A(w)dw \right)
= -A(z)g(z).
\]

So

\[
f'(z)g(z) = f'(z)g(z) + f(z)g'(z)
\]

\[
= f(z)g(z) + f(z)(-A(z)g(z))
= 0
\]

\[
\Rightarrow f(z)g(z) = \text{Constant}.
\]

\[
g(z) = e^{\int_{z_0}^{z} A(w)dw} = 1 ; \text{ let } f(z)=C
\]

\[
f(z) e^{\int_{z_0}^{z} A(w)dw} = C \Rightarrow \int_{z_0}^{z} A(w)dw = \ln C
\]

2a. \( f'(z) = 2z + f(z) \Rightarrow A(z) = 2z \ln |h| \)

So

\[
f(z) = C e^{\int_{z_0}^{z} 2zw dw} = C e^{z^2 - z^2}
= C e^{2z^2}
\]

b. \( f'(z) = -ze^{2}f(z) \Rightarrow A(z) = -e^{2}
\]

\[
f(z) = C e^{\int_{z_0}^{z} -e^{2} dw} = C e^{-e^{2z} - e^{2z}}
= C e^{-e^{2}}
\]

C. \( f'(z) = -\frac{1}{2} f(z) \) (Assume \( a \neq 0 \)).

\[
A(z) = -\frac{1}{2} \Rightarrow f(z) = C e^{\frac{1}{2} \int_{z_0}^{z} -w dw}
= C e^{-\frac{1}{2} \ln z}
= C \left( \frac{1}{z} \right)
\]

Directly: \( f'(z) + \frac{1}{2} f(z) = 0 \Rightarrow 2f'(z) + f(z) = 0 \Leftrightarrow (z - f(z)) = 0
\]

3. \( (z^2 + z - 2)^2 = (z-2)^2(z+1)^2 \)

Zero of order 2 at \( z = 2 \)
Zero of order 2 at \( z = -1 \)

5. \( z^2(1 - \cos z) = 0 \Rightarrow z = 0 \) or \( z \) is \( 0 \) or \( \pi \) or \( \cdots \)

\( \cos z = 1 \Rightarrow e^{iz} = 1 \Rightarrow 2z = 0 \Rightarrow e^{iz} = e^{i(\pi)} = i = \cos z + 1 \Rightarrow 2z = 1 \Rightarrow z = -2n \pi i, n \in \mathbb{Z} \)

At \( z = 2 \pi i, n \neq 0 \),

\[
f(z) = z^2(1 - \cos z)
= z^2 z + 2z(1 - \cos z)
\]

\[
f'(z) = 0 \text{ because } 2z(1 - \cos z) \text{ is } 0 \text{ as well.}
\]

\[
f''(z) = z^2 z + 2z(1 - \cos z)
\]

\[
f''(z) = 2z + 10 - 0 = 10
\]

So \( f \) has zero of order 2.

At \( z = 0, z^2(1 - \cos z) = 2(1 - \frac{2z^2}{2!} + \frac{2z^4}{4!} - \cdots ) \)

\[
= \frac{2z^2}{2!} \frac{2z^4}{4!} \cdots
\]

has a zero of order 4.

9. \( z(2e^{z} - 1) \) is entire, so

\[
z \left( \frac{2e^z}{n!} - 1 \right) = 2 \cdot \frac{2e^z}{n!} \cdot \frac{2e^z}{n+1} \cdot \cdots = \frac{2n}{n!} \cdot \frac{2n}{n+1} \cdot \cdots
\]

Converges for all \( n \)

10. \( \sum z \) is also entire

at \( z = \frac{1}{z} \) (let \( \frac{1}{z_0} = \sum z \))

\[
= \sum f(z)^n \left( \frac{1}{z} - 2 \right)^n
f(z) = 2 \Rightarrow \sum f(z)^n \left( \frac{1}{z} - 2 \right)^n
= \sum \frac{1}{n^2} \left( \frac{2}{z} - 1 \right)^n
f(z) = -1 \Rightarrow \sum f(z)^n \left( \frac{1}{z} - 2 \right)^n
= \sum \frac{1}{n^2} \left( \frac{2}{z} - 1 \right)^n
f(z) = -e^z \Rightarrow \sum f(z)^n \left( \frac{1}{z} - 2 \right)^n
\]

The pattern is that \( \frac{1}{z} \) cosine at 0, with powers of \( \frac{1}{z} \)