5. Binary forms and ternary quartics

We now apply the discussion of the last section to binary forms \((n = 2)\) and to ternary quartics \(((n,m) = (3,4))\). The results for binary forms are basically due to Fischer, Akhiezer and Krein, in a way made explicit below.

If \(n = 2\) and \(m = 2s\), then \(N(2,m) = 2s + 1\) and \(N(2,s) = s + 1\).

**Theorem 5.1**

Suppose \(p \in Q_{2,m} = \Sigma_{2,m}^*\).

(i) If \(w(p) \leq s\), then \(p\) has a strongly unique representation as a sum of \(w(p)\) \(m\)-th powers, and there exists \(g_p \in F_{2,w(p)}\) so that \(\mathcal{N}(p) = g_p F_{2,s-w(p)}\).

(ii) If \(w(p) = s + 1\), then \(\mathcal{N}(p) = \{0\}\) and, for every \((b,c) \neq (0,0)\), \(p\) has a unique minimal representation which includes a multiple of \((bx + cy)^m\).

**Proof**

(i) By Theorem 4.6, \(p\) has a minimal representation

\[
(5.2) \quad p(x,y) = \sum_{k=1}^{r} (b_k x + c_k y)^m,
\]

where \(r = w(p)\). We define two auxiliary forms:

\[
(5.3) \quad g_p(x,y) = \sum_{k=1}^{r} (c_k x - b_k y),
\]

\[
(5.4) \quad h(x,y) = (x^2 + y^2)^{s-r}(g_p(x,y))^2.
\]

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Then $h \in P_{2,m}$ and $Z(h) = \{(b_k, c_k) : 1 \leq k \leq r\}$ is $m$–independent by Theorem 2.10 because $r \leq s < 2s + 1$. It follows by Corollary 3.10 that every good representation of $p$ is a rearrangement of (5.2).

(ii) In this case $H_p$ is positive definite, so $\mathcal{K}(p) = \{0\}$. By Lemma 4.5, $\sigma_p(\alpha) > 0$ for $\alpha = (b, c)$, so $p$ has a minimal representation using $\tau(bx + cy)^m$.

Suppose $p$ has two such minimal representations:

\begin{equation}
(5.5) \quad p(x, y) = (\kappa bx + \kappa cy)^m + \sum_{k=2}^{s+1} (d_k x + e_k y)^m,
\end{equation}

\begin{equation}
(5.6) \quad p(x, y) = (\lambda bx + \lambda cy)^m + \sum_{k=2}^{s+1} (d_k' x + e_k' y)^m.
\end{equation}

Then by Theorem 4.9, $\sigma_p(\kappa b, \kappa c) = \sigma_p(\lambda b, \lambda c) = 1$, so $\kappa^m = \lambda^m$ by (4.4). Let $q(x, y) = p(x, y) - \kappa^m(bx + cy)^m$ and consider the two representations of $q$ arising from (5.5) and (5.6) by deletion of the common first term. Since $w(q) = \text{rank}(H_q) \leq s$, these are rearrangements of each other by (i). It follows that (5.5) and (5.6) are also rearrangements of each other; thus, $p$ covers $(b, c)$ minimally in exactly one way. \hfill \Box

We shall need the following peculiar corollary in connection with the truncated moment problem. Let

\begin{equation}
(5.7) \quad \bar{Q}_{2,m} = \{\sum (b_k x + c_k y)^m : c_k \neq 0\}
\end{equation}
denote the cone of binary m-ics which can be written as a sum of m-th powers
without using \( \lambda x^m \). It is easy to see that \( Q_{2,m} \) is the closure of \( \tilde{Q}_{2,m} \), and
that \( x^m \in Q_{2,m} \setminus \tilde{Q}_{2,m} \).

**Corollary 5.8**

If \( p \in Q_{2,m} \), then \( p \notin \tilde{Q}_{2,m} \) if and only if \( \text{rank}(H_p) \leq s \) and \( y \mid g_p \).

**Proof**

Suppose first that \( \text{rank}(H_p) = s + 1 \). Write \( p \) as a sum of \( s + 1 \) m-th
powers including \( \lambda x^m \); by Theorem 5.1(ii), this can be done in only one way.
Pick \((b,c)\) so that no multiple of \((bx + cy)^m\) occurs in this representation.
Now write \( p \) as sum of \( s + 1 \) m-th powers including \( \sigma(bx + cy)^m \). These two
representations cannot be rearrangements of each other, so the second one does
not contain a multiple of \( x^m \) and it follows that \( p \in \tilde{Q}_{2,m} \).

If \( \text{rank}(H_p) \leq s \), then \( p \) has up to rearrangement only one representation as
a sum of m-th powers, say (5.2). Thus, \( p \) can be written without using a
multiple of \( x^m \) unless \( c_k = 0 \) for some \( k \), in which case \( y \mid g_p \) by (5.3). \(\square\)

We now present an algorithm for determining whether a binary m-ic \( p \)
belongs to \( Q_{2,m} \), and for writing the elements of \( Q_{2,m} \) as explicit sums of m-th
powers. As in (1.23), write \( a_j = a(p; (j,m-j)) \) and consider the Hankel matrix
\([a_{j,k}]\), \( 0 \leq j, k \leq s \). Since \( Q_{2,m} = \Sigma_{2,m}^s \), \( p \in Q_{2,m} \) if and only if this matrix
is psd; the rank of this matrix is \( \text{rank}(H_p) = w(p) \). The isotropic vectors for
\( H_p \) (that is, the null eigenvectors of the matrix) give the coefficients of the
forms in \( \mathcal{V}(p) \leq F_{2,s} \).

If \( w(p) = r \leq s \), then we may choose a basis \( \{h_1, \ldots, h_{s+1-r}\} \) for \( \mathcal{V}(p) \) from
\( F_{2,s} \), and \( g_p = \gcd(h_1, \ldots, h_{s+1-r}) \) will be a product of \( r \) distinct linear
factors. If the factorization (5.3) holds (up to scalar multiple), then there will exist $\lambda_k > 0$ so that

$$p(x,y) = \sum_{k=1}^r \lambda_k (b_k x + c_k y)^m.$$  \hspace{1cm} (5.9)

(The structure of $\mathcal{N}(p)$ and the positivity of the solutions to (5.9) are not a priori obvious, but follow from Theorem 5.1(i).) If $w(p) = r = s + 1$, then $p$ has infinitely many inequivalent minimal representations. Select $\alpha = (b,c) \neq (0,0)$ and let $p_\nu(x,y) = p(x,y) - \nu(bx + cy)^m$. We compute $\sigma = \sigma_p(\alpha)$ by noting that $p_\nu \in \mathbb{Q}_{2,m}$ for $0 \leq \nu \leq \sigma$, rank($H_{p_\nu}$) = $s + 1$ for $0 \leq \nu < \sigma$ and rank($H_{p_\sigma}$) = $s$ by Lemma 4.5(ii). Thus $\nu = \sigma$ is a solution to the determinantal equation:

$$\det | a_{j+k} - \nu b_j c_{m-(j+k)} | = 0.$$ \hspace{1cm} (5.10)

Expanding (5.10) by columns, we obtain $2^{s+1}$ sub-determinants, leading to an equation in $\nu$ of nominal degree $s + 1$. Since any two columns containing $\nu$ are proportional, (5.10) reduces to $s + 2$ sub-determinants and a linear equation in $\nu$; see (5.16). After solving for $\nu = \sigma$, we obtain

$$p(x,y) = \sigma(bx + cy)^m + p_\sigma(x,y),$$ \hspace{1cm} (5.11)

where $w(p_\sigma) = s$; now we can write $p_\sigma$ as a sum of $s$ $m$-th powers as above, and thus obtain a minimal representation for $p$ from (5.11).
Example 5.12

We now discuss the simplest non-trivial example, the even binary quartic:
let \( p(x, y) = a_4x^4 + 6a_2x^2y^2 + a_0y^4 \). Then

\[
H_p(t_0, t_1, t_2) = a_0t_0^2 + a_2(2t_0t_2 + t_1^2) + a_4t_2^2,
\]

and it is easy to see that \( H_p \) is psd (and so \( p \in \mathbb{Q}_{2, 4} \)) if and only if

\[
a_0 \geq 0, \ a_4 \geq 0 \text{ and } (a_0a_4)^{1/2} \geq a_2 \geq 0.
\]

Suppose \( 0 \neq p \in \mathbb{Q}_{2, 4} \). Then \( w(p) = 1 \) if \((a_0, a_2, a_4) = (0, 0, >0) \) or \((>0, 0, 0)\); \( w(p) = 2 \) if \( a_0 > 0 \) and \( a_4 > 0 \), and either \( a_2 = 0 \) or \( a_2 = (a_0a_4)^{1/2} \); otherwise, \( w(p) = 3 \). If \( a_4 > 0 \), then one representation for \( p \) is:

\[
p(x, y) = \left(\frac{a_4}{2}\right)(x + \xi y)^4 + \left(\frac{a_4}{2}\right)(x - \xi y)^4 + \kappa y^4,
\]
\[
\xi = \left(\frac{a_2}{a_4}\right)^{1/2}, \ \kappa = \left(\frac{a_0a_4 - a_2^2}{a_4}\right).
\]

To compute \( \sigma_p((b, c)) \), we write out (5.10) explicitly:

\[
0 = \begin{vmatrix}
a_0 - \nu c_4^4 & -\nu bc_3^3 & a_2 - \nu b_2c_2^2 \\
-\nu bc_3^3 & a_2 - \nu b_2c_2^2 & -\nu b_3c_4 \\
a_2 - \nu b_2c_2^2 & -\nu b_2c_2^2 & a_4 - \nu b_4
\end{vmatrix}
\]

\[
= a_2(a_0a_4 - a_2^2) - \nu(a_0a_2b_4^4 + (a_0a_4 - 3a_2^2)b_2c_2^2 + a_2a_4c_4^2),
\]
from which \( \sigma_p((b,c)) \) can be computed. The reader may wish to verify this formula for the representation in (5.15), by taking in turn \((b,c) = (1,\xi), (1, -\xi)\) and \((0,1)\).

These results have already appeared in the literature in a somewhat disguised form. Suppose \( p \in \mathcal{Q}_{2,m} \) has the representation

\[
(5.17) \quad p(x,y) = \sum_{j=0}^{m} \binom{m}{j} a_j x^j y^{m-j} = \sum_{k=1}^{t} (b_k x + c_k y)^m,
\]

where \( c_1 = 0 \) (possibly \( b_1 = 0 \)) and \( c_k \neq 0 \) for \( k \geq 2 \). (This of course is not a good representation if \( b_1 = c_1 = 0 \).) Let \( \rho_k = c_k^m \) and \( \xi_k = b_k/c_k \) for \( k \geq 2 \). Then (5.17) is equivalent to the following system of equations:

\[
(5.18) \quad a_j = \sum_{k=2}^{t} \rho_k \xi_k^j, \quad 0 \leq j \leq m - 1; \quad a_m = \sum_{k=2}^{t} \rho_k \xi_k^m + b_1^m.
\]

Guided by this reinterpretation, we now present homogenized versions of two theorems proved by Fischer [F2] in 1911 and Akhiezer and Krein [A2] in 1938, in connection with moment problems. The original proofs (see [A2, pp.1-12]) used orthogonal polynomials. We happily acknowledge that a homogenization of their presentation inspired much of this paper, though our proofs are ultimately quite different than those of the originals.

**Proposition 5.19** (Fischer)

Suppose \( a_0 > 0 \), \( H_p(t) \) is psd but not positive definite and \( r < s \) is maximal so that \( H_p(t_0, \ldots, t_r, 0, \ldots, 0) \) is positive definite in \((t_0, \ldots, t_r)\).
Then there is a unique choice of \( \rho_k > 0, 1 \leq k \leq r, \xi_1 < \ldots < \xi_{r+1} \) and \( \tau \geq 0 \) so that

\[
a_j = \sum_{k=1}^{r+1} \rho_k \xi_k^j, \quad 0 \leq j \leq m - 1; \quad a_m = \sum_{k=1}^{r+1} \rho_k \xi_k^m + \tau.
\]  (5.20)

**Proposition 5.21** (Akhiezer and Krein)

Suppose \( H_p \) is positive definite.

(i) There exist infinitely many representations of the form

\[
a_j = \sum_{k=1}^{s+1} \rho_k \xi_k^j, \quad 0 \leq j \leq m, \quad \rho_k > 0, \quad \xi_1 < \ldots < \xi_{s+1}.
\]  (5.22)

(ii) There is an exceptional set \( E = \{\eta_1, \ldots, \eta_s\} \) so that, for every \( \eta \in \mathbb{R} \setminus E \), there is a unique set \( \{\xi_1, \ldots, \xi_{s+1}\} \) satisfying (5.22) and containing \( \eta \).

Using (5.18), we see that Fischer's Theorem is Theorem 5.1(i): \( H_p(u) = 0 \) if and only if \( g(x,y) = L((x,y); u) \in \mathcal{K}(p) \). Suppose \( w(p) = r \leq s \) and, following (5.3), let \( g_p(x,y) = \sum x^r y^{r-i} \). Then Theorem 5.1(i) says that \( \mathcal{E}(H_p) \) is spanned by

\[
(5.23) \quad \{(x_0, \ldots, x_r, 0, \ldots, 0), (0, x_0, \ldots, x_r, 0), \ldots, (0, \ldots, 0, x_0, \ldots, x_r)\},
\]

which corresponds to \( \{x^j y^{s-r-j} g_p\} \). If \( x_r = 0 \), then \( y|g_p \) and \( r x^m \) occurs in the representation of \( p \). If \( x_r \neq 0 \), then \( y|g_p \) and \( \tau = 0 \). (One can recover this information about \( \tau \) from the proof in [A2].)
If \( w(p) = s + 1 \), then \( \mathcal{N}(p) = \{0\} \) and \( H_p \) is positive definite. Proposition 5.21 restates Theorem 5.1(ii); the exceptional set in part (ii) consists of the \( \xi_j \)'s which arise when \( p \) is written minimally using \( \lambda x^m \). (They are also the roots of a certain orthogonal polynomial.) It should be noted that the Akhiezer–Krein Theorem contains results on \( \sum k \ell_k = m + 1 \) and the interlacing of roots which we have not discussed here, but see [K3].

We now turn to ternary quartics. Suppose

\[
(5.24) \quad p(x, y, z) = \sum_{i+j+k=4} c(i, j, k) a_{ijk} x^i y^j z^k \in \mathcal{F}_{3,4}.
\]

As \( N(3, 2) = 6 \), \( H_p(t) \) is a quadratic form in six variables, which we order as follows: \( t_1 = t((2, 0, 0)) \), \( t_2 = t((0, 2, 0)) \), \( t_3 = t((0, 0, 2)) \), \( t_4 = t((1, 1, 0)) \), \( t_5 = t((1, 0, 1)) \), \( t_6 = t((0, 1, 1)) \). The resulting matrix for \( H_p \) is

\[
\begin{bmatrix}
  a_{400} & a_{220} & a_{202} & a_{310} & a_{301} & a_{211} \\
  a_{220} & a_{040} & a_{022} & a_{130} & a_{121} & a_{031} \\
  a_{202} & a_{022} & a_{004} & a_{112} & a_{103} & a_{013} \\
  a_{310} & a_{130} & a_{112} & a_{220} & a_{211} & a_{121} \\
  a_{301} & a_{121} & a_{103} & a_{211} & a_{202} & a_{112} \\
  a_{211} & a_{031} & a_{013} & a_{121} & a_{112} & a_{022}
\end{bmatrix}.
\]

(5.25)

(This matrix is used in [E1, p.294] to illustrate Sylvester's proof of Clebsch's result, discussed in the Historical Notes to the last section.) Since \( \mathcal{Q}_{3,4} = \Sigma_{3,4} \), \( p \in \mathcal{Q}_{3,4} \) if and only if \( H_p \) is psd. By Theorem 4.6, if \( H_p \) is psd and \( w(p) = \text{rank}(H_p) = r \), then there exists \( A = \{a_{\ell}: 1 \leq \ell \leq r\} = \{(b_{\ell}, c_{\ell}, d_{\ell})\} \) so that \( p \) has a minimal representation:
(5.26) \[ p(x, y, z) = \sum_{\ell=1}^{r} (b_{\ell}x + c_{\ell}y + d_{\ell}z)^4. \]

Suppose \( H_p(u) = 0 \); i.e., \( u \) is an isotropic vector for \( H_p \) and \((u_1, \ldots, u_6)^T \) is a null eigenvector for (5.25), and let

(5.27) \[ g(x, y, z) = L((x, y, z); u) = u_1 x^2 + u_2 y^2 + u_3 z^2 + u_4 xy + u_5 xz + u_6 yz. \]

By Theorem 3.8, \( g(b_{\ell}, c_{\ell}, d_{\ell}) = 0 \) and more generally,

(5.28) \[ \mathcal{N}(p) = I(A) \cap F_{3, 2} \]

is a \((6 - r)\)-dimensional subspace of \( F_{3, 2} \). By Corollary 4.8(v), \( p \) covers \( \alpha \) if and only if \( \mathcal{N}(p) \subseteq I(\{\alpha\}) \).

We use two parallel approaches for the analysis of ternary quartics. Certainly, by (5.28), the geometry of \( A \) determines \( \mathcal{N}(p) \). The existence of strongly unique representations may follow from Corollary 3.10, so we are also interested in finding \( h \in P_{3, 4} \) so that \( I(h) = A \). The disadvantage of both approaches is that they depend on prior knowledge of a minimal representation for \( p \). Fortunately, we are eventually able to present our criteria so that they do not require having such a representation of \( p \) in hand.

It will be convenient to make (invertible) linear changes of variable for the \( \alpha_k \)'s. (If \( p = \Sigma(\alpha_k \cdot)^d \) and \( \alpha_k \mapsto T(\alpha_k) \) is expressed as \( \alpha_k \mapsto \alpha_k M \), then \( p \circ M = \Sigma(T(\alpha_k \cdot)^d \cdot \).) These changes do not affect such quantities as \( w(p) \) and \( \text{rank}(H_p) \) or the possible uniqueness of representation. We also make the simple observation that, if \( \{\beta_1, \ldots, \beta_r\} \) is a linearly independent set in \( \mathbb{R}^n \), then there is a linear change taking \( \beta_j \) into \( e_j \), the \( j \)-th unit vector, for
$1 \leq j \leq r$. Planes and lines below shall always contain the origin, and so are subspaces of $\mathbb{R}^3$; linear changes take planes and lines into planes and lines.

**Lemma 5.29**

Suppose $p \in Q_{3,4}$ has the minimal representation (5.26).

(i) If $Z(p) \neq \emptyset$, then $\text{rank}(H_p) \leq 3$ and $A \subseteq \pi$ for some plane $\pi$.

(ii) If three elements of $A$ lie in a plane $\pi$, then $p$ covers every $\alpha \in \pi$.

(iii) No four elements of $A$ lie in a plane $\pi$.

**Proof**

(i) After a linear change we may assume that $p(0,0,1) = 0$. It follows from (5.26) that $d_\ell = 0$ for all $\ell$, and $p = p(x,y) \in Q_{2,4}$. By the discussion of Example 5.12, $\text{rank}(H_p) \leq 3$ and $A$ is contained in the plane $\pi = \{(b,c,0)\}$ = $\{z = 0 \}$. 

(ii) As above, after a linear change, we may assume that $\pi = \{z = 0 \}$, so $\alpha_\ell = (b_\ell,c_\ell,0)$ for $1 \leq \ell \leq 3$. If $g \in \mathcal{N}(p)$, then $g(x,y,0)$ is a binary quadratic which has zeros at the three distinct points $\{(b_\ell,c_\ell)\}$, so $g(x,y,0) \equiv 0$. It follows that $z \mid g$ and $g(b,c,0) = 0$ for all $(b,c)$; that is, for all $\alpha \in \pi$, $g(\alpha) = 0$, so $\mathcal{N}(p) \subseteq I(\{\alpha\})$.

(iii) We again assume that $\alpha_\ell = (b_\ell,c_\ell,0)$ lies in the plane $\pi = \{z = 0 \}$ for $1 \leq \ell \leq 4$. Since $\dim(F_{2,2}) = 3$, the set $\{(b_\ell x + c_\ell y)^2\}$ is linearly dependent, and so by Theorem 2.16, (5.26) cannot be minimal. (Alternatively, a sum of four $(b_\ell x + c_\ell y)^4$'s lies in $Q_{2,4}$, and, by Theorem 4.6, can be rewritten as a sum of three fourth powers. This contradicts the purported minimality of (5.26).)

$\square$

We proceed now with our analysis of $Q_{3,4}$. 
Theorem 5.30

If \( p \in Q_{3,4} \) and \( w(p) \leq 2 \), then \( p \) has a strongly unique representation.

Proof

After renaming variables, \( p(x,y,z) \) is either \( x^4 \) or \( x^4 + y^4 \), and Theorem 5.1(i) applies. □

Theorem 5.31

If \( p \in Q_{3,4} \) and \( w(p) = 3 \), then \( p \) has a strongly unique representation if and only if \( \mathcal{I}(p) = \emptyset \). If \( \mathcal{I}(p) \neq \emptyset \), then there is a plane \( \pi \) so that \( p \) covers \( \alpha \) if and only if \( \alpha \in \pi \).

Proof

There are two cases, depending on whether \( A \) is linearly independent. If \( A \) is dependent, then it determines a plane \( \pi \), and by Lemma 5.29(ii), \( p \) covers every \( \alpha \in \pi \). After a linear change taking \( \{\alpha_1, \alpha_2\} \) into \( \{e_1, e_2\} \), we may assume that \( \pi = \{z = 0\} \), so that \( p(x,y,z) = x^4 + y^4 + (bx + cy)^4 \) and \( \mathcal{I}(p) = \{e_3\} \). (This is consistent with Theorem 5.1(ii), since \( s + 1 = 3 \).)

If \( A \) is linearly independent, there is a linear change taking it into \( \{e_1, e_2, e_3\} \), and taking \( p(x,y,z) \) into \( x^4 + y^4 + z^4 \). We now apply Corollary 3.10 with \( h(x,y,z) = x^2y^2 + x^2z^2 + y^2z^2 \in P_{3,4} \). Since \( \mathcal{I}(h) = \{e_1, e_2, e_3\} \) is 4-indepedent, \( p \) has a strongly unique representation. Here, \( \mathcal{I}(p) = \emptyset \). □

Theorem 5.32

Suppose \( p \in Q_{3,4} \) and \( w(p) = 4 \), so \( \dim N(p) = 2 \), and that \( \{g_1, g_2\} \) is a basis for \( \mathcal{N}(p) \).
(i) If \( g_1 \) and \( g_2 \) are relatively prime, then \( p \) has a strongly unique representation.

(ii) If \( g_1 \) and \( g_2 \) have a common (linear) factor, then \( p \) does not have a strongly unique representation. Write \( g_1 = L_0L_1 \) and \( g_2 = L_0L_2 \), where \( L_i = L_i(x,y,z) \) is linear; let \( \pi \) denote the plane \( \{L_0 = 0\} \) and \( \lambda \) denote the line \( \{L_1 = 0\} \cap \{L_2 = 0\} \). Then \( p \) covers \( \alpha \) if and only if \( \alpha \in \pi \) or \( \alpha \in \lambda \).

**Proof**

Observe that the condition on \( g_1 \) and \( g_2 \) is independent of choice of basis, and is unaltered by a linear change. In either case, since \( w(p) = 4 \), Lemma 5.29(iii) implies that \( A \) does not lie in a plane, and we can assume after reindexing that \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are linearly independent. After the linear change taking \( \alpha_\ell \) to \( e_\ell \) for \( \ell \leq 3 \) and \( \alpha_4 \) to \( (b,c,d) \), we have:

\[
(5.33) \quad p(x,y,z) = x^4 + y^4 + z^4 + (bx + cy + dz)^4.
\]

If no three \( \alpha_\ell \)'s are dependent, then \( bcd \neq 0 \); otherwise, \( bcd = 0 \), and we can take \( d = 0 \), without loss of generality. We designate these possibilities as case (a) and case (b) respectively. By (5.28), the ternary quadratic \( g \) belongs to \( \mathcal{N}(p) \) if and only if \( e_1, e_2, e_3 \) and \( (b,c,d) \) belong to \( \mathcal{E}(g) \), so

\[
(5.34) \quad \mathcal{N}(p) = \{s_0xy + s_1xz + s_2yz: \ bcs_0 + bds_1 + cds_2 = 0\}.
\]

Let

\[
(5.35) \quad h(x,y,z) = x^2(dy - cz)^2 + y^2(bz - dx)^2 + z^2(cx - by)^2 \in P_{3,4}.
\]
In case (a), \( bcd \neq 0 \), and it is easy to see that

\[(5.36) \quad Z(h) = \{(1,0,0),(0,1,0),(0,0,1),(b,c,d)\}.\]

Since \( x^4, y^4, z^4 \) and \((bx + cy + dz)^4\) are linearly independent, \( Z(h) \) is 4-independent, and \( p \) has a strongly unique representation by Corollary 3.10. In this case, we take \( g_1(x,y,z) = x(dy - cz) \) and \( g_2(x,y,z) = y(dx - bz) \) as the basis for \( \mathcal{N}(p) \) from (5.34); \( g_1 \) and \( g_2 \) are relatively prime.

In case (b), where \( d = 0 \), (5.36) is incomplete: \( h(r,s,0) = 0 \) for all \( r \) and \( s \). Furthermore, \( q(x,y) = x^4 + y^4 + (bx + cy)^4 \) has infinitely many representations, as we have seen in Theorem 5.1(ii), as does \( p(x,y,z) = q(x,y) + z^4 \). In this case, we may take \( g_1(x,y,z) = xz \) and \( g_2(x,y,z) = yz \) from (5.34); these have the common linear factor \( z \). Further, \( p \) covers \( \alpha = (b,c,d) \) if and only if \( g_j(\alpha) = 0 \); that is, \( bd = cd = 0 \): this happens if \( b = c = 0 \) (\( \alpha \in \lambda \)), or \( d = 0 \) (\( \alpha \in \tau \)).

The preceding arguments show that cases (a) and (b) coincide with parts (i) and (ii) of the theorem, respectively. \( \Box \)

Note that, in the second case,

\[(5.37) \quad p(x,y,z) = x^4 + y^4 + (bx + cy)^4 + z^4,\]

and all minimal representations of \( p \) have the shape

\[(5.38) \quad p(x,y,z) = (b_1x + c_1y)^4 + (b_2x + c_2y)^4 + (b_3x + c_3y)^4 + z^4.\]
Theorem 5.39

If \( p \in Q_{3,4} \) and \( w(p) = 5 \), then \( p \) has infinitely many minimal representations; there exists \( g \in F_{3,2} \) so that \( p \) covers \( \alpha \) if and only if \( g(\alpha) = 0 \).

Proof

Choose one minimal representation (5.26) for \( p \). Since \( r = 5 \), \( \mathcal{N}(p) \) has dimension \( 6 - 5 = 1 \); let \( g \) generate \( \mathcal{N}(p) \). Then \( g(\alpha_\ell) = 0 \) for \( 1 \leq \ell \leq 5 \), so \( g \) is the (unique) ternary quadratic determined by these five points. (Any five points determine at least one quadratic; it is unique provided no four lie in a plane, c.f. Lemma 5.29(iii).) Since \( |\mathcal{Z}(g)| > 1 \), \( g \) is indefinite, hence \( |\mathcal{Z}(g)| = \infty \). By Corollary 4.8(v), \( p \) covers \( \alpha \) if and only if \( g(\alpha) = 0 \), so \( p \) has infinitely many different minimal representations. \( \square \)

Theorem 5.40

If \( p \in Q_{3,4} \) and \( w(p) = 6 \), then \( p \) covers minimally every \( \alpha \neq 0 \) and \( p \in \text{int} \, Q_{3,4} \).

Proof

Since \( \text{rank}(H_p) = N(3,2) = 6 \), this is Corollary 4.8(ii),(iii). \( \square \)

Example 5.41

Let \( p(x,y,z) = (x^2 + y^2)^2 + (x^2 + z^2)^2 \); since \( c(4,0,0) = 1 \) and \( c(2,2,0) = 6 \), in the notation of (5.24) we have: \( a_{400} = 2 \), \( a_{040} = a_{004} = 1 \), \( a_{220} = a_{202} = 1/3 \) and \( a_{ijk} = 0 \) otherwise. It follows from (5.25) that \( \text{rank}(H_p) = 5 \) and \( H_p(0,\ldots,0,1) = 0 \), so \( \mathcal{N}(p) \) is generated by \( (yz) \). By Theorem 5.39, \( p \) covers \( \alpha = (b,c,d) \) if and only if \( cd = 0 \). We now indicate how to write \( p \) as a
sum of 4-th powers which covers such an \( \alpha \). By symmetry, it suffices to assume \( c = 0 \) and \( \alpha = (b,0,d) \). Write \( p(x,y,z) = q_1(x,y) + q_2(x,z) \), where

\begin{align*}
(5.42) \quad q_1(x,y) &= (1/9)x^4 + 2x^2y^2 + y^4 - \frac{1}{2} ((3^{-1/2}x + y)^4 + (3^{-1/2}x - y)^4), \\
(5.43) \quad q_2(x,z) &= (17/9)x^4 + 2x^2z^2 + z^4.
\end{align*}

The argument in Example 5.12 shows that \( w(q_1) = 2 \) and \( w(q_2) = 3 \). In fact, \( q_1 \) has the unique representation given above, and \( q_2 \) can be written as a sum of three 4th powers, covering any preassigned \((b,0,d)\).

The general behavior of the minimal representations of ternary sextics when \( r = 5 \) or 6 seems to be hard to describe explicitly. (This last example illustrates a degenerate case in which \( \mathcal{N}(p) \) is generated by a product of two linear forms.) For two more examples of the representations of ternary quartics for \( r = 5 \) and 6, see the discussion of \( h_{3,4} \) and \( q \) (c.f. (9.15)) in section nine. However, both \( h_{3,4} \) and \( q \) possess symmetries which manifest themselves in their representations. The general case is still open.