3. Cones of n-ary m-ics and their duals

In this section, we shall discuss the cones $P_{n,m}$, $\Sigma_{n,m}$, and $Q_{n,m}$ (recall that $m$ is even.) We begin with a unified proof that these cones are closed and convex; these results are due to Hilbert [H7] and R. M. Robinson [R7]. We shall need two intermediate definitions: for $r \geq 1$, let

$$\Sigma_{n,m}^r = \{ \sum_{k=1}^{r} h_k^2 : h_k \in F_{n,s} \},$$

(3.1)

$$Q_{n,m}^r = \{ \sum_{k=1}^{r} (\alpha_k \cdot x)^m : \alpha_k \in \mathbb{R}^n \}. $$

(3.2)

Lemma 3.3

For all $r \geq 1$, $\Sigma_{n,m}^r$ and $Q_{n,m}^r$ are closed sets in $F_{n,m}$.

Proof

Suppose $p_j \in Q_{n,m}^r$ and $p_j \to p$. Then,

$$p_j(x) = \sum_{k=1}^{r} (\alpha_{j1} x_1 + \cdots + \alpha_{jn} x_n)^m = \sum_{i \in I} c(i) a(p_j;i)x^i$$

(3.4)

and $a(p_j;i) \to a(p;i)$. For $1 \leq b \leq n$, let $e_b$ denote the $b$-th unit vector and reindex so that $i_b = me_b$. Then $x^i_b = x^m_b$ and, since $c(i_b) = 1,$

$$\sum_{k=1}^{r} (\alpha_{jkb})^m = a(p_j;i_b) \to a(p;i_b) = p(e_b).$$

and $a(p;j_i) \to a(p;i).$
Thus there is a uniform upper bound \( M \) for \( |\alpha_{jkb}|, 1 \leq k \leq r, 1 \leq b \leq n, j \geq 1 \), and there is a subsequence \((j_v)\) so that for each \((k,b)\), \(\{\alpha_{j_vkb}\} \) converges to some \(\alpha_{kb}\). Hence,

\[
(3.5) \quad p(x) = \sum_{k=1}^{r} (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^m \in \mathbb{Q}_{n,m}^r.
\]

Similarly, suppose \(p_j(x) = \sum_{k=1}^{r} (h_{k,j}(x))^2 \in \Sigma_{n,m}^r\) and \(p_j(x) \to p(x)\); also suppose \(\{\alpha_\ell\}\) is a basic set of nodes for \(F_{n,s}\) and \(M = \max\{p(\alpha_\ell)\}\). Then for \(j\) sufficiently large, \(0 \leq p_j(\alpha_\ell) \leq 2M\), so \(|h_{k,j}(\alpha_\ell)| \leq (2M)^{1/2}\). In the language of (2.9), \(|a(h_{j,k};i)| \leq (2M)^{1/2}\Sigma_j \lambda_k(i) | \leq T\) uniformly in \(i\). Since the coefficients in \(h_{k,j}\) are uniformly bounded, we may select a convergent subsequence, \(h_{k,j_v}(x) \to h_k(x)\), so that \(p(x) = \sum_{k=1}^{r} (h_{k}(x))^2 \in \Sigma_{n,m}^r\). \(\square\)

Proposition 3.6

\(P_{n,m}, Q_{n,m}\) and \(\Sigma_{n,m}\) are closed convex cones.

Proof

If \(p_j \in P_{n,m}\) and \(p_j \to p\), then for all \(x\), \(0 \leq p_j(x) \to p(x)\), and so \(p \in P_{n,m}\); thus \(P_{n,m}\) is closed. Carathéodory's Theorem implies that, for \(r \geq N = N(n,m)\), \(\Sigma_{n,m}^r = \Sigma_{n,m}^N\) and \(Q_{n,m}^r = Q_{n,m}^N\), and so \(\Sigma_{n,m} = \Sigma_{n,m}^N\) and \(Q_{n,m} = Q_{n,m}^N\) are both closed by the last theorem. (As noted earlier, Hilbert had proved that \(\Sigma_{3,6} = \Sigma_{3,6}^{28}\), essentially by Carathéodory's argument.) \(\square\)
We now compute the duals of these cones, and study their interiors.

**Theorem 3.7**

\[ P_{n,m}^* = Q_{n,m} \] and \[ Q_{n,m}^* = P_{n,m} \].

**Proof**

As \( P_{n,m} \) and \( Q_{n,m} \) are both closed convex cones, these assertions are equivalent. Since the elements of \( Q_{n,m} \) are the finite sums \( \Sigma(\alpha_k x)^m \), \( p \) belongs to \( Q_{n,m}^* \) if and only if \([p, \Sigma(\alpha_k x)^m] = \Sigma p(\alpha_k) \geq 0 \) for all sets \( A = \{\alpha_1, \ldots, \alpha_r\} \subseteq \mathbb{R}^n \). This occurs of course if and only if \( p(\alpha) \geq 0 \) for all \( \alpha \); that is, if and only if \( p \) is psd. \( \square \)

**Theorem 3.8**

Suppose \( A = \{\alpha_1, \ldots, \alpha_r\} \) and \( p(x) = \Sigma(\alpha_k x)^m \).

(i) If \( h \in P_{n,m} \), then \([p,h] = 0 \) if and only if \( A \subseteq \mathcal{Z}(h) \); i.e., \( h \in I(A) \).

(ii) If \( B = \{\beta_1, \ldots, \beta_t\} \) and \( p(x) = \Sigma(\beta_x^t x)^m \), then \( I(A) \cap P_{n,m} = I(B) \cap P_{n,m} \).

(iii) \( H_p(u) = 0 \) if and only if \( L(\alpha_k; u) = 0 \) for all \( k \); i.e., \( A \subseteq \mathcal{Z}(L; u) \).

(iv) \( \mathcal{N}(p) = I(A) \cap F_{n,s} \).

(v) \( \mathcal{N}(p) \) is an \((N(n,s) - \text{rank}(H_p))\)-dimensional subspace of \( F_{n,s} \).

**Proof**

In (i), since \([p,h] = \Sigma h(\alpha_k) \) and \( h \) is psd, \([p,h] \geq 0 \), with equality if and only if \( h(\alpha_k) = 0 \) for all \( k \). For (ii), observe that

\[ \{h \in P_{n,m} : [p,h] = 0\} = I(A) \cap P_{n,m} = I(B) \cap P_{n,m}. \]
The proofs of (iii), (iv), and (v) are immediate upon recalling (1.21), setting \( h(x) = L^2(u) = g^2(u) \) in (i) and identifying \( \mathcal{N}(p) \) with \( \mathcal{E}(\mathcal{H}_p) \). \( \square \)

**Corollary 3.10**

Suppose \( h \in P_{n,m} \), \( \mathcal{E}(h) = \{\alpha_1, \ldots, \alpha_r\} \) is \( m \)-independent and

\[
(3.11) \quad p(x) = \sum_{k=1}^{r} (\lambda_k \alpha_k \cdot x)^m,
\]

where \( \lambda_k \neq 0 \). Then (3.11) is a strongly unique representation for \( p \). In particular, \( w(p) = r \).

**Proof**

By Theorem 3.8(i), \( [p,h] = 0 \). Suppose

\[
(3.12) \quad p(x) = \sum_{j=1}^{b} (\beta_j \cdot x)^m
\]

is any good representation of \( p \); then \( h(\beta_j) = 0 \) by Theorem 3.8(i). Since (3.12) is good, no two \( \beta_j \)'s are proportional, and thus we may assume that \( \beta_j = \nu_j \alpha_j \) after reindexing, \( 1 \leq j \leq b < r \). But if

\[
(3.13) \quad p(x) = \sum_{k=1}^{r} (\lambda_k \alpha_k \cdot x)^m = \sum_{k=1}^{b} (\nu_k \alpha_k \cdot x)^m,
\]

then \( \nu_k^m = \lambda_k^m \) (and so \( b = r \)) by the \( m \)-independence of \( \{\alpha_k\} \). Since \( \nu_k = \pm \lambda_k \), (3.12) is a rearrangement of (3.11), completing the proof. \( \square \)
Theorem 3.14

(i) $p \in \text{int } P_{n,m}$ if and only if $p$ is strictly definite.

(ii) $p \in \text{int } Q_{n,m}$ if and only if $p \in Q_{n,m}$, and in every representation $p = \Sigma(\alpha_k \cdot)^m$, we have $I(\{\alpha_k\}) \cap P_{n,m} = \{0\}$.

(iii) If $\{\alpha_k\}$ is a basic set of nodes for $F_{n,m}$, then $\Sigma(\alpha_k \cdot)^m \in \text{int } Q_{n,m}$.

(iv) If $p \in \text{int } Q_{n,m}$, then $\omega(p) \geq N(n,s)$.

Proof

(i) By Lemma 2.2, $p \in \text{int } P_{n,m}$ if and only if for every non-empty $A = \{\alpha_1, \ldots, \alpha_r\} \subseteq \mathbb{R}^n \setminus \{0\}$, $[p, \Sigma(\alpha_k \cdot)^m] = \Sigma p(\alpha_k) > 0$; that is, if and only if $p(\alpha) > 0$ for all $\alpha \neq 0$.

(ii) Similarly, $p = \Sigma(\alpha_k \cdot)^m \in \text{int } Q_{n,m}$ if and only if $0 \neq q \in P_{n,m}$ implies that $[\Sigma(\alpha_k \cdot)^m, q] = \Sigma q(\alpha_k) > 0$; that is, $\alpha_\ell \notin \mathcal{Z}(q)$ for some $\ell$.

(iii) If $q = 0$ on a basic set of nodes for $F_{n,m}$, then $q = 0$ by (2.14). Thus, $\{0\} = I(\{\alpha_k\}) \cap F_{n,m} = I(\{\alpha_k\}) \cap P_{n,m}$, and $\Sigma(\alpha_k \cdot)^m \in \text{int } Q_{n,m}$ by (ii).

(iv) Since $\Sigma_{n,m} \subseteq P_{n,m}$, $Q_{n,m} \subseteq \Sigma_{n,m}^*$, thus if $p \in \text{int } Q_{n,m}$, then $p \in \text{int } \Sigma_{n,m}^*$; for the rest of the proof, see Theorem 3.16(iv). \qed

Corollary 3.15

$Q_{n,m}$, $P_{n,m}$ and $\Sigma_{n,m}$ have non-empty interiors (in $\mathbb{R}^{N(n,m)}$).

Proof

Since $Q_{n,m} \subseteq \Sigma_{n,m} \subseteq P_{n,m}$, it suffices to show that $Q_{n,m}$ has a nonempty interior; combine Proposition 2.7 and Theorem 3.14(iii). \qed

Theorem 3.16

(i) $\Sigma_{n,m}^* = \{p \in F_{n,m} : H_p \text{ is psd}\}$. 


(ii) Suppose \( p \in F_{n,m} \). Then \( p \in \text{int} \Sigma_{n,m}^* \) if and only if \( H_p \) is positive definite (i.e., \( \mathcal{N}(p) = \{0\} \)).

(iii) If \( p \in \text{int} \Sigma_{n,m}^* \), then \( w(p) \geq N(n,s) \).

(iv) If \( \{\alpha_k\} \) is a basic set of nodes for \( F_{n,s} \) (not \( F_{n,m} \)), then \( p = \Sigma(\alpha_k \cdot)^m \in \text{int} \Sigma_{n,m}^* \).

**Proof**

(i) By definition, \( p \in \Sigma_{n,m}^* \) if and only if \( [p, \Sigma f_k^2] \geq 0 \); that is, if and only if \( [p, f^2] \geq 0 \) for every \( f \in F_{n,s} \). Since \( F_{n,s} = \{L(u); u \in \mathbb{R}^N(n,s)\} \), \( p \in \Sigma_{n,m}^* \) if and only if \( [p, L(u)^2] = H_p(u) \geq 0 \) for all \( u \in \mathbb{R}^N(n,s) \).

(ii) This is immediate from Lemma 2.2 and \( H_p(t) = [p, L(t)^2] \).

(iii) By (1.29) and (ii), \( w(p) \geq \text{rank}(H_p) = N(n,s) \).

(iv) If \( [p, g^2] = 0 \), then \( \Sigma(g(\alpha_k))^2 = 0 \), so \( g \in F_{n,s} = 0 \) on a basic set of nodes for \( F_{n,s} \) and so \( g = 0 \) by (2.14); thus \( \mathcal{N}(p) = \{0\} \) and \( p \in \text{int} \Sigma_{n,m}^* \) by part (ii).

\( \Box \)

Any \( p \), as described in (iv), will be a sum of \( N(n,s) \) \( m \)-th powers, and have width \( N(n,s) \). Alternate descriptions of the interiors of \( Q_{n,m} \) and \( \Sigma_{n,m}^* \) are given in Corollary 4.8.

We may combine Theorem 3.16 with Hilbert's Theorem.

**Corollary 3.17**

If \( (n,m) \in \mathcal{X} \), then \( p \in Q_{n,m} \) if and only if \( H_p \) is a psd quadratic form. If \( (n,m) \notin \mathcal{X} \), then there exists \( q \notin Q_{n,m} \) for which \( H_q \) is psd.

This corollary is trivial if \( m = 2 \): the quadratic form \( p = H_p \) is a sum of squares if and only if it is psd. The implications for \( n = 2 \) and \( (n,m) = (3,4) \)
are discussed in the next two sections. If \( q \not\in Q_{n,m} \) and \( H_q \) is psd, then \( H_q \) is a sum of squares of linear forms in \( \{t(\ell)\} \) which cannot be written as a sum of squares from the family of linear forms \( \{L(\alpha; t)\} \). In section six, we shall interpret a dehomogenized version of strict inclusion in terms of the Haviland moment problem.

By Lemma 2.2, Theorem 3.16(i) and duality, we have the following:

**Corollary 3.18**

(i) \( p \in \Sigma_{n,m} \) if and only if \( H_q \) psd implies that \( [p,q] \geq 0 \).

(ii) \( p \in \text{int} \Sigma_{n,m} \) if and only if \( H_q \) psd and \( q \neq 0 \) implies that \( [p,q] > 0 \).

A direct description of \( \Sigma_{n,m} \) can be derived from the expansion of the relevant squares. The following criterion is given in [C5] and [R4] and more fully described in [C8]:

\[
p \in \Sigma_{n,m} \text{ if and only if there is a psd quadratic form } h(t) = \\
\left(3.19\right) \sum_{\ell, \ell'} \gamma(\ell, \ell') t(\ell) t(\ell') \text{ so that } c(i)a(p;i) = \sum_{\ell+\ell'=i} \gamma(\ell, \ell') \text{ for all } i \in I(n,s).
\]

A comparison of Theorems 3.14(ii) and 3.16(ii) raises a natural question: if \( A = \{\alpha_k\} \) is a basic set of nodes for \( F_{n,s} \), is \( I(A) \cap P_{n,m} = \{0\} \)? The answer is "sometimes." We show now that the answer is "yes" if we take \( A = I(n,s) \) as the basic set of nodes (c.f. Proposition 2.11, Theorem 3.22), but give a counterexample in \( P_{3,6} \) based on an example due to R. M. Robinson.
Lemma 3.20

Let \( J_n \) denote the \( n \times n \) matrix of 1's. Then \( M = I_n + s^{-1}J_n \) is invertible, and, for \( i \in I(n,s) \),

\[
(3.21) \quad M_i^T = i^T + (1, \ldots, 1)^T.
\]

Proof

Since \( J_n^2 = nJ_n \), \( M^{-1} = I_n - (n+s)^{-1}J_n \) exists; (3.21) follows from the fact that each component of \( s^{-1}J_n i^T \) is \( s^{-1}\sum k = 1 \).

Theorem 3.22

\( I(I(n,s)) \cap P_{n,m} = \{0\} \).

Proof

Suppose \( n = 2 \) and \( p \in P_{2,m} \). If \( p(j,s-j) = 0 \) for \( 0 \leq j \leq s \), then

\[
(3.23) \quad \prod_{j=0}^{s} ((s-j)x - jy)^2 \mid p(x,y).
\]

Since \( \deg p = m < 2(s+1) = m + 2 \), (3.23) implies that \( p = 0 \).

Suppose now that the theorem holds for \( n - 1 \) and suppose \( p \in P_{n,m} \), where \( m < 2n \). Let

\[
(3.24) \quad h(x_1, \ldots, x_{n-1}) = p(x_1, \ldots, x_{n-1}, 0) \in P_{n-1,m}.
\]

Since \( h(i) = 0 \) for \( i \in I(n-1,s) \), the induction hypothesis implies that \( h \) is the zero form; that is, \( x_n \mid p \). Since \( p \) is psd, \( x_n^2 \mid p \). Applying this argument to
the other variables, we see that \( \prod_{j} x_j^2 \mid p \); since \( \deg p = m < 2n \), it follows
that \( p = 0 \).

Finally, suppose the theorem holds for \((n, m-2n)\). We shall show that it
holds for \((n, m)\). Suppose \( p \in P_{n,m} \) and \( I(n,s) \in \mathcal{I}(p) \). The previous argument
may be applied to show that

\[(3.25) \quad p(x_1, \ldots, x_n) = x_1^2 \cdots x_n^2 \bar{p}(x_1, \ldots, x_n),\]

where \( \bar{p} \in P_{n, m-2n} \) and \( \bar{p}(i) = 0 \) if \( i \in I(n,s) \) and \( i_k \geq 1 \) for all \( k \).
We now choose \( \mathcal{M} \) as in Lemma 3.20 and let \( q = \bar{p} \circ \mathcal{M} \). Then
\( q \in P_{n, m-2n} \) and \( q(i) = 0 \) for \( i \in I(n, s-n) \). By the induction hypothesis, \( q = 0 \), hence \( \bar{p} = 0 \), and by (3.25), \( p = 0 \).

\[\square\]

**Corollary 3.26**

For every \((n, m)\) there exists \( p \in \text{int } Q_{n,m} \) with \( w(p) = N(n,s) \).

**Proof**

Let \( \{ \alpha_k \} = I(n,s) \) and \( p = \Sigma(\alpha_k \cdot) \mathcal{M} \). Since \( p \) is given as a sum of \( N(n,s) \)
m-th powers, \( w(p) \leq N(n,s) \). On the other hand, \( p \in \text{int } Q_{n,m} \) and so \( w(p) \geq \)
\( N(n,s) \) by Theorems 3.14(iii), 3.14(iv) and 3.22.

\[\square\]

**Example 3.27**

We now show that there is a non-zero form in \( P_{3,6} \) whose zero set is a
basic set of nodes for \( F_{3,3} \). The following example in \( P_{3,6} \setminus \Sigma_{3,6} \) was
constructed by R. M. Robinson [R7] in 1969, using Hilbert's method; see also
[C5]:
\[ R(x, y, z) = \]
\[ x^6 + y^6 + z^6 - (x^4 y^2 + x^2 y^4 + x^4 z^2 + x^2 z^4 + y^4 z^2 + y^2 z^4) + 3x^2 y^2 z^2 \]
\[ = \Xi x^6 - \Xi x^4 y^2 + 3x^2 y^2 z^2, \]

(3.29) \[ \mathcal{I}(R) = \{(1, \pm 1, 0), (0, 1, \pm 1), (1, 0, \pm 1), (1, \pm 1, \pm 1)\} := \{\beta_k: 1 \leq k \leq 10\}. \]

The fact that \( R \) is psd follows from Schur's inequality [H3,p.64]. We show below that \( \mathcal{I}(\mathcal{I}(R)) \cap F_{3,3} = \{0\}; \) or equivalently that \( \mathcal{I}(R) \) is a basic set of nodes for \( F_{3,3} \), as \( N(3,3) = 10 \). Thus, if \( R = \Xi h_j^2 \) were sos, then \( \mathcal{I}(R) \subseteq \mathcal{I}(h_j) \) would imply \( h_j = 0 \), a contradiction.

We show that \( \mathcal{I}(\mathcal{I}(R)) \cap F_{3,3} = \{0\}. \) Suppose \( h(x, y, z) = \Sigma a_{ijk} x^i y^j z^k \) is cubic and \( h(\beta) = 0 \) for \( \beta \in \mathcal{I}(R) \). Then \( h(1,1,0) = h(1,-1,0) = 0 \), so \( a_{120} = -a_{300} \) and \( a_{210} = -a_{030} \). By symmetry, it follows that for some \( \{c_j\}, \)

\[ h(x, y, z) = c_0 xyz + c_1 (x^3 - xy^2 - xz^2) + c_2 (y^3 - yz^2 - yx^2) \]
\[ + c_3 (z^3 - zx^2 - zy^2). \]

Finally, \( h(1, \pm 1, \pm 1) = 0 \) gives four linear equations which imply \( c_j = 0 \).

We use \( R \) to construct a form which is interior to \( \Sigma^*_{3,6} \) but not interior to \( Q_{3,6} \). Let \( \beta_k = (a_k, b_k, c_k) \) and define

\[ \tilde{R}(x, y, z) = \sum_{k=1}^{10} (a_k x + b_k y + c_k z)^6 = 8\Xi x^6 + 90\Xi x^4 y^2 + 360x^2 y^2 z^2. \]

Since \([\tilde{R}, R] = \Sigma \mathcal{R}(\beta_k) = 0, \tilde{R} \not\in \text{int } Q_{3,6} \) by Theorem 3.14(ii). But \([\tilde{R}, h^2] = \Sigma h^2(\beta_k) > 0 \) for any \( 0 \neq h \in F_{3,3} \) (since \( \{\beta_k\} \) is a basic set of nodes for \( F_{3,3} \)), hence \( \tilde{R} \in \text{int } \Sigma^*_{3,6} \) by Theorem 3.16(ii), and \( \text{rank}(H_{\tilde{R}}) = 10 \). It is easy
to check that $\mathcal{E}(R)$ is, in fact, 6–independent, so (3.31) is a strongly unique representation of $\hat{R}$ by Corollary 3.10.

The four families of cones we have discussed share the property that they are invariant under linear changes of variable. We say that a closed convex cone $C \subseteq F_{n,m}$ is a blender if $p \in C$ implies $p \circ M \in C$ for every $n \times n$ matrix $M$. (This is equivalent to asserting only that $p \circ M \in C$ for $M \in \text{GL}_n(R)$, since every $n \times n$ matrix $M$ is a limit of invertible matrices $M_j$ and $C$ is closed.)

**Lemma 3.32**

If $C$ is a blender, then so is $C^*$.

**Proof**

We already know that $C^*$ is a closed convex cone. Suppose $p \in C^*$, $q \in C$ and $M$ is an $n \times n$ matrix. Then $[p \circ M, q] = [p, q \circ M^T]$ by Theorem 2.15, and $q \circ M^T \in C$ (since $C$ "blends"). It follows that $[p \circ M, q] \geq 0$. Since $q \in C$ is arbitrary, $p \circ M \in C^*$. Thus, $C^*$ is a blender. \(\square\)

**Theorem 3.33**

$P_{n,m}$, $Q_{n,m}$, $\Sigma_{n,m}$ and $\Sigma^*_{n,m}$ are all blenders.

**Proof**

If $p(x) \geq 0$ for all $x$, then $p(Mx) \geq 0$ for all $x$, so $P_{n,m}$ is a blender; by the lemma, $Q_{n,m}$ is also a blender. Similarly, $p(x) = \Sigma_{2k}^k(x)$ implies $p(Mx) = \Sigma_{2k}^k(Mx) \in \Sigma_{n,m}$, so $\Sigma_{n,m}$ is a blender, and hence so is $\Sigma^*_{n,m}$. \(\square\)
Finally, we remark that $Q_{n,m}$ and $\Sigma_{n,m}$ can be regarded as two instances of a family of cones. Suppose $r$ divides $m$, and let

\[(3.34) \quad W(n;m,r) = \{\Sigma h_k^r : h_k \in F_{n,m/r}\}.
\]

(In this notation, $Q_{n,m} = W(n;m,m)$ and $\Sigma_{n,m} = W(n;m,2)$.) We shall say that $W(n;m,r)$ is a Waring blender. Waring blenders were defined, but not named, by Ellison in [E2].

**Theorem 3.35**

(i) If $r$ is odd, then $W(n;m,r) = F_{n,m}$.

(ii) If $r$ is even, then $W(n;m,r)$ is a blender and $W(n;m,r)^*$ is the set of those $p$ for which the following $r$-ic form in $N(n,m/r)$ variables is psd:

\[(3.36) \quad \sum_{\ell_1, \ldots, \ell_r \in I(n,m/r)} a(p; \ell_1+\cdots+\ell_r) T(\ell_1) \cdots T(\ell_r).
\]

**Sketch of Proof**

(i) Let $\{\alpha_k\}$ be a basic set of nodes for $F_{n,m}$. Every $p \in F_{n,m}$ can be written as $p = \Sigma \lambda_k^m (\alpha_k \cdot)$ for suitable $\{\lambda_k\}$. Since $r$ is odd and a factor of $m$, each $\lambda_k^r (\alpha_k \cdot)^m$ is the $r$-th power of a form in $F_{n,m/r}$.

(ii) Clearly, $W(n;m,r)$ is a convex cone which blends. The proof that it is closed is very similar to that for $\Sigma_{n,m}$ and is omitted. The expression (3.36) is simply $[p,h^r]$ for $h(x) = \Sigma t(\ell)x^\ell \in F_{n,m/r}$.

Part (i) was proved by Ellison [E2,p.667]. When $r = 2$, (ii) restates Theorem 3.16(i); when $r = m$, it is a disguised version of Theorem 3.7. (As
before, \( I(n,1) \) consists of the \( n \) unit vectors, and \( i \) can be written as \( \Sigma j \) in exactly \( c(i) \) different ways. Under the identification \( t(e_j) = t_j \), (3.36) becomes \( p(t) \), and we find that \( p \in Q_{n,m}^* \) if and only if \( p \in P_{n,m} \).

We shall return to blenders in section ten.

**Historical Notes**

Hilbert [H7] was apparently the first person to make a systematic study of psd and sos forms. R. M. Robinson [R7] identified the psd and sos \( n \)-ary \( m \)-ics as closed cones and showed that \( \Sigma x_1^m \) is in the interior of the subcone of \( \Sigma_{n,m} \) consisting of the sums of squares of monomials and binomials. The notations \( P_{n,m} \) and \( \Sigma_{n,m} \) were introduced by Choi and Lam [C3],[C4]; see also Holbrook's [H10]. Hilbert [H8] used the fact (if not the notation) that \( Q_{5,m} \) is a closed convex cone in solving Waring's problem, and showed that \( h_{5,m} \in \text{int} \ Q_{5,m} \) (c.f. (1.45)); see also [E3] and section eight. The cone \( Q_{2,m} \), under another name, appears in [R3] as the set of binary \( m \)-ics \( p \) such that \( p(x,y) = \|xu + yv\|^m \) for \( u, v \in L_m(X,\mu) \). The cone \( Q_{3,4} \) appears there with a similar interpretation; see the discussion at the end of section eight.

There is an elegant explicit construction by Ellison [E2] of a definite form in \( P_{n,m} \setminus Q_{n,m} \) for \( m \geq 4 \). (If \( m = 2 \), then \( P_{n,2} = Q_{n,2} = \Sigma_{n,2} \) is the cone of psd quadratic forms.) An easier example follows from the observation that, if \( L \mid \Sigma h_j^m \), where \( L \) is linear, \( m \) is even and the \( h_j \)'s are forms, then \( L \mid h_j \) for each \( j \); so \( L^m \mid \Sigma h_j^m \). Thus \( \hat{p}(x) = x_1^2x_2^{m-2} \), which belongs to \( P_{n,m} \) if \( n \geq 2 \) and \( m \geq 2 \), does not belong to \( Q_{n,m} \) for \( m \geq 4 \). Let \( p_\lambda = (1 - \lambda)\hat{p} + \lambda h_{n,m} \) for \( 0 \leq \lambda \leq 1 \). Then \( p_\lambda \) is definite for \( 0 < \lambda \), and \( I = \{ \lambda : p_\lambda \in Q_{n,m} \} \) is closed (Proposition 3.6) and convex. Thus \( I = [\tau,1] \) for some \( \tau > 0 \), and \( p_{\tau/2} \) is a definite form in \( P_{n,m} \setminus Q_{n,m} \).
The fact that psd quadratic forms are sos goes back to Lagrange. A psd binary form \( p \) factors into a product of linear forms to even degree and definite quadratic factors, each of which is a sum of two squares. It follows that \( p \) is a sum of two squares. This was known to Hilbert in 1888, but the first published proof apparently was Landau's [L2]. (These historical remarks are found in Delzell [D6,p.67].)

Hilbert established the strict inclusion of \( \Sigma_{n,m} \) in \( P_{n,m} \) for \( (n,m) \notin \mathcal{X} \) by indicating the existence of two examples for \( (n,m) = (3,6) \) and \( (4,4) \) via techniques from classical algebraic geometry. (Modern English-language versions of Hilbert's arguments can be found in Gel'fand and Vilenkin [61], as well as in [E2] and [R7].) The first explicit forms in \( P_{n,m} \setminus \Sigma_{n,m} \) were given by Motzkin [M2] in 1967 and R. M. Robinson [R7] in 1969. Other examples have been given by Choi and Lam ([C3], [C4]), Choi, Lam and Reznick (e.g. [C5], [C6]), Reznick (e.g. [R4]) and Schmüdgen [S4].

The duality of \( P_{n,m} \) and \( Q_{n,m} \) is embedded in the solution of the Hamburger moment problem \( (n = 2) \) and the Haviland moment problem \( (n \geq 3) \) under the identification of \( q \in Q_{n,m} \) with \( F_{\mathbb{R}^n,\mu} \); see section 6.

For \( n = 2 \) and 3 and a complex n–ary m–ic \( p \), the determinant of the matrix associated to \( H_p \), \( C(p) \), was called the "catalecticant" by Sylvester [S13]:
\[ C(p) = 0 \] if and only if \( \text{rank}(H_p) < N(n,s) \). In [R5] we give this a differential interpretation: \( C(p) = 0 \) if and only if there is \( g \in F_{n,s} \) so that \( g(D)p = 0 \), in the sense of (1.31). Sylvester was an expert prosodist, and a "catalectic" line of verse is one which is lacking part of the last foot. A form which is a sum of fewer \( m \)–th powers than canonically required thereby exhibits catalecticism [S13,p.268]. The vanishing of the catalecticant is a necessary condition for a binary form of degree \( m \) to be expressible as a sum of as few as
s m-th powers. Sylvester was slightly defensive about his terminology [S14,p.293n]: "Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant." In fairness, we note that the paper in which Sylvester denominated the catalecticant also introduced the term "unimodular" in its modern meaning [S14,p.284n1].