1. Introduction and overview

An \( n \)-ary \( m \)-ic form is a polynomial \( p(x_1, \ldots, x_n) \) which is homogeneous of degree \( m \). In this paper we study the representations of real \( n \)-ary \( m \)-ic forms of even degree as sums of \( m \)-th powers of linear forms. Our work can be regarded as a partial generalization to higher degree of the familiar theory of real psd quadratic forms and their representations as a sum of squares. (A companion paper [R5] will discuss forms of odd degree and, more generally, forms with complex coefficients.) To each \( n \)-ary \( m \)-ic \( p \), we associate a quadratic form \( H_p \) in \( \binom{n + m/2 - 1}{n - 1} \) variables so that, if \( p \) is a sum of \( m \)-th powers of linear forms, then \( H_p \) is psd. The converse is false in general, and, in a rather precise way, is dual to Hilbert's classical result that for "sufficiently large" \( n \) and \( m \), there are psd \( n \)-ary \( m \)-ic forms which cannot be written as a sum of squares of forms.

We use the familiar multinomial notation in describing real \( n \)-ary \( d \)-ic forms, where \( d \) is not assumed to be even. Let \( \mathbb{Z}_+^n \) denote the set of \( n \)-tuples of non-negative integers. For \( i = (i_1, \ldots, i_n) \in \mathbb{Z}_+^n \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we write \( x^i \) for \( \Pi_{k=1}^n x_k^{i_k} \) and define \( |i| = \Sigma_{k=1}^n i_k \) and \( c(i) = |i|! / \Pi(i_k!) \), the associated multinomial coefficient. (For small values of \( n \), the variables will be called \( x, y, z, w, \ldots \)) For \( 1 \leq d \in \mathbb{Z} \), let

\[
(1.1) \quad I(n,d) = \{ i \in \mathbb{Z}_+^n : |i| = d \},
\]

so \( |I(n,d)| = N(n,d) = \binom{n + d - 1}{n - 1} \). If \( d \) is even, we write \( d = m = 2s \).

Throughout the paper, the symbols "\( d \)", "\( i \)" , "\( m \)" , "\( n \)" and "\( s \)" are reserved for

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these roles. A typical \( n \)-ary \( d \)-ic form \( p \) is written:

\[
(1.2) \quad p(x_1, \ldots, x_n) = \sum_{i \in I} c(i)a(p;i)x^i,
\]

where \( I = I(n,d) \) and \( a(p;i) \in \mathbb{R} \). (Of course, there is no loss of generality in scaling the coefficients of \( p \) by the \( c(i) \)'s; this notation was standard in older algebra texts, and is the usual way of writing quadratic forms.)

Let \( F_{n,d} \) denote the vector space of real \( n \)-ary \( d \)-ics. By identifying the form \( p \) with the ordered set \( \{ a(p;i) : i \in I \} \) of its coefficients, we see that \( F_{n,d} \) is isomorphic to \( \mathbb{R}^{N(n,d)} \). For \( \alpha \in \mathbb{R}^n \), define the \( n \)-ary \( d \)-ic \( (\alpha,)^d \) by:

\[
(1.3) \quad (\alpha,)^d(x) = (\alpha\cdot x)^d = \left( \sum_{j=1}^{n} \alpha_j x_j \right)^d = \sum_{i \in I} c(i)\alpha^i x^i,
\]

the last equality following from the multinomial theorem. We introduce the following inner product on \( F_{n,d} \):

\[
(1.4) \quad [p,q] = \sum_{i \in I} c(i)a(p;i)a(q;i).
\]

Since \([p,q]\) is bilinear and \([p,p] \geq 0\) with equality if and only if \( p = 0 \), the pair \( (F_{n,d},[\cdot,\cdot]) \) is an \( N(n,d) \)-dimensional real inner product space. In the resulting topology on \( F_{n,d} \), \( p_k \rightarrow p \) means that \( a(p_k;i) \rightarrow a(p;i) \) for all \( i \), or, equivalently, that \( p_k(x) \rightarrow p(x) \) pointwise on all compact subsets of \( \mathbb{R}^n \).

There are many justifications, both practical and historical, for this inner product. Of particular importance to us is the fact that \([\cdot,\cdot]\) is a reproducing kernel: it follows from the foregoing that for all \( \alpha \) and \( p \),
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\[(1.5) \quad [p, (\alpha \cdot)^d] = \sum_{i \in I} c(i) a(p; i) \alpha^i = p(\alpha).\]

This choice of inner product will allow us to classify those subspaces of \( F_{n,d} \) generated by sets of \( d \)-th powers in a particularly convenient way.

Fix \( d \), let \( A \) be a (usually finite) subset of \( \mathbb{R}^n \) and let

\[(1.6) \quad \text{span}_d(A) = \{ \Sigma \lambda_k (\alpha_k \cdot)^d : \alpha_k \in A, \lambda_k \in \mathbb{R} \}\]

denote the subspace of \( F_{n,d} \) spanned by \( \{ (\alpha_k \cdot)^d \} \). The orthogonal complement of \( \text{span}_d(A) \) is also a subspace of \( F_{n,d} \):

\[(1.7) \quad \text{span}_d(A)^\perp = \{ p \in F_{n,d} : 0 = [p, \Sigma \lambda_k (\alpha_k \cdot x)^d] = \Sigma \lambda_k p(\alpha_k) \text{ for all } \{ \lambda_k \} \} = \{ p \in F_{n,d} : p(\alpha) = 0 \text{ for all } \alpha \in A \}.\]

For \( A \subseteq \mathbb{R}^n \), we define as usual the ideal of \( A \):

\[(1.8) \quad I(A) = \{ p \in \mathbb{R}[x_1, \ldots, x_n] : p(\alpha) = 0 \text{ for } \alpha \in A \}.\]

We see from (1.7) and (1.8) that

\[(1.9) \quad \text{span}_d(A)^\perp = I(A) \cap F_{n,d}.\]

Since the forms in \( F_{n,d} \) have fixed degree, it is possible to have \( \text{span}_d(A) = \text{span}_d(B) \), but \( A \neq B \). It follows from (1.9) that \( I(A) \cap F_{n,d} = \{0\} \) if and only if \( \text{span}_d(A) = F_{n,d} \).

For \( h \in F_{n,d} \), we define the zero-set of \( h \):
(1.10) \[ \mathcal{Z}(h) = \{ x \in \mathbb{R}^n : h(x) = 0 \}. \]

Since \( h \) is a form, \( \mathcal{Z}(h) \) can be viewed projectively. We shall write \( \mathcal{Z}(h) = \{ \alpha_1, \ldots, \alpha_r \} \) to mean that no two \( \alpha_j \)'s are proportional and note that \( h(y) = 0 \) if and only if \( y = \lambda \alpha_j \) for some \( j \) and \( \lambda \); \( \mathcal{Z}(h) = \emptyset \) means that \( h \) has no non-trivial zeros. Of course, \( h \in \mathcal{I}(\mathcal{Z}(h)) \), and \( h \in \mathcal{I}(A) \) if and only if \( A \subseteq \mathcal{Z}(h) \).

We shall be especially interested in \([p,q]\) when \( q(x) = \Sigma(\alpha_k \cdot x)^d \) and \( p(x) \geq 0 \) for all \( x \) (necessarily, \( d = m \) is even). In this case,

\[
(1.11) \quad [p,q] = [p,\Sigma(\alpha_k \cdot)^m] = \Sigma p(\alpha_k) \geq 0.
\]

If \( A = \{ \alpha_k \} \subseteq \mathcal{Z}(p) \), then \([p,q] = 0\); conversely, if \([p,q] = 0\), then \( A \subseteq \mathcal{Z}(p) \) since \( p(\alpha_k) \geq 0 \) for all \( k \) in (1.11). This duality has important consequences for the representation of forms as sums of powers of linear forms.

We now define three convex cones in \( F_{n,m} \) (recall that \( m = 2s \)):

\[
(1.12) \quad P_{n,m} = \{ p \in F_{n,m} : p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \},
\]

\[
(1.13) \quad \Sigma_{n,m} = \{ p \in F_{n,m} : p = \sum_{j=1}^r h_j^2 \text{ where } h_j \in F_{n,s} \text{ and } r < \omega \},
\]

\[
(1.14) \quad Q_{n,m} = \{ p \in F_{n,m} : p = \sum_{k=1}^r (\alpha_k \cdot)^m, \text{ where } \alpha_k \in \mathbb{R}^n \text{ and } r < \omega \}.
\]

It is clear that these are cones and that \( P_{n,m} \supseteq \Sigma_{n,m} \supseteq Q_{n,m} \). The forms in \( P_{n,m} \), \( \Sigma_{n,m} \) and \( Q_{n,m} \) are called positive semidefinite (psd), a sum of squares.
(sos) and a sum of \( m \)-th powers, respectively. If \( p \in P_{n,m} \) and \( \mathcal{Z}(p) = \{0\} \), then \( p \) is positive definite (pd). We have seen in (1.11) that \([p,q] \geq 0\) for \( p \in P_{n,m} \) and \( q \in Q_{n,m} \); we shall prove in section three that \( P_{n,m} \) and \( Q_{n,m} \) are actually dual cones with respect to \([\cdot, \cdot]\).

In 1888, Hilbert [H7] determined the necessary and sufficient conditions on \((n,m)\) for the inclusion \( P_{n,m} \supset \Sigma_{n,m} \) to be an equality:

\[ P_{n,m} = \Sigma_{n,m} \text{ if and only if } (n,m) \in \mathcal{K}, \text{ where} \]

\[ \mathcal{K} = \{(n,m) : n = 2 \text{ or } m = 2 \text{ or } (n,m) = (3,4)\}. \]

We shall refer to this as "Hilbert’s Theorem".

(Hilbert’s 17th Problem asks whether a psd form is a sum of squares of rational functions; that is, for each \( p \in P_{n,m} \) must there always exist a "denominator" \( f \in F_{n,d} \) (for some \( d \)) so that \( f^2 p \in \Sigma_{n,m+2d} \)? Artin’s affirmative answer, in a more general setting, inaugurated the study of real algebra. More recently, Becker [B8] has determined necessary and sufficient conditions on a psd form for it to be a sum of \( m \)-th powers of rational functions. See also Bradley [B14]. A more concrete version of Becker’s work, especially for \( n = 2 \), will be presented in [C7]. Other discussions of sums of powers of forms of higher degree have been presented for real forms by Ellison [E2] and for complex polynomials by Newman and Slater [N1]. We have not tried to generalize our work in this paper to forms over a real closed field \( \mathbb{R} \); at a minimum, this would require the existence of separating hyperplanes for closed convex cones over real closed fields. Robson [R8] has recently shown that for any ordered field \( \mathbb{R} \), there exists a continuous \( \mathbb{R} \)-valued linear function which
separates a closed convex set from a point outside. (Here, \( \bar{R} = \text{Sper} \, R[x] \) is the real spectrum of \( R[x] \).) Craven has shown by counterexample that, for every \( R \neq \mathbb{R} \), \( \bar{R} \) above cannot be replaced by \( R \).

Suppose \( m = 2s \) is even. We introduce a crucial family of auxiliary quadratic forms, inspired by the computation of \([p, g^2]\) for \( g \in F_{n,s} \). Let \( t = (t(\ell)) \) be an \( N(n,s) \)-tuple of variables indexed by \( \{\ell \in I(n,s)\} \), and let

\[
(1.17) \quad L(x;t) = \sum_{\ell} x^t(\ell).
\]

As \( t \) ranges over \( \mathbb{R}^{N(n,s)} \), \( L(\cdot;t) \) ranges over \( F_{n,s} \); at the same time, each \( L(x;) \) is a linear form in \( t \) whose coefficients depend on \( x \). Since

\[
(1.18) \quad L(x;t)^2 = \sum_{\ell} \sum_{\ell'} t(\ell)t(\ell')x^{\ell+\ell'} = \sum_{i} \sum_{\ell+\ell'=i} t(\ell)t(\ell')x^i \in F_{n,m},
\]

we can compute \([p, L(\cdot;t)^2]\) for \( p \in F_{n,m} \) by (1.4), and define:

\[
(1.19) \quad H_p(t) = [p, L(\cdot;t)^2] = \sum_{\ell} \sum_{\ell'} a(p;\ell+\ell')t(\ell)t(\ell') \in F_{N(n,s),2}
\]

We stress that \( H_p \) is one of the central objects of study in this paper. Observe that we may recover \( p \) from \( H_p \), and that the mapping \( p \rightarrow H_p \) is a bijection between \( F_{n,m} \) and the quadratic forms of shape \( \sum b(\ell+\ell')t(\ell)t(\ell') \).

Suppose \( p = \Sigma (\alpha_k \cdot)^m \) and \( A = \{\alpha_k\} \). We have already noted in (1.11) that \([p,q] = 0 \) for \( q \in F_{n,m} \) if and only if \( A \subseteq \mathcal{Z}(q) \). In particular, suppose

\([p, g^2] = 0 \), where \( g(x) = L(x;u) \in F_{n,s} \). Then,
(1.20) \[ 0 = [p, g^2] = \sum_{k=1}^{r} g^2(\alpha_k) = \sum_{k=1}^{r} (L(\alpha_k; u))^2, \]

so \( g(\alpha_k) = L(\alpha_k; u) = 0 \) for all \( k \). This suggests a dual definition:

(1.21) \[ \mathcal{N}(p) = \{ g \in F_{n,s} : [p, g^2] = 0 \} = \{ L(\cdot; u) : H_p(u) = 0 \}. \]

Even though \( \mathcal{N}(p) \) is defined without reference to any particular representation of \( p \) as a sum of \( m \)-th powers, we see from the above that \( g \in F_{n,s} \) belongs to \( \mathcal{N}(p) \) if and only if \( A \subseteq \mathcal{E}(g) \); in view of (1.9),

(1.22) \[ \mathcal{N}(p) = I(A) \cap F_{n,s} = \text{span}_S(A)^\perp. \]

If \( p = \Sigma(\beta_{\ell})^m \) is another representation of \( p \) and \( B = \{ \beta_{\ell} \} \), then \( \mathcal{N}(p) = \text{span}_S(B)^\perp \) as well, so \( I(A) \cap F_{n,s} = I(B) \cap F_{n,s} \); see also Theorem 3.8.

We may, of course, interpret \( \mathcal{N}(p) \) in terms of quadratic form theory. If \( 0 \neq u \in \mathbb{R}^N(n,s) \), \( H_p \) is psd and \( H_p(u) = 0 \), then \( u^T \) is an isotropic vector for the Hankel form \( H_p \). (Alternatively, the matrix representing \( H_p \) is singular, and \( u^T \) is an eigenvector associated to 0.)

In the binary case, \( I(2,s) = \{(j,s-j) : 0 \leq j \leq s\} \) and \( N(2,s) = s + 1 \); for \( \ell_j = (j,s-j) \), we write \( t(\ell) = t_j \) and \( a(p, \ell) = a_j \), so that

(1.23) \[ p(x,y) = \sum_{j=0}^{m} (j^m) a_j x^j y^{m-j} \Leftrightarrow H_p(t_0, \ldots, t_s) = \sum_{j=0}^{m} \sum_{k=0}^{m} a_{j+k} t_j t_k. \]

(That is, \( H_p \) is just the usual Hankel form associated to \( (a_0, \ldots, a_m) \).) If \( p \in F_{n,2} \) is a quadratic form, then \( I(n,s) = I(n,1) \) consists of the \( n \) unit vectors.
Writing $t(e_j) = t_j$, we see that $H_p(t_1, \ldots, t_n) = p(t_1, \ldots, t_n): H_p = p$. Finally, for all $n$ and $m$, if $\alpha \in \mathbb{R}^n$ and $p = (\alpha \cdot)^m$, then $a(p; \ell, \ell') = \alpha^\ell \ell'$ and

$$H_p(t) = [(\alpha \cdot)^m, L(t) ; t]^2 = (L(\alpha \cdot t))^2 = \sum_{\ell} \alpha^\ell \ell t(\ell))^2.$$  

(1.24)  

(Sylvester discussed the determinant of the matrix associated to $H_p$ for $n = 2$ and $(n,m) = (3,4)$; see the historical notes at the end of sections three and four. Psd quadratic forms of the shape (1.19), in which the variables are indexed by the semigroup $\mathbb{Z}_+^n$ rather than the finite set $I(n,s)$, have been extensively studied by Berg, Christensen and Ressel [B11],[B12].)

We turn to representations of $p$ as a sum of $m$-th powers. If $\alpha = \kappa \beta$, $\kappa \in \mathbb{R}$, then $(\alpha \cdot x)^m = \kappa^m (\beta \cdot x)^m$; thus there is no loss of generality in writing a non-negative linear combination of $m$-th powers with non-negative coefficients simply as a sum of $m$-th powers (as we did in (1.14)). Suppose $p \in Q_{n,m}$ and

$$p(x) = \sum_{k=1}^{r} (\alpha_k \cdot x)^m = \sum_{k=1}^{r} (\alpha_k x_1 + \cdots + \alpha_k x_n)^m.$$  

(1.25)

We say that (1.25) is a good representation of $p$ if no two $\alpha_k$'s are proportional. Since $(\alpha \cdot)^m + (\kappa \alpha \cdot)^m = (\lambda \alpha \cdot)^m$, where $\lambda^m = 1 + \kappa^m$, any representation reduces to a good one; note also that $\alpha_k \neq 0$ in any good representation.

The width of $p \in Q_{n,m}$, $w(p)$, is the smallest number of $m$-th powers which sum to $p$; (1.25) is a minimal representation of $p$ if $r = w(p)$. Every minimal representation is good and $w(p) \leq N(n,m)$ by Carathéodory's Theorem (Proposition 2.3). (We shall reserve the more familiar term "length" for the minimal
representations of \( p \) as a linear combination of \( m \)-th powers, allowing negative coefficients; see [R5].)

We say that \( p \in Q_{n,m} \) has a unique minimal representation if, in any two minimal representations

\[
(1.26) \quad p(x) = \sum_{k=1}^{w(p)} (\alpha_k \cdot x)^m = \sum_{k=1}^{w(p)} (\beta_k \cdot x)^m,
\]

we have \( \beta_k = \epsilon_k \alpha_{\pi(k)} \) for some permutation \( \pi \in S_\Gamma \) and choice of \( \epsilon_k \in \{1,-1\} \). Since \( u^2 + v^2 = (cu + sv)^2 + (su - cv)^2 \) for \( c^2 + s^2 = 1 \), unique minimal representations are impossible when \( m = 2 \), unless \( w(p) = 1 \). Uniqueness is more common in higher degrees. A theorem of Fischer (see Proposition 5.19) implies that any \( p \in Q_{2,m} \) satisfying \( w(p) \leq m/2 \) has a unique minimal representation.

An even more restrictive condition is possible. We say that \( p \) has a strongly unique representation as a sum of \( m \)-th powers if every good (not necessarily minimal) representation of \( p \) is a rearrangement of a single fixed minimal representation. In Fischer's Theorem, mentioned above, \( p \) has a strongly unique representation (see Theorem 5.1(i).)

This result has interesting algebraic consequences when \( m \geq 4 \). Note that

\[
(1.27) \quad q(x,y) = (x + \sqrt{2} \ y)^m + (x - \sqrt{2} \ y)^m
\]

evidently is a binary \( m \)-ic form with integral coefficients. Since (1.27) is the only good representation of \( q \) as a real sum of \( m \)-th powers by Fischer's theorem, it follows that \( q \) is not a linear combination of \( m \)-th powers with non-negative rational coefficients. This contrasts with the well-known properties of quadratic forms over fields; see Theorem 10.2.
There is an important connection between representations of $p$ as a sum of $m$-th powers and representations of $H_p$ as a sum of squares; by (1.24),

$$p(x) = \sum_{k=1}^{r} (\alpha_k \cdot x)^m \iff H_p(t) = \sum_{k=1}^{r} (L(\alpha_k; t))^2.$$  

Thus if $p \in Q_{n,m}$ is a sum of $r$ $m$-th powers, then $H_p$ is a psd quadratic form whose rank is at most $r$; it follows immediately that

$$w(p) \geq \text{rank}(H_p) \text{ for all } p \in Q_{n,m}.$$  

We shall see (Theorem 4.6) that $w(p) = \text{rank}(H_p)$ for $p \in Q_{n,m}$ if $(n,m) \in \mathcal{K}$ (c.f. (1.16)), but strict inequality in (1.29) may occur for other $(n,m)$.

Finally, for $\ell \in I(n,s)$, let $D_\ell$ denote $\partial/\partial t(\ell)$. Since $H_p$ is psd, $H_p(u) = 0$ implies that $0 = D_\ell H_p(u) = \sum_{\ell'} a(p; \ell + \ell') u(\ell')$. Thus, $[p, x^\ell g] = 0$ for all $g \in \mathcal{K}(p)$ and all $\ell \in I(n,s)$. This orthogonality has arisen in some of the analytic applications discussed below, but we shall not pursue it in this paper.

Another fact about the inner product shows its relationship to invariant theory. For any $n \times n$ matrix $M$ and $p \in F_{n,d}$, we may change variables using $M$ by writing $y = Mx$ (where $x$ and $y$ are column vectors). Define $p \circ M \in F_{n,d}$ by $(p \circ M)(x) = p(Mx) = p(y)$. Then for all $p$ and $q \in F_{n,d}$,

$$[p \circ M, q] = [p, q \circ M^T].$$

(This is easy to prove when $p = (\alpha \cdot)^d$ and $q = (\beta \cdot)^d$; see Theorem 2.15.)
There have been several other previous appearances of the inner product \([p,q]\); some of these will be discussed in detail in [R5]. We can associate to \(p \in F_n, d\), the differential operator \(p(D)\):

\[(1.31) \quad p(D) = \sum_{i \in I(n,d)} c(i)a(p;i)D^i,\]

where \(D = (D_1, \ldots, D_n)\) and \(D_j = \partial/\partial x_j\). If \(j \in I(n,d)\), then \(D^i(x^j)\) vanishes if \(i \neq j\) and equals \(\Pi(i, k)\) otherwise. Since \(c(i)\Pi(i, k) = d!\), it follows that \(p(D)(q) = d! [p, q]\). Complex versions of this definition, in which \(p\) and \(q\) need not have the same degree, were used in 19th century projective geometry (to define apolarity); modern harmonic analysis (see Helgason [H6], Stein and Weiss [S11]); and in spherical designs (see e.g. [G3]). Recently, Beauzamy, Bombieri, Enflo and Montgomery [B7] proved an inequality for complex polynomials which can easily be expressed in terms of \(\|p\|^2 = p(D)p : \|pq\|^2 \geq \|p\| \cdot \|q\|\). Froil and Millour [F4] and the author [R6] have, independently and using different techniques, shown that equality holds above if and only if, after a unitary linear change, \(p\) and \(q\) involve disjoint sets of variables. In the binary case, the inner product is closely related to the "lineo-linear invariant" of 19th century algebra; see [K11] for a modern reappraisal: let \(\tilde{q}(x, y) = q(-y, x)\), so \(ax + by\) is a factor of \(q\) if and only if \(\tilde{q}(a, b) = 0\). The lineo-linear invariant of \(p\) and \(q\) equals \([p, q]\) in our notation; its invariance implies (1.30) for binary forms.

Our inner product is familiar to students of the classical moment problem. Many of the results we present regarding the moment and quadrature problems are not new; the novelty in our presentation lies in the generating function and in the absence of explicit orthogonal polynomials. Suppose \(C \subseteq \mathbb{R}^n\) and \(\mu\) is a
measure on $\mathbb{C}$. (All measures in this paper are assumed to be non-negative Borel measures on a subset of $\mathbb{R}^n$.) Suppose that for all $i \in I(n,d)$, $\int_{u \in \mathbb{C}} |u|^i d\mu$ is finite. Then the moments of $\mu$:

\[(1.32) \quad M(i) = \int_{u \in \mathbb{C}} u^i d\mu \]

exist and are finite for all $i \in I(n,d)$. By (1.3) and the usual interchange of integral and sum, we obtain the following representation for $F_{\mathbb{C},\mu}$, the $d$-th moment generating function for $(\mathbb{C},\mu)$:

\[(1.33) \quad \mathbb{E}_{\mathbb{C},\mu}^{(d)}(x) = F_{\mathbb{C},\mu}(x) = \sum_{i \in I} c(i) M(i)i^i = \int_{u \in \mathbb{C}} (x \cdot u)^d d\mu. \]

By (1.4) and (1.33), it follows that for all $p \in F_{n,d}$,

\[(1.34) \quad [p, F_{\mathbb{C},\mu}] = \sum_{i \in I(n,d)} c(i)a(p;i)M(i) = \int_{u \in \mathbb{C}} p(u)d\mu. \]

It is often desirable to dehomogenize forms into polynomials. Let $F_{n,d}^*$ denote the set of polynomials in $n-1$ variables with degree $\leq d$. We identify $p^* \in F_{n,d}^*$ with $p \in F_{n,d}$ via $p^*(x_1, \ldots, x_{n-1}) = p(x_1, \ldots, x_{n-1}, 1)$.

Let $I^*(n,d)$ denote the projection of the elements of $I(n,d)$ onto their first $n-1$ coordinates, with the identification

\[(1.35) \quad i = (i_1, \ldots, i_n) \in I(n,d) \leftrightarrow i^* = (i_1, \ldots, i_{n-1}) \in I^*(n,d). \]
(Of course, \( i^* \) only determines \( i \) when \( d \) is given.) For \( u \in \mathbb{R}^{n-1} \), we define \( \bar{u} = (u_1, \ldots, u_{n-1}, 1) \), so that \( \bar{u}^i = u^i \). Suppose \( K \subseteq \mathbb{R}^{n-1} \) is a closed set, \( \mu \) is a measure on \( K \) and let \( C = \bar{K} = \{ \bar{u} : u \in K \} \) be the suspension of \( K \) at \( \{ x_n = 1 \} \) in \( \mathbb{R}^n \). We now define another generating function, \( G_{K,\mu} \):

\[
G_{K,\mu}(x) = F_{\bar{K},\mu}(x) = \int_{u \in K} \left( \cdots \int \left( \sum_{1} \cdots \sum_{n-1} u_1 x_1 + \cdots + u_{n-1} x_{n-1} + x_n \right)^d d\mu. \]

For \( i^* \in I^*(n,d) \), we have

\[
a(G_{K,\mu}; i) = \int_{u \in \bar{K}} u_1^i d\mu = M(i),
\]

so for \( p \in F_{n,d} \), it follows from (1.32) that

\[
[p, G_{K,\mu}] = \int_{u \in \bar{K}} p(u_1, \ldots, u_{n-1}, 1) d\mu.
\]

If \( d = m \) is even and \( p(x) \geq 0 \) for \( x \in \bar{K} \), then \( [p, G_{K,\mu}] \geq 0 \) by (1.38).

The moment problems of interest to us involve the converse. Suppose we are given a prospective infinite moment set: \( \{ M(i^*) : i^* \in \mathbb{Z}_{n-1}^+ \} \). For each even \( m \), define \( G_m \in F_{n,m} \) by \( a(G_m; i) = M(i) \) for \( i^* \in I^*(n,m) \). If \( [p, G_m] \geq 0 \) for all \( p \in F_{n,m} \) which are \( \geq 0 \) on \( \bar{K} \), then is there a measure \( \mu \) on \( K \) so that \( G = G_{K,\mu} \)? The answer is classical: if \( K \) is a closed subset of \( \mathbb{R}^{n-1} \), then such a measure exists. This is the Stieltjes moment problem for \( K = [0,\alpha) \), the Hausdorff moment problem for \( K = [0,1] \), the Hamburger moment problem for \( K = (-\infty,\alpha) \), and the Haviland moment problem for \( K = \mathbb{R}^{n-1} \). (In the classical theory, a different inner product is used: a separate Hilbert space is given for each
\((K, \mu)\) and \((f_1, f_2) = [f_1 f_2, G_{K, \mu}]\) in the previous notation. However, integration is still viewed as a functional on the space of polynomials.)

Numerical analysts are interested in a different question about the same situation. Suppose \(G_{K, \mu}\) can be written as follows:

\[(1.39) \quad G_{K, \mu}(x) = \sum_{k=1}^{r} \lambda_k (\alpha_{k,1} x_1 + \cdots + \alpha_{k,n-1} x_{n-1} + x_n)^d,\]

where \(\lambda_k \in \mathbb{R}\) need not be positive and \(\alpha_{k,E} \in \mathbb{R}^{n-1}\) need not lie in \(K\). For \(p \in F_{n,d}\), we compute \([p, G_{K, \mu}]\) in two ways, obtaining

\[(1.40) \quad [p, G_{K, \mu}] = \int_{u \in K} \cdots \int p(u,1) d\mu = \sum_{k=1}^{r} \lambda_k p(\alpha_k).\]

Since any polynomial \(f \in F_{n,d}^*\) can be realized as \(p(x_1, \ldots, x_{n-1}, 1)\) for some \(p \in F_{n,d}\), it follows that the equation

\[(1.41) \quad \int_{u \in K} \cdots \int f(u) d\mu = \sum_{k=1}^{r} \lambda_k f(\alpha_{k,1}, \ldots, \alpha_{k,n-1})\]

is valid for all \(f \in F_{n,d}^*\). This is called a quadrature formula for \((K, \mu)\) of precision \(d\) with \(r\) nodes. Conversely, if (1.41) holds for all \(f \in F_{n,d}^*\), then in particular it holds for \(f(u) = u^{i^*}\), \(i^* \in I^*(n,d)\), so (1.41) implies (1.39).

In (1.36), if \(K\) has a non-empty interior and \(\mu(u) = \varphi(u) dx\), where \(\varphi(u) > 0\), then \(H_{G_{K, \mu}}\) is strictly definite, so \(w(G_{K, \mu}) \geq N(n, s)\) by (1.29). It
is already known in this case (see [D4,p.366]) that the smallest possible \( r \) in (1.41) satisfies the inequality \( N(n,s) \leq r \leq N(n,m) \).

Sections two through five develop more of the theory of sums of \( m \)-th powers and contain a rather complete description of \( Q_{n,m} \) when \( (n,m) \in \mathcal{X} \). Sections six and seven contain applications of this theory to moment problems and numerical analysis. Sections eight and nine are devoted to the representations of \( (\sum x_j^2)^S \). Section ten presents some open questions.

Section two begins with some simple observations about convex cones (including Carathéodory's Theorem) and real psd quadratic forms. We discuss \( \text{span}_d(A) \) and prove, in this language, the well-known result that \( \text{span}_d(\mathbb{R}^n) = F_{n,d} \). Say that \( A \subseteq \mathbb{R}^n \) is \( d \)-independent if \( \{(\alpha_k \cdot x)^d\} \) is linearly independent; duality with respect to our inner product gives an old geometric criterion for \( d \)-independence which is originally due to Serret. If \( A \subseteq \mathbb{R}^n \) is \( d \)-independent and \( |A| = N(n,d) \), then \( \{(\alpha_k \cdot x)^d\} \) is a basis for \( F_{n,d} \). In this case, \( A \) is called a basic set of nodes for \( F_{n,d} \); we present a general construction based on ideas of Biermann. By duality, Lagrange interpolation for \( n \)-ary \( d \)-ic forms at \( A \) is possible if and only if \( A \) is a basic set of nodes for \( F_{n,d} \). Given \( A = \{\alpha_1, \ldots, \alpha_r\} \) and \( p = \Sigma(\alpha_k \cdot)^m \), we show that \( \text{rank}(\mathcal{H}_p) = r \) if and only if \( \{\alpha_k\} \) is \( s \)-independent.

Section three is the heart of the paper. We give a unified proof of the known result that \( P_{n,m}, \Sigma_{n,m} \) and \( Q_{n,m} \) are closed cones with non-empty interior, and characterize the forms in their interiors. We compute the duals with respect to the inner product: \( P_{n,m} \) and \( Q_{n,m} \) are dual cones and \( \Sigma_{n,m}^* = \{p: \mathcal{H}_p \text{ is psd}\} \). The existence of psd forms which are not sos manifests itself here in the fact that some psd quadratic forms (the \( \mathcal{H}_p \)'s) cannot be written as a sum of squares from a particular class of linear forms (the \( L(\alpha_i) \)'s). The
duality of $P_{n,m}$ and $Q_{n,m}$ ($Q_{n,m}$ is identified with the integration functional) is basic to classical moment theory. This duality is fully explored for the Hausdorff moment problem in the work of Karlin and Shapley [K2]; for a more general setting, see Rogosinski [R10]. Hilbert's Theorem implies that the inclusion $\Sigma^*_{n,m} \supseteq Q_{n,m}$ is strict if $(n,m) \not\in \mathcal{K}$; the strictness of the inclusion $(\cup_m \Sigma^*_{n,m}) \supset (\cup_m Q_{n,m})$ for $n \geq 3$ has been recently analyzed in multi-dimensional moment theory and is discussed in section six.

We also discuss further implications of duality. If $p = \Sigma(\alpha_k \cdot)^m \in Q_{n,m}$ and $h \in P_{n,m}$, then $[p,h] = 0$ if and only if $h \in I(\{\alpha_k\}) \cap P_{n,m}$. This leads to two of our principal results (see Theorem 3.8 and Corollary 3.10):

If $p = \Sigma(\alpha_k \cdot)^m = \Sigma(\beta_k \cdot)^m$, then $I(\{\alpha_k\}) \cap P_{n,m} = I(\{\beta_k\}) \cap P_{n,m}$.

If $h \in P_{n,m}$ and $\mathcal{I}(h) = \{\alpha_1, \ldots, \alpha_r\}$ is $m$-independent, then $p = \Sigma(\alpha_k \cdot)^m$ has a strongly unique representation.

Furthermore, if $\{\alpha_k\}$ is a basic set of nodes for $F_{n,s}$, then $p = \Sigma(\alpha_k \cdot)^m \in \text{int} \Sigma^*_{n,m}$ and if $\{\alpha_k\}$ satisfies the stronger condition that $I(\{\alpha_k\}) \cap P_{n,m} = \{0\}$, then $p$ is interior to the smaller cone $Q_{n,m}$. We show that the zero set of the "Robinson form" (given by (3.28)) is a basic set of nodes for $F_{3,3}$. The fact that the zero sets of pd forms in $P_{n,m}$ are not the same as the zero sets of forms in $F_{n,s}$ goes back to Hilbert and was an integral part of his original construction of forms in $P_{3,6} \setminus \Sigma_{3,6}$.

We say that a blender is a closed cone $C$ of $n$-ary $m$-ics which "blends": specifically, if $p \in C$ and $M$ is an $n \times n$ matrix, then $p \circ M \in C$. We show, using (1.30), that the cones $P_{n,m}$, $Q_{n,m}$, $\Sigma_{n,m}$, $\Sigma^*_{n,m}$ are blenders and that the dual of a blender is a blender. The blenders $Q_{n,m}$ and $\Sigma_{n,m}$ are instances of a general
class of blenders, \( \{ \Sigma f^r : f \in F_{n,m/r} \} \), where \( r \mid m \). We briefly discuss these blenders, introduced by Ellison, and their duals. We conclude the section with some historical notes.

In section four, we study representations of \( p \in Q_{n,m} \), particularly when \( (n,m) \in \mathcal{K} \). Using a perturbation lemma for quadratic forms, we prove two more general results (see Theorems 4.6 and 4.9):

If \( (n,m) \in \mathcal{K} \) and \( p \in Q_{n,m} \), then \( w(p) = \text{rank}(H_p) \).

If \( w(p) = \text{rank}(H_p) \) and \( p \) has two minimal representations,
\[
p = \Sigma(\alpha_k \cdot)^m = \Sigma(\beta_{k'} \cdot)^m,
\]
with \( \beta_1 = \lambda \alpha_1 \), then \( \lambda = \pm 1 \).

(The first may fail if \( (n,m) \not\in \mathcal{K} \); the second may fail if \( w(p) > \text{rank}(H_p) \).) We do some (rigorous) constant-counting in the spirit of 19th century algebra to show that there exists \( p \in Q_{n,m} \) with \( w(p) = n^{-1}N(n,m) \). If \( P_{n,m} = \Sigma_{n,m} \), then for \( p \in Q_{n,m} \), \( N(n,s) \geq \text{rank}(H_p) = w(p) \geq n^{-1}N(n,m) \). Since \( N(n,s) \geq n^{-1}N(n,m) \) only for \( (n,m) \in \mathcal{K} \) and five other cases, it follows that there are psd forms which are not sos for "most" \( (n,m) \not\in \mathcal{K} \). This proof is independent of Hilbert's. This section also concludes with some historical notes.

In section five, we study binary forms and ternary quartics; since \( (n,m) \in \mathcal{K} \) in those cases, \( \Sigma_{n,m}^* = Q_{n,m} \), so \( w(p) = \text{rank}(H_p) \) for all \( p \in Q_{n,m} \). There is a dichotomy in the representations of \( p \in Q_{2,m} \) depending on \( r = \text{rank}(H_p) \).

If \( r \leq s \), then \( p \) has a strongly unique representation, which is the sum of \( r \) \( m \)-th powers as dictated by Theorem 4.6. If \( r = s + 1 \), then \( p \) has infinitely many minimal representations; for every \( (a,b) \neq (0,0) \), \( p \) has exactly one such representation including a positive multiple of \( (ax + by)^m \). The foregoing are restatements of old theorems of Fischer \( (r \leq s) \) and Akhiezer and Krein \( (r = \)
s + 1), see [A2]. We also describe in this section an explicit algorithm for writing \( p \in Q_{2,m} \) as a sum of \( w(p) \) \( m \)-th powers.

The distinctions are less clearcut for ternary quartics. We are able to give an exhaustive classification of \( p \in Q_{3,4} \), based on \( \text{rank}(H_p) \) and the structure of \( Z(p) \) and \( \mathcal{N}(p) \). We also determine those \( p \) with a strongly unique representation.

In section six, we discuss two truncated Hamburger moment problems, viz.: find necessary and sufficient conditions on \( A = (a_0, \ldots, a_m) \) so that there exists a (non-negative, Borel) measure \( \mu \) on \( \mathbb{R} \) for which:

\[
(1.42) \quad a_j = \int_{-\infty}^{\infty} t^j d\mu, \quad 0 \leq j \leq m-1 \text{ and } a_m \geq \int_{-\infty}^{\infty} t^m d\mu;
\]

\[
(1.43) \quad a_j = \int_{-\infty}^{\infty} t^j d\mu, \quad 0 \leq j \leq m.
\]

If such a measure \( \mu \) exists, we say that \( A \) satisfies the appropriate set of relations. Given \( A \), define \( G^{A,m}(x,y) = \Sigma_j (m) a_j x^j y^{m-j} \). We reinterpret the classical solution of the truncated Hamburger moment problem by showing that \( A \) satisfies (1.42) if and only if \( G^{A,m} \in Q_{2,m} = \Sigma^*_{2,m} \) and that \( A \) satisfies (1.43) if and only if \( G^{A,m} \) can be written as a sum of \( m \)-th powers without using a term of the form \( \lambda x^m \), \( \lambda > 0 \). These results are obtained by dehomonogenizing the binary representation results from section five. We also interpret the classical "maximum mass" problem in terms of \( Q_{2,m} \).

Berg, Christensen, Jensen, Schmüdgen and others have studied moment problems in several variables and the consequences of the existence of psd forms in \( n \geq 3 \) variables which are not sos. We discuss this work, and give an apparently new truncated moment problem in two variables.
In section seven, we turn to quadrature formulas in numerical analysis. For a closed set $C \subseteq \mathbb{R}^n$, we define the cones $P_{n,m}(C)$ and $Q_{n,m}(C)$: the n-ary m-ics which are $\geq 0$ on $C$ and the non-negative linear combinations of the m-th powers $\{(\alpha \cdot x)^m, \alpha \in C\}$. We show that $P_{n,m}(C)$ and $Q_{n,m}(C')$ are dual, where $C'$ denotes the closure of the projection of $C$ onto the unit sphere. We use these cones to study the quadrature problem described earlier, and give another proof of what is known as Tchakaloff's Theorem. Suppose $\mu$ is a measure on $\mathbb{R}^{n-1}$, $K = \text{supp}(\mu)$ and suppose $f|_K = 0$ for $f \in P_{n,m}^*(K)$ implies $f = 0$. Then $G_{K,\mu}$ is interior to $P_{n,m}^*(K)$, and an analysis of $Q_{n,m}(K)$ shows that there is a quadrature formula for $(K,\mu)$ of precision m with r nodes:

$$
\int_{u \in K} \cdots \int_{u \in K} f(u) d\mu = \sum_{j=1}^{r} \lambda_j f(u_j),
$$

(1.44)

such that $\lambda_j > 0$, $u_j \in K$ and $r \leq N(n,m)$. No such formula can hold if $r < N(n,s)$. We also modify the constant-counting argument of section four to demonstrate the existence of a continuous $\varphi > 0$ on $K$ so that, taking $d\mu = \varphi(u) du$ in (1.44), every quadrature formula for $(K,\mu)$ of precision m must have at least $[n^{-1}N(n,m)]$ nodes, where $[x]$ denotes the smallest integer $\geq x$.

In sections eight and nine, we consider the representations of

$$
h_{n,m}(x) = (x_1^2 + \cdots + x_n^2)^s = \|x\|^m = \sum_{\ell \in I(n,s)} c(\ell)x^{2\ell}
$$

(1.45)

in two equivalent formulations:
(1.46)(i) \[ h_{n,m} = \sum_{k=1}^{r} (\alpha_k^m), \]

(1.46)(ii) \[ h_{n,m} = \sum_{k=1}^{r} \lambda_k (\beta_k^m). \]

The two versions of (1.46), related by \( \alpha_k = \lambda_k^{1/m} \beta_k \), of course, are useful in different contexts. By taking the dot product of both sides of (1.46) with \( h_{n,m} \) (see (8.3) and Corollary 8.18), we show that they imply

\[ \sum_{k=1}^{r} \| \alpha_k \|^m = \sum_{k=1}^{r} \lambda_k \| \beta_k \|^m = \frac{s}{2j + n} \frac{2j + 1}{2j + 1} . \]

The **caliber** of a representation is the number of distinct values taken by \( \| \alpha_k \| \) (or \( \lambda_k \| \beta_k \|^m \)), and is the number of different spheres on which the \( \alpha_k \)'s lie. We show that \( w(h_{n,m}) \geq N(n,s) \); a representation of \( h_{n,m} \) in which \( r = N(n,s) \) is called **tight**, following the notation of spherical designs. We also prove (Corollary 8.17) that every tight representation is first-caliber.

Representations of \( h_{n,m} \) have already appeared in the literature in several guises. In his solution of Waring's Problem, Hilbert [H8] proved that for all \( m \), there exists \( r \leq N(5,m), 0 < \lambda_k \in \mathbb{Q} \) and \( \beta_k \in \mathbb{Q}^5 \) so that (1.46)(ii) holds for \( h_{5,m} \). His proof, which amounts to showing that \( h_{5,m} \) is interior to \( \mathbb{Q}_{5,m} \), is valid for general \( n \). We present specific instances of such formulas due to Liouville, Lucas, Hurwitz and Kempner.

The key step in Hilbert's proof is equivalent to the assertion that \( h_{n,m} = B(n,m) \cdot F_{n^{-1}}^{\mu}, \) for a suitable constant \( B(n,m) \), where \( S^{n-1} \) is the unit
sphere in $\mathbb{R}^n$ and $\mu$ is Lebesgue measure. This gives a quadrature formula on
the sphere: (1.46)(ii) is equivalent to

$$\int_{u \in S^{n-1}} \cdots \int p(u) du = B(n,m) \sum_{k=1}^{r} \lambda_k p(\beta_k) \text{ for all } p \in F_{n,m}. \quad (1.48)$$

If (1.46) is first-caliber, then the constants in (1.48) simplify. Let $A(n)$
denote the surface area of $S^{n-1}$ and write $v_k = \alpha_k/\|\alpha_k\| \in S^{n-1}$. Then for
all $p \in F_{n,m}$, the average of $p$ on $S^{n-1}$ is the average of $p$ on $\{v_k\}$:

$$\frac{1}{A(n)} \int_{u \in S^{n-1}} \cdots \int p(u) du = \frac{1}{r} \sum_{k=1}^{r} p(v_k). \quad (1.49)$$

Stroud [S12] contains many quadrature formulas for $S^{n-1}$ with Lebesgue
measure, and we translate some of these into representations of $h_{n,m}$.

A spherical $t$-design is a set of points $\{v_k\} \subset S^{n-1}$ with the property
that (1.49) holds for all polynomials $p$ (not necessarily homogeneous) with
degree at most $t$. Spherical designs were defined somewhat differently by
Delsarte, Goethals and Seidel [D5] in 1977, although this equivalent definition
appeared soon afterwards [G3]. It is easy to derive from their work an
equivalence theorem (see Proposition 8.38). Let $m = 2\lceil t/2 \rceil$ and $d$ denote
the largest even and odd integers $\leq t$. Then $\{v_k\}$ is a spherical $t$-design if
and only if $\Sigma(v_k \cdot)^d = 0$ and $\Sigma(v_k \cdot)^m$ is a suitable multiple of $h_{n,m}$. Most
known spherical designs consist of antipodal pairs, so the first condition
above is automatic. A spherical $(2s + 1)$-design is tight if it has
cardinality $2N(n,s)$; it was shown in [D5] that any such tight design consists
of $N(n,s)$ antipodal pairs. By taking one of each pair, we obtain a tight representation of $h_{n,m}$. Since tight representations of $h_{n,m}$ must be first-caliber, there is a one-to-one correspondence between tight spherical $(m+1)$-designs in $\mathbb{R}^n$ and tight representations of $h_{n,m}$. These are known to exist for $(n,m) \in \mathcal{K}$ (see below), and for $(n,m) = (7,4), (23,4), (8,6), (23,6)$ and $(24,10)$. In the last case, the minimal vectors of the well-known Leech lattice in $\mathbb{R}^{24}$ give a representation of $h_{24,10}$ as a sum of $N(24,5) = 98,280$ tenth powers of linear forms, and we know that $h_{24,10}$ cannot be written as a sum of $98,279$ tenth powers!

There are two applications to Banach space theory. Figiel, Lindenstrauss and Milman [F1] have given asymptotic estimates on the smallest $r$ so that $\ell_p^r$ contains a subspace "close" to $\ell_2^n$. It is easy to see that $\ell_p^r$ contains an isometric copy of $\ell_2^n$ if and only if $r \geq w(h_{n,m})$. Let $N_{n,m}$ denote the set of $p \in P_{n,m}$ for which there exists a Banach space $X = \langle u_1, \ldots, u_n \rangle$ with $\|\sum_j u_j\| = p(x)$; if $X$ is isometric to a subspace of $L_m^n$, then $p \in Q_{n,m}$. Results from the author's thesis are reinterpreted to show that $N_{n,m}$ is a blender and $Q_{n,m} \subseteq N_{n,m} \subseteq P_{n,m}$.

In section nine we discuss minimal representations of $h_{n,m}$ for certain $(n,m)$. There are some very strong theorems on the nonexistence of tight spherical designs (due to Bannai and Damerell [B4], [B5] and Delsarte, Goethals, Lemmens and Seidel [D5], [L5]) which we interpret for $h_{n,m}$. Aside from the ones mentioned above, the only other values of $(n,m)$ for which $h_{n,m}$ might have a tight representation are $(u^2 - 2,4)$, where $u \geq 7$ is an odd integer, and $(3v^2 - 4,6)$, where $v \geq 4$ is an integer.

Suppose $(n,m) \in \mathcal{K}$; by Theorem 4.6, $w(h_{n,m}) = N(n,s)$, so the minimal representations of $h_{n,m}$ are tight. These are trivial for $m = 2$; when $n = 2$, we show that
(1.50) \[(x^2 + y^2)^s = \sum_{k=1}^{s+1} (a_k x + b_k y)^m\]

if and only if the points \{±(a_k, b_k)\} are the vertices of a regular \((m+2)\)-gon inscribed in a circle centered at the origin, with radius \(2((s+1)(\frac{m}{2}))^{-1/m}\). The minimal representations of \(h_{3,4}\) are also unique up to rotation:

(1.51) \[(x^2 + y^2 + z^2)^2 = \sum_{k=1}^{6} (a_k x + b_k y + c_k z)^4\]

if and only if the twelve points \{±(a_k, b_k, c_k)\} (= \{±\(\alpha_k\)\}) are the vertices of a regular icosahedron inscribed in a sphere centered at the origin, with radius \((5/6)^{1/4}\). One direction of the proof comes from the explicit Schönenmann coordinates for the vertices of an icosahedron. Conversely, any tight representation of \(h_{3,4}\) must be first-caliber, and \(\|\alpha_k\| = (5/6)^{1/4}\) by (1.47). Let \(q = h_{3,4} - (5/6)(\alpha_1 \cdot \cdot \cdot)^4\). By analyzing \(H_q\), we show that the angle between \(\alpha_j\) and \(\alpha_k \ (j \neq k)\) must be \(\arccos(\pm 1/\sqrt{5})\). It follows from a purely geometric result, first proved by Haantjes [H1], that \{±\(\alpha_k\)\} are the vertices of a regular icosahedron. (Under the additional assumption that a tight representation of \(h_{3,4}\) is first-caliber, this result is an immediate consequence of the uniqueness of the icosahedron as a tight spherical 5-design in \(\mathbb{R}^3\) [D5].)

The connection between tight spherical 5-designs and maximal sets of lines in \(\mathbb{R}^n\) which have a common mutual angle was discussed in [D5]. We give an alternate proof of these results using analogues of \(H_q\), as described in the last paragraph. We also show that \(w(h_{n,m}) = N(n,s) + 1\) for \((n,m) = (4,4), (5,4), (6,4), (3,6)\) and \(3,8)\).
In section ten, we indicate four areas of future study. We say that 
$p \in Q_{n,m}$ has a rational representation if it can be written as $(1.46) (ii)$ with 
$0 < \lambda_k \in Q$ and $\alpha_k \in \mathbb{Z}^n$. In this case, 
$p \in Q_{n,m} \cap Q[x_1, \ldots, x_n]$; as $(1.27)$ shows, not every such $p$ has a rational representation. As noted above, 
Hilbert showed that $h_{n,m}$ always has a rational representation. We ask when 
h_{n,m} has a minimal representation which is rational. For $(n,m) \in \mathcal{X}$, this is 
the case if and only if $m = 2$ (a trivial case) and $(n,m) = (2,6)$. The 
proofs require some elementary number theory. The known tight representations 
of $h_{n,m}$ for $(n,m) \not\in \mathcal{X}$ are all rational, as is the (non-tight) minimal 
representation for $h_{6,4}$.

Next, we prove that, if $C$ is a blender (and $C \neq \{0\}, F_{n,m}$), then 
$Q_{n,m} \subseteq C \subseteq P_{n,m}$. We discuss the subsets of $F_{n,m}$ which are invariant under 
groups $G \subseteq GL_n(\mathbb{R})$, and their intersections with blenders. Using joint work 
with Choi and Lam, we describe the even symmetric sextics in the various cones. 
Let $h_\lambda(x,y,z) = h_{3,6}(x,y,z) - \lambda (x^6 + y^6 + z^6)$. We show that $h_\lambda \in Q_{3,6}$ for 
$\lambda \leq 2/3$ and $h_\lambda \in X_{3,6}^*$ for $\lambda \leq 7/10$. Finally, we present a few reflections 
on forms of large width in $Q_{n,m}$. We give a formula for $h_{n,4}$ as linear 
combination of $n^2$ fourth powers, with some negative coefficients. For large 
n, either $h_{n,4}$ has width much smaller than $n^{-1}N(n,4)$ or the natural 
generalization of Sylvester’s Law of Inertia to higher even powers is false.

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Finally, the reader will observe that the bibliography is unusually rich in old research articles. This is a tribute to the magnificent Mathematics Library at the University of Illinois.