8. Representations of $h_{n,m}$

This section discusses representations of $h_{n,m}$ which have occurred in the literature, either directly or indirectly, and some of the resulting applications. The next section is devoted to a systematic analysis of some minimal representations of $h_{n,m}$. Many of these results are already known, upon translation into the language of spherical designs. We give several proofs of known results using the methods of this paper.

It is convenient to consider simultaneously two notations:

\[(8.1) (i) \quad h_{n,m}(x) = \|x\|^m = (x_1^2 + \ldots + x_n^2)^s = \sum_{k=1}^{r} (\alpha_k \cdot x)^m.\]

\[(8.1) (ii) \quad h_{n,m}(x) = \sum_{k=1}^{r} \lambda_k (\beta_k \cdot x)^m, \quad \lambda_k > 0.\]

These are used interchangeably with the implicit understanding that $\alpha_k = \pm \lambda_k^{1/m} \beta_k$. We shall say that \((8.1)\) is a \(j\)-th caliber representation of $h_{n,m}$ if there are \(j\) distinct values taken by $\|\alpha_k\|^m$ (resp. $\lambda_k \|\beta_k\|^m$). In this case, the $\alpha_k$'s lies on spheres of $j$ different radii centered at the origin.

The number $[h_{n,m}, h_{n,m}]$ will loom large in the next two sections. (It is explicitly evaluated in Corollary 8.18(i).) By taking the dot product of the representation \((8.1)\) with $h_{n,m}$, we see that

\[(8.2) \quad C(n,m) := [h_{n,m}, h_{n,m}] = \sum_{k=1}^{r} h_{n,m}(\alpha_k) = \sum_{k=1}^{r} \|\alpha_k\|^m = \sum_{k=1}^{r} \lambda_k \|\beta_k\|^m\]

100
Thus, if (8.1) is first-caliber, then \( \| \alpha_k \| = R = (C(n,m)/r)^{1/m} \), c.f. (8.20). We show in Proposition 8.38 that, in this case, the set \( \{ \pm \alpha_k/R \} \) is a combinatorial object known as a spherical \((m+1)\)-design; conversely, every spherical \( t \)-design in \( \mathbb{R}^n \) leads to a first-caliber representation of \( h_{n,m} \) with \( m = 2\lceil t/2 \rceil \). This equivalence is implicit in work of Goethals and Seidel, e.g. [G5]. (Our proof follows from the role of \( h_{n,m} \) in quadrature formulas for integration on \( S^{n-1} \).) Most spherical designs are the vertices of one or two regular polytopes or the union of the orbits of a finite subgroup of the orthogonal group \( O(n) \). Given such a set, there are group-theoretic methods for finding the maximal degree to which it is a spherical design. These are beyond the scope of this paper, but see [G5].

Each representation of \( h_{n,m} \) determines a family of representations of \( h_{n,m-2k} \), \( 1 \leq k \leq s \). If we apply the Laplacian \( \Delta = \Sigma(\partial^2/\partial x_j) \) to both sides of (8.1)(ii), then a routine calculation gives:

\[
(8.3) \quad m(m+n-2)h_{n,m-2}(x) = m(m-1) \sum_{k=1}^{r} \lambda_k \| \beta_k \|^2 (\beta_k \cdot x)^{m-2}.
\]

It is easy to see that a derived representation has the same caliber as the original. Repeated application of the Laplacian to \( h_{n,m} \), combined with (8.2), gives another way to compute \( C(n,m) \).

We prove below that \( w(h_{n,m}) \geq N(n,s) \), and following the notation of spherical designs, (8.1) will be called \textit{tight} if \( r = w(h_{n,m}) = N(n,s) \). We shall show that every tight representation is also first-caliber, and so is associated with a tight spherical design. Tight spherical designs have been extensively analyzed and give many interesting representations of \( h_{n,m} \).
Results from this subject imply that no tight representations exist for "most" $h_{n,m}$; for more details, see Proposition 9.2.

In 1984, Seymour and Zaslavsky [S8] proved a very general result on "averaging sets" which implies that, for all $(n,m)$, there exists $r(n,m)$ so that $h_{n,m}$ has a first-caliber representation of width $r$ for all $r \geq r(n,m)$. Recently, Wagner [W1] has shown that one can take $r(n,m) = c_{n,m}^{12n^4}$. It follows from (8.29) that for $n = 2$, the growth is $O(m)$, not $O(m^{192})$.

There are at least two other applications of the representations of $h_{n,m}$. In number theory, they have been used to study Waring's problem. Some of the first such representations are given below, as (8.4)–(8.8). In functional analysis, a representation of $h_{n,m}$ is equivalent to an isometric embedding of the Banach space $l_2^n$ in $l_m^r$; we defer our discussion of this topic to the end of the section, since no new representations of $h_{n,m}$ have arisen in this way.

Waring's problem asks whether for every positive integer $k$, there exists $g(k)$ so that every positive integer is a sum of at most $g(k)$ $k$-th powers of integers. (See Ellison's beautiful article [E3] for references and details.)

The key to Hilbert's solution of Waring's problem is a consequence of Carathéodory's Theorem: for all $(n,m)$, there exists $r \leq N(n,m)$ and suitable $\beta_k \in \mathbb{R}^n$ and $0 < \lambda_k \in \mathbb{Q}$ so that (8.1)(ii) is valid. Hurwitz [H12] had conjectured (based in part on (8.4)–(8.8) below) that such a formula can always be found, without estimating the number of terms.

Dickson's "History" [D10, pp.717–724] gives many representations of $h_{n,m}$, of which we present five. We use two summation conventions, which are combined in the obvious way: multiple appearances of $\pm$ are independent and $\Sigma_r^p(x_1, \ldots, x_n)$ is the sum of the $r$ distinct expressions $p(x_\sigma(1), \ldots, x_\sigma(n))$.
arising as $\sigma$ ranges over the symmetric group $S_n$. We refer the reader to [D10] for detailed bibliographic information.

(8.4) $12h_{3,4} = 8\sum (x_1)^4 + \sum (x_1 \pm x_2 \pm x_3)^4$
(Lucas, 1877)

(8.5) $24h_{4,4} = \sum (2x_1)^4 + \sum (x_1 \pm x_2 \pm x_3 \pm x_4)^4$
(Liouville, 1859)

(8.6) $6h_{4,4} = \sum (x_1 \pm x_2)^4$
(Lucas, 1876)

(8.7) $120h_{4,6} = \sum (2x_1)^6 + 8\sum (x_1 \pm x_2)^6 + \sum (x_1 \pm x_2 \pm x_3 \pm x_4)^6$
(Kempner, 1912)

(8.8) $5040h_{4,8} = 6\sum (2x_1)^8 + 60\sum (x_1 \pm x_2)^8 + \sum (2x_1 \pm x_2 \pm x_3)^8$
$+ 6\sum (x_1 \pm x_2 \pm x_3 \pm x_4)^8$ (Hurwitz, 1908)

(Apparently, (8.4) and (8.6) were originally set as school algebra exercises.)

It is not hard to show that (8.5) and (8.6) are related by a linear change. It is remarkable that the middle three of these are first-caliber, since this was not a criterion in their discovery. We shall see in the next section that, with the possible exception of (8.7), these are not minimal. In fact, $w(h_{3,4}) = 6$, $w(h_{4,4}) = 11$ and $w(h_{4,8}) \leq w(h_{4,10}) \leq 60$ by Theorem 9.13, Theorem 9.28 and (8.32), respectively. Dickson also lists some formulas for
these $h_{n,m}$ with greater width than those given, as well as Schur’s width 72 representation of $22680h_{4,10}$. There are also wider representations by Fleck (1907) and Kempner (1912) which are uncited in [D10].

The other representations of $h_{n,m}$ in the literature ultimately rely on the fact (Theorem 8.15(i) below) that $h_{n,m}$ is $B(n,m)F_{C,\mu}$ (in the sense of (1.33)) for a suitable constant $B(n,m)$, where $C = S^{n-1}$ and $\mu$ is Lebesgue measure on $\mathbb{R}^n$. A key step in this argument is Proposition 8.12, which was used by Hilbert [H8] for $n = 5$ in solving Waring’s problem, and can also be found for $n = 4$ in an 1870 paper of Th. Reye [R2,p.296]. It also suggests that $h_{n,m}$ is the "center" of the cone $Q_{n,m}$.

The following integral formula is standard (see e.g. [D4,p.347]). Suppose $i = 2j \in \mathbb{Z}_+^n$ is even. Then

$$
(8.9) \quad \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} u_i^i \, du = \frac{2\Gamma(j_1 + 1/2) \cdots \Gamma(j_n + 1/2)}{\Gamma(j_1 + \cdots + j_n + n/2)}.
$$

If $i$ is not even, then the integral is 0 by symmetry. We need two special cases of (8.9), corresponding to $i = 0$ and $i = me_1$. Let $A(n)$ denote the surface area of $S^{n-1}$. Then from (8.9) and the usual properties of $\Gamma(x)$,

$$
(8.10) \quad A(n) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} du = \frac{2\pi^{n/2}}{\Gamma(n/2)} ,
$$

$$
(8.11) \quad B(n,m) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} u_i^m \, du = A(n) \frac{\Gamma(s + 1/2)\Gamma(n/2)}{\Gamma(1/2)\Gamma(s + n/2)} = A(n) \prod_{j=0}^{s-1} \left(\frac{2j + 1}{2j + n}\right).
$$
Proposition 8.12 (Hilbert)

(8.13) \[ \int \cdots \int (u \cdot x)^m du = B(n,m) h_{n,m}(x). \]
\[ \text{u} \in S^{n-1} \]

Proof

We regard x as a constant and make an orthogonal change of variables in (8.13). Let \( v_1 = \|x\|^{-1} \sum u_j x_j = \|x\|^{-1} (u \cdot x) \) and choose \( v_2, \ldots, v_n \) to complete the orthogonal basis. The value of the integral is unchanged and, by (8.11), equals:

(8.14) \[ \int \cdots \int \|x\|^m v_1^m dv = \left( \int \cdots \int v_1^m dv \right) \|x\|^m = B(n,m) \|x\|^m. \]
\[ \text{v} \in S^{n-1} \]

Theorem 8.15

(i) For \( C = S^{n-1} \) and \( \mu \) Lebesgue measure, \( F_{C,\mu}(x) = B(n,m) h_{n,m}(x). \)

(ii) For all \( (n,m), h_{n,m} \in \text{int } Q_{n,m} \), so \( N(n,m) \geq w(h_{n,m}) \geq N(n,s) \).

(iii) If \( (n,m) \in \mathcal{N} \), then \( w(h_{n,m}) = N(n,s) \).

(iv) For suitable \( \gamma_{n,m} > 0 \), \( \sigma_{h_{n,m}}(\alpha) = \gamma_{n,m}(\|\alpha\|)^{-m} \).

Proof

(i) By dotting both sides of (8.13) with \( p \) (c.f. (1.34)), we see that

(8.16) \[ \int \cdots \int p(u) du = B(n,m) [p, h_{n,m}]. \]
\[ \text{u} \in S^{n-1} \]
(ii) If \( p \in P_{n,m} \) and \([p, h_{n,m}] = 0\), then \( p|_{S^{n-1}} = 0\), so \( p = 0\) by homogeneity.

By Theorem 3.14(ii) and (iv), \( h_{n,m} \in \text{int} \ Q_{n,m} \), and so \( w(h_{n,m}) \geq N(n,s)\). The upper bound is Carathéodory's Theorem.

(iii) Combine (ii) with Theorems 3.16(ii) and 4.6.

(iv) By (4.4), \( \sigma_p(\kappa \alpha) = \kappa^{-m} \sigma_p(\alpha) \), so it suffices to show that \( \sigma_p(\alpha) \) is constant for \( \alpha \in S^{n-1} \). If \( \|\alpha\| = \|\beta\| = 1 \), then there is an orthogonal \( M \) with \( \alpha M = \beta \). For \( q_\nu(x) = h_{n,m}(x) - \nu(\alpha \cdot x)^m \), we have \( (q_\nu \circ M)(x) = h_{n,m}(Mx) - \nu(\alpha \cdot Mx)^m = h_{n,m}(x) - \nu(\beta \cdot x)^m \). But \( \Sigma_{n,m}^* \) is a blender by Theorem 3.33, so \( q_\nu \in \Sigma_{n,m}^* \) if and only if \( q_\nu \circ M \in \Sigma_{n,m}^* \). Since \( \sigma_{h_{n,m}}(\alpha) \) is the largest value of \( \nu \) for which \( q_\nu \in \Sigma_{n,m}^* \), it follows that \( \sigma_{h_{n,m}}(\alpha) = \sigma_{h_{n,m}}(\beta) \).

\[ \square \]

**Corollary 8.17**

Every tight representation of \( h_{n,m} \) is first-caliber.

**Proof**

Suppose for concreteness that (8.1) is tight, so that \( r = N(n,s) \). By Theorem 8.15(ii), \( h_{n,m} \in \text{int} \ Q_{n,m} \subseteq \text{int} \ \Sigma_{n,m}^* \), so \( \text{rank}(H_{h_{n,m}}) = N(n,s) \) by Theorem 3.16(ii). Thus, Theorem 4.9 applies to (8.1), so that \( \sigma_{h_{n,m}}(\alpha_k) = 1 \) for \( 1 \leq k \leq r \). It follows from Theorem 8.15(iv) that \( \|\alpha_k\|^m = \gamma_{n,m} \) for all \( k \); that is, (8.1) is first-caliber.

\[ \square \]

**Corollary 8.18**

(i) For all \((n,m)\), we have
(8.19) \[ C(n,m) = [h_{n,m}, h_{n,m}] = A(n)/B(n,m) = \prod_{j=0}^{s-1} \left( \frac{2j + n}{2} \right). \]

(ii) If (8.1)(i) is first-caliber, then

(8.20) \[ \|\alpha_k\| = \left[ \frac{1}{r} \prod_{j=0}^{s-1} \left( \frac{2j + n}{2} \right) \right]^{1/m} = (C(n,m)/r)^{1/m}. \]

(iii) If \( h_{n,m} \) has a tight representation, then \( \gamma_{n,m} = C(n,m)/N(n,s) \) in Theorem 8.15(iv).

Proof

Since \( h_{n,m} = 1 \) on \( S^{n-1} \), the various parts of (8.19) follow from (8.10), (8.11) and (8.16). For (ii), see (8.2). Finally, (iii) implements the last sentence of the proof of Corollary 8.17 with \( r = N(n,s) \).

We conjecture that the value of \( \gamma_{n,m} \) given in (iii) above holds for all \((n,m)\). As noted earlier, the value of \( C(n,m) \) can be computed via (8.3). It can also be computed combinatorially. Since

(8.21) \[ h_{n,m}(x) = \sum_{\ell} c(\ell)x^{2\ell} \]

where the sums are taken over \( \ell \in I(n,s) \), it follows from (1.4) that

(8.22) \[ C(n,m) = \sum_{\ell} \frac{(c(\ell))^2}{c(2\ell)} = \sum_{\ell} \frac{(s!)^2}{(m!)^2} \frac{(2\ell_1)! \ldots (2\ell_n)!}{(\ell_1)!^2 \ldots (\ell_n)!^2}. \]
By standard generating function techniques, (8.22) implies that \( \binom{m}{s}C(n,m) \) is the coefficient of \( z^s \) in \( F(z)^n \), where \( F(z) = \sum (\binom{2j}{j})z^j = (1 - 4z)^{-1/2} \). The generalized binomial theorem can be used to check that the resulting formula reduces to (8.19).

**Corollary 8.23**

(i) The following are equivalent:

\[
\begin{align*}
\text{(8.24)(i)} & \quad h_{n,m} = \sum_{k=1}^{r} \lambda_k \beta_k^m, \\
\text{(8.24)(ii)} & \quad \int_{u \in S^{n-1}} \cdots \int_{u \in S^{n-1}} p(u)du = B(n,m) \sum_{k=1}^{r} \lambda_k \beta_k \quad \text{for all } p \in F_{n,m}.
\end{align*}
\]

(ii) Suppose \( h_{n,m} = \Sigma(\alpha_k \cdot)^m \) is first-caliber and \( \gamma_k = \alpha_k/\|\alpha_k\| \). Then for all \( p \in F_{n,m} \), the average of \( p \) on \( S^{n-1} \) is the average of \( p \) on \( \{\gamma_k\} \):

\[
\text{(8.25)} \quad \frac{1}{A(n)} \int_{u \in S^{n-1}} \cdots \int_{u \in S^{n-1}} p(u)du = \frac{1}{r} \sum_{k=1}^{r} p(\gamma_k).
\]

(iii) If \( Mh_{n,m} = \Sigma \lambda_k \beta_k^m \), then \( M = (\Sigma \lambda_k \|\beta_k\|^m)/C(n,m) \).

**Proof**

(i) For \( p \in F_{n,m} \), (8.24)(i) implies \( B(n,m)[p,h_{n,m}] = B(n,m)\Sigma \lambda_k p(\beta_k) \); thus, (8.24)(ii) follows from (8.16). Conversely, if (8.24)(ii) holds, then by (8.16), \( [p,h_{n,m}] = \Sigma \lambda_k p(\beta_k) \) for \( p \in F_{n,m} \). Let \( q = h_{n,m} - \Sigma \lambda_k p(\beta_k)^m \); as
[p,q] = 0 for all \( p \in F_{n,m} \), it follows that \( q = 0 \).

(ii) Apply (i) with \( \lambda_k = \|a_k\|_m = C(n,m)/r \) and compare with (8.19).

(iii) Divide by \( M \) and apply (8.2). \( \Box \)

If (8.24)(ii) holds for \( p \in F_{n,m} \), then it holds for \( p = (\Sigma x_j^2)q \), where \( q \in F_{n,m-2} \). As \( p = q \) on \( S^{n-1} \), \( p(\beta_k) = \|\beta_k\|^2 q(\beta_k) \) and \( (m+1)B(n,m) = (m+n)B(n,m-2) \), we obtain another proof of (8.3).

We return to (dehomogenized) polynomials and numerical analysis, but rehomogenize differently than in the last section. Recall that

\[
(8.26) \quad \int \cdots \int_{u \in S^{n-1}} f(u) du = \sum_{k=1}^{r} \lambda_k f(\beta_k)
\]

is called a quadrature formula of precision \( t \) if it is valid for all polynomials \( f \) in \( n \) variables of degree \( \leq t \). A quadrature formula is called antipodal if it consists of antipodal pairs \( (\beta,-\beta) \) with the same weight.

**Corollary 8.27**

Let \( m = 2\lfloor t/2 \rfloor \) and \( d = 2\lfloor (t-1)/2 \rfloor + 1 \) be the largest even and odd integers \( \leq t \). Then (8.26) is a quadrature formula of precision \( t \) if and only if

\[
(8.28)(i) \quad \Sigma \lambda_k(\beta_k \cdot)^m = B(n,m)h_{n,m},
\]

\[
(8.28)(ii) \quad \Sigma \lambda_k(\beta_k \cdot)^d = 0.
\]
Proof

Write \( f \) as a sum of forms of degree at most \( t \). Multiplicative factors of \((\sum x_j^2)^i\) may be ignored on \( S^{n-1} \). Thus (8.26) is true if and only if it holds for forms of the highest two degrees. By Corollary 8.23(i), (8.26) holds for \( f \in F_{n,m} \) if and only if (8.28)(i) is true. Since \( f \in F_{n,d} \) is homogeneous of odd degree, (8.26) holds if and only if \( 0 = \Sigma f(\beta_k) = [\Sigma f(\beta_k)]^d, f \) for all \( f \), which implies (8.28)(ii), as in the last proof. \( \square \)

If (8.26) is antipodal, then (8.28)(ii) is trivial. Thus, any antipodal quadrature formula for \( S^{n-1} \) is equivalent to a representation of \( h_{n,m} \).

Our main source of quadrature formulas is Stroud [S12, pp.267–303], to which we refer the reader for more detailed references. (Note that \( S^{n-1} \) and the solid \( n \)-ball \( B^n \) are denoted there by \( U_n \) and \( S_n \).) We may have scanted the extensive Russian-language literature. Sobolev [S10] and Salihov [S1] introduced group theory techniques to the subject and made a comprehensive survey of formulas based on regular polytopes. Their work has also been incorporated into the theory of spherical designs.

Here are some more representations of \( h_{n,m} \) which are derivable from quadrature formulas. We retain the earlier summation conventions and add a new one: \( \Sigma^r_{e} p(x_1, \ldots, x_n) \) is the sum of the \( r \) distinct expressions \( p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \), but now as \( \sigma \) ranges over the even permutations in the symmetric group \( S_n \).

For \( n = 2 \), there is a quadrature formula whose true provenance we have been unable to pin down; the resulting first-caliber representations of \( h_{2,m} \) have the feel of a much older formula (see also [R3,p.229]). If \( 2t > m \) is an even integer, then for all angles \( \varphi \),
(8.29) \[ \frac{1}{t} \sum_{k=0}^{t-1} (\cos(\varphi + \frac{k}{t}) \cdot x + \sin(\varphi + \frac{k}{t}) \cdot y)^m = 2^{-m}(m)(x^2 + y^2)^s. \]

We prove this identity as part of Theorem 9.5. Note that (8.29) is tight when \( t = s + 1 \). An instance of (8.29) for \( m = 8 \) is (10.7).

Ljusternik and Ditkin (1948) gave a quadrature formula for \( S^2 \) of precision five using the vertices of a regular icosahedron. This leads to a tight representation:

(8.30) \[ 6\varphi^2 h_{3,4} = \Sigma_e^6 (\varphi x \pm y + 0z)^4, \]

where \( \varphi = (\sqrt{5} + 1)/2 \). The even permutations in this case give a cyclic sum: \((\varphi x + y)^4 + (\varphi y + z)^4 + \ldots\). This identity is easy to verify by hand, since \( 6\varphi^2 = 2 + 2\varphi^4 \). We prove in Theorem 9.13 that, up to a rotation of the \( \alpha_k \)'s, (8.30) gives the unique tight representation for \( h_{3,4} \).

The next representation arises from combining the vertices of a regular icosahedron and dodecahedron with weights in ratio 25/27.

(8.31) \[ 140h_{3,8} = 3\varphi^{-4} \Sigma_e^6 (\varphi x \pm y + 0z)^8 + \Sigma_e^6 (\varphi^{-1}x \pm \varphi y + 0z)^8 \]

\[ + \Sigma^4 (x \pm y \pm z)^8. \]

(The values of \( \lambda||\beta||^8 \) in (8.31) are \( 3\varphi^{-4} (\varphi^2 + 1)^4 = 75 \) and \( (\varphi^2 + \varphi^{-2})^4 = 3^4 \)
\( = 81 \), and \( 140 = (6.75 + 10.81)/C(3,8) \), confirming Corollary 8.23(iii) in this case.) Stroud attributes the underlying quadrature formula to Finden (1961), Sobolev (1962) and McLaren (1963). We show in Proposition 9.37 that (8.31) is minimal; we do not know about its uniqueness.
The regular polytope \( \{3,3,5\} \) in \( \mathbb{R}^4 \) (see [C13, p.153]) has 120 vertices. In 1975, Salihov [S1] proved that these points give a quadrature formula of precision 11. The associated representation is:

\[
(8.32) \quad 2520 h_{4,10} = \Sigma^4 (2x_1)^{10} + \Sigma^8 (x_1 \pm x_2 \pm x_3 \pm x_4)^{10} + \Sigma^48 (x_1 \pm x_2 \pm x_3 \pm x_4)^{10}.
\]

Although \( x^2 + x^{-2} = 3 \), so (8.32) is first-caliber, it is not tight: \( N(4,5) = 56 < 60 \). We do not know whether it is minimal. In the same paper, Salihov constructs a quadrature formula of precision 19 by taking 120 vertices of \( \{3,3,5\} \) and the 600 vertices of \( \{5,3,3\} \) with weights in ratio 25/16. This gives a representation of \( h_{4,18} \) of length 360 (\( N(4,9) = 220 \)), which is too involved to give here.

Stroud (1967) gave a family of quadrature formulas of precision 7, which lead to the following family of identities for \( n \geq 3 \):

\[
15 h_{n,6} = (16 - 2n) \Sigma^n x_1^6 + \Sigma^n (x_1 \pm x_2)^6 + \Sigma^{n-1} (x_1 \pm x_2)^6.
\]

\[
(8.33)_n
\]

This is a representation of \( h_{n,6} \) for \( n \leq 8 \), of width \( n^2 + 2^{n-1} \) for \( n \leq 7 \) and width 184 for \( n = 8 \) (since the first summands vanish). Observe that (8.33)\(_4\) is first-caliber and, on multiplication by 8, becomes (8.7).

McLaren (1963) gave a non-antipodal quadrature formula of precision 7 leading to a representation of \( h_{3,6} \) which has greater width than (8.33)\(_3\) (24 vs. 13), but is first-, rather than third-caliber:
Sums of Even Powers of Real Linear Forms

\[(8.34) \quad 24h_{3,6} = C(3,6)\Sigma^2 (ax_1 + bx_2 + cx_3)^6,\]

where \(a^2, b^2\) and \(c^2\) are the three (positive) roots of the cubic \(105t^3 - 105t^2 + 21t - 1 = 0\). A minimal representation of \(h_{3,6}\) is given in (9.36).

Finally, Stroud (1967) also gave a family of quadrature formulas of precision 5, based on solving for a set of undetermined constants. This gives a family of identities:

\[(8.35)_n \quad Mh_{n,4}(x) = a(\Sigma x_j)^4 + \Sigma^n (b\Sigma x_j + cx_1)^4 + \Sigma^{\binom{n}{2}} (d\Sigma x_j + e(x_1 + x_2))^4.\]

There are two solutions to \((8.35)_n\), which give a representation only if \(n \leq 7\). The solutions are related by a linear change of variables. We introduce the parameter \(g = (8 - n)^{1/4}\), and solve \((8.35)_n\):

\[(8.36) \quad M = 3e^4, \quad a = 8(g^4 - 1)(g^2 \pm 2\sqrt{2})^4, \quad b = 2g^2 \pm 2\sqrt{2}, \quad c = \mp 2\sqrt{2} g^4 - 8g^2, \quad d = 2g, \quad e = \mp 2\sqrt{2} g^3 - 8g.\]

When \(n \leq 6\), \((8.35)_n\) has width \(1 + n + n(n-1)/2 = N(n,2) + 1\). When \(n = 7\), \(g = 1\), so \(a = 0\). Thus, \((8.35)_7\) is tight, and so is first-caliber by Corollary 8.17. We prove in Proposition 9.26 and Theorem 9.28 that \((8.35)_n\) is minimal for \(4 \leq n \leq 7\). These representations are all minimal; for particular values of \(n\), they may not be the most attractive, but see (8.40) and (9.27).

Spherical t-designs were first defined in 1977 by Delsarte, Goethals and Seidel [D5]; see also [B1], [C11], [G3], [G4], [G5], [H9], [S5] and [S6] for more on the subject. There are several equivalent definitions; the one of greatest interest to us can be given in terms of (8.25). A set of points
\( X = \{v_k\} \subset S^{n-1} \) is a spherical \( t \)-design if, for every polynomial \( f \) of degree \( \leq t 
olinebreak\):

\[
\frac{1}{A_n} \int_{u \in S^{n-1}} \cdots \int_{u \in S^{n-1}} f(u) du = \frac{1}{|X|} \sum_{k=1}^{|X|} f(v_k).
\]

As before, \( X \) is antipodal if \( v \in X \) implies \( -v \in X \). We now show that there is a one-to-one correspondence between antipodal spherical designs and first-caliber representations of \( h_{n,m} \): if (8.1)(i) is first-caliber, then \( \{ \pm \alpha_k/\|\alpha_k\| \} \) is a spherical \((m+1)\)-design. Every tight spherical design is antipodal [D5,p.375], thus the study of tight representations of \( h_{n,m} \) is equivalent to the study of tight spherical \((m+1)\)-designs in \( \mathbb{R}^n \).

**Proposition 8.38**

Let \( m \) and \( d \) be the largest even and odd integers \( \leq t \). Then \( X = \{v_k\} \subset S^{n-1} \) is a spherical \( t \)-design if and only if

\[
\begin{align*}
(8.39)(i) & \quad |X|^{-1} \Sigma(v_k \cdot)^{m} = C(n,m)^{-1} h_{n,m}, \\
(8.39)(ii) & \quad \Sigma(v_k \cdot)^{d} = 0.
\end{align*}
\]

**Proof**

Apply Corollary 8.27 with \( \lambda_k = A_n/|X| \).

This result is implicit, if unstressed, in the literature. By taking Theorem 3.9(i) and (iii) in [G5,p.208], and applying Lemmas 3.5 and 3.3, we find that \( X \) is a spherical \( t \)-design if and only if, for every \( \xi \in S^{n-1} \),
\[(8.39)(i)' \quad |x|^{-1} \sum (v_k \cdot \xi)^m = C(n,m)^{-1},\]
\[(8.39)(ii)' \quad \sum (v_k \cdot \xi)^d = 0.\]

It is easy to see by homogeneity that \((8.39)'\) is equivalent to \((8.39)\).

Spherical designs have been sought by considering the orbits under a finite subgroup of the orthogonal group in \(n\) dimensions. This idea had been introduced to quadrature theory by Sobolev [S10]. There is a detailed and concrete application of these methods in [G5].

The known spherical designs seem to have been described in [D5] and [G5], and many of the representations we have already presented can also be derived from them: \((8.5)\), \((8.6)\), \((8.7)\), \((8.29)\), \((8.30)\), \((8.32)\), \((8.34)\) and \((8.35)\)\(_7\). Liouville's \((8.5)\), and its rotated version \((8.6)\), correspond to the fact that the regular polytope \(\{3,4,3\}\) in \(\mathbb{R}^4\) (see [C13,p.298]) is a spherical 5–design. These two representations are neither tight \((N(4,4) = 10 < 12)\) nor minimal -- \((8.35)_4\) has width 11.

In the following formulas, we follow the previous conventions, with another new one: \(\Sigma_p\) denotes a sum over \(\ast\)'s taken so that the product is 1. The representations of \(h_{n,m}\) associated to every known tight spherical design has either already been presented, or will be mentioned below.

As noted earlier, \((8.35)_7\) is tight. A more aesthetically pleasing version can be derived using projective planes. Recall that

\[(8.40) \quad \mathcal{L} = \{(k,k+1,k+3) : 1 \leq k \leq 7\},\]

with the indices reduced mod 7, is the set of lines in the projective plane of order 2. By using \(\mathcal{L}\) to select the coordinates, we obtain a simple tight representation for \(h_{7,4}\):
(8.41) \[ 12h_{7,4} = \Sigma_k^7 (\sum (x_k \pm x_{k+1} \pm x_{k+3})^4). \]

The tight spherical 5–design associated to (8.41) is discussed in [D5] and has an interesting alternative characterization (see also the beginning of the next section): it is the (maximal) set of 28 lines in \( \mathbb{R}^7 \) with mutual angle \( \arccos(1/3) \). A set of 276 lines in \( \mathbb{R}^{23} \) with mutual angle \( \arccos(1/5) \) gives a tight representation of \( h_{23,4} \) of width \( N(23,2) = 276 \); we do not give it explicitly here.

The minimal vectors in the root system \( E_8 \) are a tight spherical 7–design, and give a tight representation of \( h_{8,6} \) of width \( N(8,3) = 120 \):

(8.42) \[ 960h_{8,6} = \Sigma_p^{56} (2x_1 \pm 2x_2)^6 + \Sigma_p^{64} (x_1 \pm x_2 \pm \cdots \pm x_8)^6. \]

There is a linear change of this representation which also has integral coefficients, and sums to \( 120h_{8,6} \). However, the pattern of signs is somewhat more difficult to describe in this case, and we omit it.

The most spectacular representation arising from a spherical design occurs in \( \mathbb{R}^{24} \). The \( 2N(24,5) = 2 \binom{28}{5} = 196,560 \) minimal vectors in the Leech lattice are a spherical 11–design. It follows that

(8.43) \[ 185,794,560h_{24,10} = \Sigma_p^{98,280} (a_k \cdot)^{10}, \]

where the terms on the right–hand side fall into three classes:

(8.44)

<table>
<thead>
<tr>
<th>Terms</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>48,576</td>
<td>((2x_1 \pm \cdots \pm 2x_8)^{10})</td>
</tr>
<tr>
<td>49,152</td>
<td>((3x_1 \pm x_2 \pm \cdots \pm x_{24})^{10})</td>
</tr>
<tr>
<td>552</td>
<td>((4x_1 \pm 4x_2)^{10})</td>
</tr>
</tbody>
</table>
The selection of coordinates to receive non-zero entries and the choices of sign follow delicate combinatorial rules. Note that \( \|\alpha_k\|^2 = 8.2^2 = 3^2 + 23 = 24^2 = 32 \) in (8.43). A certain subset of these minimal vectors leads to a tight representation for \( h_{23,6} \), which we do not present \( (N(23,3) = 2300) \).

Finally, we turn to functional analysis. The \( r \)-dimensional Banach space \( \mathbb{L}_p^r \), \( 1 \leq p < \infty \) consists of \( \mathbb{R}^r \), under the \( p \)-norm:

\[
\|u\|_p = \|(u_1, \ldots, u_r)\| = \left( \sum_{j=1}^r |u_j|^p \right)^{1/p}.
\]

(8.45)

The infinite-dimensional space \( \mathbb{L}_p^\infty \) is defined similarly, consisting of those infinite sequences \( u \) for which \( \Sigma|u_j|^p \) is finite; \( \mathbb{L}_2^\infty \) is a Hilbert space. Let \( p = m \) be an even integer, and for \( u_k = (u_{k1}, \ldots, u_{kr}) \in \mathbb{R}^r \), let \( \langle u_1, \ldots, u_n \rangle \) denote the subspace of \( \mathbb{L}_m^r \) spanned by the \( u_k \)'s. Then

\[
\Phi(x_1, \ldots, x_n) = \|x_1u_1 + \ldots + x_nu_n\|_m^m = \sum_{j=1}^r (x_1u_{1j} + \ldots + x nu_{nj})^m,
\]

(8.46)

so \( \Phi \in \mathbb{Q}_m^{r,n} \). Conversely, if \( p \in \mathbb{Q}_m^{r,n} \), then we can work backwards from the representation of \( p \) to construct an \( n \)-dimensional subspace \( \langle u_1, \ldots, u_n \rangle \) of \( \mathbb{L}_m^r \) so that \( p(x) = \|\Sigma x_iu_i\|_m^m \).

If \( X \) and \( Y \) are two Banach spaces, and there is an isometric isomorphism between \( X \) and a subspace \( Z \subseteq Y \), then \( X \) is said to embed in \( Y \), and \( Y \) is said to contain an isometric copy of \( X \). If \( \mathbb{Q}_2 \) embeds in \( \mathbb{Q}_m^r \), and \( u_1, \ldots, u_n \) are the images of the \( n \) unit vectors \( e_1, \ldots, e_n \), then
\[(8.47) \quad \Phi(x) = \|\Sigma x_j u_j\|^m = \|\Sigma x_j e_j\|^m = (\sum_{j=1}^n x_j^2)^{m/2} = h_{n,m}(x).\]

Thus $\ell_2^n$ embeds in $\ell_m^n$ if and only if $r \geq w(h_{n,m})$. Since $q_{n,m}^r = q_{n,m}$ for $r \geq N(n,m)$ by Carathéodory's Theorem, if an $n$-dimensional Banach space $X$ embeds in $\ell_m$, then it embeds in $\ell_m^n(n,m)$.

Every infinite dimensional Banach space contains an isometric copy of $\ell_2$ by Dvoretzky's Theorem. This leads to a natural question: What conditions can be placed on $n$ to ensure that $X$, an $r$-dimensional Banach space, contains an $n$-dimensional subspace $Y$ which is "close" to $\ell_2^n$? (Closeness is measured by the Banach–Masur distance; if $X_1$ and $X_2$ are two isomorphic Banach spaces, then $d(X_1, X_2)$ is the infimum of $\|T\| \cdot \|T^{-1}\|$, as $T$ ranges over the isomorphisms from $X_1$ onto $X_2$.) Figiel, Lindenstrauss and Milman [F1] have proved that there exist constants $c_i > 0$ so that $\ell_m^n$ must contain an $n$-dimensional subspace $Y$ with $d(Y, \ell_2^n) \leq 2$ if $n \leq c_1 r 2^m$, and $\ell_m^n$ cannot contain such a subspace if $n \geq c_2 r 2^m$.

For fixed $m$, $N(n,m) \approx (m!)^{-1} n^m$ and $N(n,s) \approx (s!)^{-1} n^s$. Theorem 8.15(ii) implies that $\ell_m^n(n,m)$ contains an isometric copy of $\ell_2^n$ with $n \approx (m!)^{-1} N(n,m)^{1/m}$, as compared with the guaranteed subspace "close" to $\ell_2^n$ with dimension $c_1 N(n,m)^{2/m}$. Further, $\ell_m^n(n,s)^{-1}$ cannot contain an isometric copy of $\ell_2^n$ with $n \approx (s!)^{-1} N(n,s)^{2/m}$ and it cannot even contain a subspace "close" to $\ell_2^n$ with $n \approx c_2 N(n,s)^{2/m}$.

As noted earlier, the author's thesis contained an explicit embedding of $\ell_2^n$ in $\ell_m^{n+1}$ based on (8.29). Milman [M1] (in collaboration with Gromov) gave the connection described above between representations of $h_{n,m}$ and embeddings of $\ell_2^n$ into $\ell_m^n$; they also made some of these asymptotic estimates and used (8.6) to embed $\ell_2^n$ in $\ell_4^{12}$. (One consequence of Theorem 9.28 is that $\ell_4^n$ embeds in $\ell_4^{11}$,
but not in $\ell_{4}^{10}$. They report that the existence of such embeddings, without a bound on $r$, was noted by Figiel.

Given $p \in F_{n,m}$ and an $n$-dimensional vector space $X = \langle u_{1}, \ldots, u_{n} \rangle$, one can define a prospective norm on $X$ by:

$$
(8.48) \quad \|x_{1}u_{1} + \ldots + x_{n}u_{n}\|^{m} = p(x_{1}, \ldots, x_{n}).
$$

By the homogeneity of $p$, $(X, \| \cdot \|)$ is a Banach space provided $p \geq 0$ and the triangle inequality is satisfied. Let $N_{n,m}$ denote the set of all $p$ for which $(X, \| \cdot \|)$ is a Banach space. By (8.46), $Q_{n,m}$ is the set of those $p$ which are the norm functions of an $\ell_{m}^{n}$ space, under a suitable choice of basis. Thus $Q_{n,m} \subseteq N_{n,m} \subseteq P_{n,m}$, so $N_{n,2} = P_{n,2} = Q_{n,2}$.

Proposition 8.49

(i) \quad $p \in N_{n,m}$ if and only if $p \in P_{n,m}$ and, for all fixed $x$ and $y \in \mathbb{R}^{n}$, $(p(x + ty))^{1/m}$ is convex in the real variable $t$.

(ii) \quad $N_{2,4} \subseteq Q_{2,4}$.

(iii) \quad $N_{2,m} \subseteq Q_{2,m}$ for $m \geq 6$.

(iv) \quad $N_{3,4} \subseteq Q_{3,4}$.

Proof

We refer the reader to [R3], where (i) through (iv) are Theorems 1, 3, 4 and 8 respectively. Counterexamples for (iii) and (iv) are $p(x,y) = x^{m} + x^{m-2}y^{2} + y^{m}$ and $p(x,y,z) = x^{4} + 6x^{2}y^{2} + 6x^{2}z^{2} + y^{4} + 2y^{2}z^{2} + z^{4}$. □
It follows from (ii) and (iv) that there exists a three-dimensional Banach space \( X \) which has the property that \( X \) is not embeddable in \( \ell_4 \), but all two-dimensional subspaces of \( X \) are embeddable in \( \ell_4 \). Neyman [N2] has generalized this result to all \( \ell_p \) spaces: for \( p \in [1,\infty) \), and all \( k \) and \( n \), \( 2 < k \leq n \), there exists an \( n \)-dimensional Banach space \( X_{n,k} \) so that every \((k-1)\)-dimensional subspace of \( X_{n,k} \) is embeddable in \( \ell_p \), but there is a \( k \)-dimensional subspace of \( X_{n,k} \) which is not embeddable in \( \ell_p \).

We conclude with a result which was inchoate in [R3].

**Theorem 8.50**

\( N_{n,m} \) is a blender.

**Proof**

It is clear that \( N_{n,m} \) is closed topologically and that \( p \in N_{n,m} \) implies that \( \lambda p \in N_{n,m} \) for \( \lambda > 0 \). By Theorem 2 in [R3], \( N_{n,m} \) is closed under addition, and so \( N_{n,m} \) is a closed convex cone. If \( q = p_0 M \), then \( q(x + ty) = p((Mx) + t(My)) \) is also convex in \( t \), so \( q \in N_{n,m} \) by Proposition 8.49(i); hence \( N_{n,m} \) blends. \( \square \)

We conjecture that an independent characterization of \( N_{n,m}^* \) would have some analytic significance.