we use the fact that \( y = e^{\ln y} \):

\[
\lim_{x \to 0^+} (1 + \sin 4x)^{\ln x} = \lim_{x \to 0^+} e^{\ln y} = e^4
\]

**EXAMPLE 9** Find \( \lim_{x \to 0^+} x^x \).

**SOLUTION** Notice that this limit is indeterminate since \( 0^0 = 0 \) for any \( x > 0 \) but \( x^0 = 1 \) for any \( x \neq 0 \). We could proceed as in Example 8 or by writing the function as an exponential:

\[
x^x = (e^{\ln x})^x = e^{x \ln x}
\]

In Example 6 we used l'Hospital's Rule to show that

\[
\lim_{x \to 0^+} x \ln x = 0
\]

Therefore

\[
\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1
\]

---

### 4.4 EXERCISES

1–4 Given that

\[
\lim_{x \to a} f(x) = 0 \quad \lim_{x \to a} g(x) = 0 \quad \lim_{x \to a} h(x) = 1 \\
\lim_{x \to a} p(x) = \infty \quad \lim_{x \to a} q(x) = \infty
\]

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) \( \lim_{x \to a} \frac{f(x)}{g(x)} \)  
   (b) \( \lim_{x \to a} \frac{f(x)}{p(x)} \)
   (c) \( \lim_{x \to a} \frac{h(x)}{p(x)} \)
   (d) \( \lim_{x \to a} \frac{p(x)}{f(x)} \)
   (e) \( \lim_{x \to a} \frac{p(x)}{q(x)} \)

2. (a) \( \lim_{x \to a} [f(x)p(x)] \)  
   (b) \( \lim_{x \to a} [h(x)p(x)] \)
   (c) \( \lim_{x \to a} [p(x)q(x)] \)

3. (a) \( \lim_{x \to a} [f(x) - p(x)] \)  
   (b) \( \lim_{x \to a} [p(x) - q(x)] \)
   (c) \( \lim_{x \to a} [p(x) + q(x)] \)

4. (a) \( \lim_{x \to a} [f(x)]^{p(x)} \)  
   (b) \( \lim_{x \to a} [f(x)]^{q(x)} \)  
   (c) \( \lim_{x \to a} [h(x)]^{p(x)} \)
   (d) \( \lim_{x \to a} [p(x)]^{q(x)} \)  
   (e) \( \lim_{x \to a} [p(x)]^{q(x)} \)  
   (f) \( \lim_{x \to a} \sqrt[p(x)]{q(x)} \)

5–64 Find the limit. Use l'Hospital's Rule where appropriate there is a more elementary method, consider using it. If l'H Rule doesn't apply, explain why.

5. \( \lim_{x \to 1} \frac{x^2 - 1}{x^3 - x} \)

6. \( \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} \)

7. \( \lim_{x \to 1} \frac{x^9 - 1}{x - 1} \)

8. \( \lim_{x \to 0^+} \frac{x^{1/2} - 1}{x^5 - 1} \)

9. \( \lim_{x \to \infty} \frac{\cos x}{1 - \sin x} \)

10. \( \lim_{x \to 0^+} e^{\frac{x^2}{t^3}} \)

11. \( \lim_{x \to 0} \frac{\tan px}{\tan qx} \)

12. \( \lim_{x \to 0} \frac{e^{1/t} - 1}{t} \)

13. \( \lim_{x \to 0} \frac{\ln x}{\sqrt{x}} \)

14. \( \lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{\csc \theta} \)

15. \( \lim_{x \to 0^+} \frac{x + x^2}{1 - 2x^2} \)

16. \( \lim_{x \to 0} \frac{x + x^2}{1 - 2x^2} \)

17. \( \lim_{x \to 0} \frac{\ln x}{x} \)

18. \( \lim_{x \to 0} \frac{\ln x}{x} \)

19. \( \lim_{x \to 0} \frac{e^x}{x^3} \)

20. \( \lim_{x \to 0} \frac{\ln x}{x^2} \sin \pi x \)

21. \( \lim_{x \to 0} \frac{e^x - 1 - x}{x^3} \)

22. \( \lim_{x \to 0} \frac{e^x - 1 - x}{x^3} \)
23. \[ \lim_{x \to 0} \frac{x - \sin x}{x - \tan x} \]

24. \[ \lim_{x \to 0} \frac{x - \sin x}{x - \tan x} \]

25. \[ \lim_{t \to 0} \frac{t^2 - 3t}{t} \]

26. \[ \lim_{x \to 0} \frac{\sin x - x}{x^3} \]

27. \[ \lim_{x \to 0} \frac{\sin x}{x} \]

28. \[ \lim_{x \to 0} \frac{(\ln x)^2}{x} \]

29. \[ \lim_{x \to 0} \frac{\cos \pi x - \cos \pi x}{x^2} \]

30. \[ \lim_{x \to 0} \frac{x + \sin x}{x + \cos x} \]

31. \[ \lim_{x \to 0} \frac{1 - \cos x}{x^2} \]

32. \[ \lim_{x \to 0} \frac{x}{\tan^{-1}(4x)} \]

33. \[ \lim_{x \to 0} \frac{1 - \sin x}{x^2} \]

34. \[ \lim_{x \to 0} \frac{\sqrt{x^2 + 2}}{x} \]

35. \[ \lim_{x \to 0} \frac{x^3 - ax + a - 1}{(x - 1)^2} \]

36. \[ \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \]

37. \[ \lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^2} \]

38. \[ \lim_{x \to 0} \frac{\cos x \ln(x - a)}{x^2} \]

39. \[ \lim_{x \to 0} \frac{x}{x \sin(\pi x)} \]

40. \[ \lim_{x \to 0} \frac{x^2e^x}{x^2 + 2x} \]

41. \[ \lim_{x \to 0} \frac{\ln x}{x} \]

42. \[ \lim_{x \to 0} \frac{x}{x \ln x} \]

43. \[ \lim_{x \to 0} \frac{\cos x}{x} \]

44. \[ \lim_{x \to 1} \frac{x - \tan x - \sec x}{x - \cos x} \]

45. \[ \lim_{x \to 0} \frac{x^2}{x \tan(\pi x/2)} \]

46. \[ \lim_{x \to 0} \frac{\ln x}{x \tan(1/x)} \]

47. \[ \lim_{x \to 1} \frac{x - 1}{x - \ln x} \]

48. \[ \lim_{x \to 0} \frac{\csc x - x}{\cot x} \]

49. \[ \lim_{x \to 0} \frac{x}{x^2 + x - x} \]

50. \[ \lim_{x \to 0} \frac{\cot x}{x - \ln x} \]

51. \[ \lim_{x \to 0} \frac{x - 1}{x - x \ln x} \]

52. \[ \lim_{x \to 0} \frac{x}{x} \]

53. \[ \lim_{x \to 0} \frac{x^3}{x} \]

54. \[ \lim_{x \to 0} \frac{(x^2 + x)}{x^2} \]

55. \[ \lim_{x \to 0} \frac{(1 - 2x)^{1/2}}{x} \]

56. \[ \lim_{x \to 0} \frac{1 + a}{x^{1/2}} \]

57. \[ \lim_{x \to 0} \frac{(1 + x^2 + x^3)}{x^{1/2}} \]

58. \[ \lim_{x \to 0} \frac{\ln(2x + 1)}{x} \]

59. \[ \lim_{x \to 0} \frac{\ln(\ln x)}{x} \]

60. \[ \lim_{x \to 0} \frac{(e^x + x)^{1/2}}{x} \]

61. \[ \lim_{x \to 0} \frac{(1 - x)^{\cos x}}{x} \]

62. \[ \lim_{x \to 0} \frac{(2 - x)^{\cos x}}{x} \]

63. \[ \lim_{x \to 0} \frac{x - \sin x}{x - \tan x} \]

64. \[ \lim_{x \to 0} \frac{x - 3}{x + 5} \]

65. \[ \lim_{x \to 0} \frac{2x - 3}{x + 5} \]

66. \[ \lim_{x \to 0} \frac{x^3 - 4}{3x - 2} \]

67. \[ f(x) = e^x - 1, \quad g(x) = x^3 + 4x \]

68. \[ f(x) = 2x \sin x, \quad g(x) = \sec x - 1 \]

69. Prove that

\[ \lim_{x \to \infty} \frac{e^x}{x^n} = \infty \]

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

70. Prove that

\[ \lim_{x \to \infty} \frac{\ln x}{x^p} = 0 \]

for any number p > 0. This shows that the logarithmic function approaches 0 more slowly than any power of x.

71. What happens if you try to use l'Hopital's Rule to evaluate

\[ \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} \]

Evaluate the limit using another method.

72. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

\[ v = \frac{mg}{c} - \frac{1}{e^{ct/m}} \]

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate \( \lim_{t \to \infty} v \). What is the meaning of this limit?

(b) For fixed t, use l'Hopital's Rule to calculate \( \lim_{t \to 0^+} v \). What can you conclude about the velocity of a falling object in a vacuum?
4.5 Exercises

1–52. Use the guidelines of this section to sketch the curve.

1. \( y = x^3 + x \)
2. \( y = x^3 + 6x^2 + 9x \)
3. \( y = 2 - 15x + 9x^2 - x^3 \)
4. \( y = 8x^2 - x^4 \)
5. \( y = x^4 + 4x^3 \)
6. \( y = x(x + 2)^3 \)
7. \( y = 2x^3 - 5x^2 + 1 \)
8. \( y = (4 - x^2)^3 \)
9. \( y = \frac{x}{x - 1} \)
10. \( y = \frac{x^2 - 4}{x^3 - 2x} \)
11. \( y = \frac{1}{x^2 - 9} \)
12. \( y = \frac{x}{x^2 - 9} \)
13. \( y = \frac{x}{x^2 + 9} \)
14. \( y = \frac{x}{x^2 + 9} \)
15. \( y = \frac{x - 1}{x^2} \)
16. \( y = 1 + \frac{1}{x} + \frac{1}{x^2} \)
17. \( y = \frac{x^2}{x^2 + 3} \)
18. \( y = \frac{x}{x^2 - 1} \)
19. \( y = x\sqrt{x} - x \)
20. \( y = 2\sqrt{x} - x \)
21. \( y = \sqrt{x^2 + x - 2} \)
22. \( y = \sqrt{x^2 + x} - x \)
23. \( y = \frac{x}{\sqrt{x^2 + 1}} \)
24. \( y = x\sqrt{2 - x^2} \)
25. \( y = \sqrt{1 - x^2} \)
26. \( y = \frac{x}{\sqrt{x^2 - 1}} \)
27. \( y = x - 3x^{1/3} \)
28. \( y = x^{5/3} - 5x^{1/3} \)
29. \( y = \sqrt{x^2 - 1} \)
30. \( y = \sqrt{x^2 + 1} \)
31. \( y = 3\sin x - \sin^3 x \)
32. \( y = x + \cos x \)
33. \( y = x \tan x, \quad -\pi/2 < x < \pi/2 \)
34. \( y = 2x - \tan x, \quad -\pi/2 < x < \pi/2 \)
35. \( y = \frac{1}{2}x - \sin x, \quad 0 < x < 3\pi \)
36. \( y = \sec x + \tan x, \quad 0 < x < \pi/2 \)
37. \( y = \frac{\sin x}{1 + \cos x} \)
38. \( y = \frac{\sin x}{2 + \cos x} \)
39. \( y = e^{\sin x} \)
40. \( y = e^{-x} \sin x, \quad 0 \leq x \leq 2\pi \)
41. \( y = \frac{1}{(1 + e^{-x})} \)
42. \( y = e^{2x} - e^x \)
43. \( y = x - \ln x \)
44. \( y = e^{x}/x \)
45. \( y = (1 + e^{-x})^{-2} \)
46. \( y = \ln(x^2 - 3x + 2) \)
47. \( y = \ln(x) \)
48. \( y = \frac{\ln x}{x^2} \)
49. \( y = xe^{-x} \)
50. \( y = (x^2 - 3)e^{-x} \)
51. \( y = e^{1/x} + e^{-2x} \)
52. \( y = \tan^{-1}\left(\frac{x - 1}{x + 1}\right) \)

53. In the theory of relativity, the mass of a particle is

\[ m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \]

where \( m_0 \) is the rest mass of the particle, \( m \) is the mass when the particle moves with speed \( v \) relative to the observer, and \( c \) is the speed of light. Sketch the graph of \( m \) as a function of \( v \).

54. In the theory of relativity, the energy of a particle is

\[ E = \sqrt{m_0^2 c^4 + h^2 \omega^2 / \lambda^2} \]

where \( m_0 \) is the rest mass of the particle, \( \lambda \) is its wavelength, and \( h \) is Planck's constant. Sketch the graph of \( E \) as a function of \( \lambda \). What does the graph say about the energy?

55. The figure shows a beam of length \( L \) embedded in concrete walls. If a constant load \( W \) is distributed evenly along its length, the beam takes the shape of the deflection curve

\[ y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2 \]

where \( E \) and \( I \) are positive constants. (\( E \) is Young's modulus of elasticity and \( I \) is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.

56. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of their charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position \( x \) between them. It follows from Coulomb's Law that the net force acting on the middle particle

\[ F(x) = -\frac{k}{x^2} + \frac{k}{(x - 2)^2} \quad 0 < x < 2 \]

where \( k \) is a constant positive. Sketch the graph of the net function. What does the graph say about the force?
4.7 Exercises

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.
   
<table>
<thead>
<tr>
<th>First number</th>
<th>Second number</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>42</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

   (b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m² whose perimeter is as small as possible.

7. A model used for the yield $Y$ of an agricultural crop as a function of the nitrogen level $N$ in the soil (measured in appropriate units) is
   
   $Y = \frac{kN}{1 + N^2}$
   
   where $k$ is a positive constant. What nitrogen level gives the best yield?

8. The rate (in mg carbon/m²/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function
   
   $P = \frac{100I}{I^2 + I + 4}$
   
   where $I$ is the light intensity (measured in thousands of foot-candles). For what light intensity is $P$ a maximum?

9. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
   (a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the total area.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the total area as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

10. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
   (a) Draw several diagrams to illustrate the situation, some with boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
   (c) Write an expression for the volume.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the volume as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

11. A farmer wants to fence an area of 1.5 million square feet in rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

12. A box with a square base and open top must have a volume of 32,000 cm³. Find the dimensions of the box that minimize the amount of material used.

13. If 1200 cm² of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

14. A rectangular storage container with an open top is to have volume of 10 m³. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of material for the cheapest such container.

15. Do Exercise 14 assuming the container has a lid that is made from the same material as the sides.

16. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
   (b) Show that of all the rectangles with a given perimeter, one with greatest area is a square.

17. Find the point on the line $y = 4x + 7$ that is closest to the origin.

18. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

19. Find the points on the ellipse $x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$. 

20. Find, correct to two decimal places, the coordinates of the point on the curve \( y = \tan x \) that is closest to the point \((1, 1)\).

21. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius \( r \).

22. Find the area of the largest rectangle that can be inscribed in the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

23. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side \( L \) if one side of the rectangle lies on the base of the triangle.

24. Find the dimensions of the rectangle of largest area that has its base on the \( x \)-axis and its other two vertices above the \( x \)-axis and lying on the parabola \( y = 8 - x^2 \).

25. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius \( r \).

26. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

27. A right circular cylinder is inscribed in a sphere of radius \( r \). Find the largest possible volume of such a cylinder.

28. A right circular cylinder is inscribed in a cone with height \( h \) and base radius \( r \). Find the largest possible volume of such a cylinder.

29. A right circular cylinder is inscribed in a sphere of radius \( r \). Find the largest possible surface area of such a cylinder.

30. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 56 on page 23.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

31. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm\(^2\), find the dimensions of the poster with the smallest area.

32. A poster is to have an area of 180 in\(^2\) with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

33. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

34. Answer Exercise 33 if one piece is bent into a square and the other into a circle.

35. A cylindrical can without a top is made to contain \( V \) cm\(^3\) of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

36. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest lader that will reach from the ground over the fence to the wall of the building?

37. A cone-shaped drinking cup is made from a circular piece of paper of radius \( R \) by cutting out a sector and joining the edges \( CA \) and \( CB \). Find the maximum capacity of such a cup.

38. A cone-shaped paper drinking cup is to be made to hold 27 cm\(^3\) of water. Find the height and radius of the cup that will use the smallest amount of paper.

39. A cone with height \( h \) is inscribed in a larger cone with height \( H \) so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when \( h = \frac{1}{3} H \).

40. An object with weight \( W \) is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle \( \theta \) with a plane, then the magnitude of the force is

\[
F = \frac{\mu W}{\mu \sin \theta + \cos \theta}
\]

where \( \mu \) is a constant called the coefficient of friction. For what value of \( \theta \) is \( F \) smallest?

41. If a resistor of \( R \) ohms is connected across a battery of \( E \) volts with internal resistance \( r \) ohms, then the power (in watts) in the external resistor is

\[
P = \frac{E^2R}{(R + r)^2}
\]

If \( E \) and \( r \) are fixed but \( R \) varies, what is the maximum value of the power?

42. For a fish swimming at a speed \( v \) relative to the water, the energy expenditure per unit time is proportional to \( v^3 \). It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current \( u \) (\( u < v \)), then the time required to swim a distance \( L \) is \( L/(v - u) \) and the total energy \( E \) required to swim the distance is given by

\[
E(v) = av^3 \cdot \frac{L}{v - u}
\]

where \( a \) is the proportionality constant.

(a) Determine the value of \( v \) that minimizes \( E \).

(b) Sketch the graph of \( E \).

Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.
57. A manufacturer has been selling 1000 television sets a week at $450 each. A market survey indicates that for each $10 rebate offered to the buyer, the number of sets sold will increase by 100 per week. 
(a) Find the demand function.
(b) How large a rebate should the company offer the buyer in order to maximize its revenue?
(c) If its weekly cost function is \( C(x) = 68,000 + 150x \), how should the manufacturer set the size of the rebate in order to maximize its profit?

58. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is $800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each $10 increase in rent. What rent should the manager charge to maximize revenue?

59. Show that if all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

60. The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?

61. A point \( P \) needs to be located somewhere on the line \( AD \) so that the total length \( L \) of cables linking \( P \) to the points \( A, B, \) and \( C \) is minimized (see the figure). Express \( L \) as a function of \( x = |AP| \) and use the graphs of \( L \) and \( dL/dx \) to estimate the minimum value.

63. Let \( v_1 \) be the velocity of light in air and \( v_2 \) the velocity of light in water. According to Fermat’s Principle, a ray of light will travel from a point \( A \) in the air to a point \( B \) in the water by a path \( ACB \) that minimizes the time taken. Show that
\[
\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}
\]
where \( \theta_1 \) (the angle of incidence) and \( \theta_2 \) (the angle of refraction) are as shown. This equation is known as Snell’s Law.

64. Two vertical poles \( PQ \) and \( ST \) are secured by a rope \( PRS \) going from the top of the first pole to a point \( R \) on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when \( \theta_1 = \theta_2 \).

65. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose \( x \) to minimize \( y \)?