The inverse trigonometric functions that occur most frequently are the ones that we have just discussed. The derivatives of the remaining four are given in the following table. The proofs of the formulas are left as exercises.

**DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS**

\[
\begin{align*}
\frac{d}{dx} (\sin^{-1}x) &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx} (\csc^{-1}x) &= -\frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx} (\cos^{-1}x) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx} (\sec^{-1}x) &= \frac{1}{x\sqrt{x^2-1}} \\
\frac{d}{dx} (\tan^{-1}x) &= \frac{1}{1+x^2} \\
\frac{d}{dx} (\cot^{-1}x) &= -\frac{1}{1+x^2}
\end{align*}
\]

### 3.5 EXERCISES

1-4
(a) Find \( y' \) by implicit differentiation.

(b) Solve the equation explicitly for \( y \) and differentiate to get \( y' \) in terms of \( x \).

(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for \( y \) into your solution for part (a).

1. \( xy + 2x + 3x^2 = 4 \)
2. \( 4x^2 + 9y^2 = 36 \)
3. \( \frac{1}{x} + \frac{1}{y} = 1 \)
4. \( \cos x + \sqrt{y} = 5 \)

5-20 Find \( dy/dx \) by implicit differentiation.

5. \( x^2 + y^3 = 1 \)
6. \( 2\sqrt{x} + \sqrt{y} = 3 \)
7. \( x^2 + xy - y^2 = 4 \)
8. \( 2x^2 + x^2y - xy^3 = 2 \)
9. \( (x+y) = y^2(3x-y) \)
10. \( x^3 + x^2y + 1 + ye^y \)
11. \( x^3y^2 + x \sin y = 4 \)
12. \( 1 + x = \sin(xy^3) \)
13. \( 4 \cos x \sin y = 1 \)
14. \( y \sin(x^2) = y \sin(y^2) \)
15. \( e^{xy} = x - y \)
16. \( \sqrt[3]{x+y} = 1 + x^3y^2 \)
17. \( \sqrt[3]{xy} = 1 + x^2y \)
18. \( \tan(x-y) = \frac{y}{1+xy} \)
19. \( e^{-\cos x} = 1 + \sin(xy) \)
20. \( \sin x + \cos y = \sin x \cos y \)

21. If \( f(x) + f^2(x) = 10 \) and \( f(1) = 2 \), find \( f'(1) \).

22. If \( g(x) + x \sin g(x) = x^2 \), find \( g'(0) \).

23-24 Regard \( y \) as the independent variable and \( x \) as the dependent variable and use implicit differentiation to find \( dx/dy \).

23. \( x^3y^2 - x^3y + 2xy^3 = 0 \)
24. \( y \sec x = x \tan y \)

25-30 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

25. \( x^2 + xy + y^2 = 3 \), \( (1, 1) \) (ellipse)
26. \( x^2 + 2xy - y^2 + x = 2 \), \( (1, 2) \) (hyperbola)
27. \( x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \), \( (0, \frac{1}{2}) \) (cardiod)
28. \( x^{2/3} + y^{2/3} = 4 \), \( (-3\sqrt{3}, 1) \) (astroid)
29. \( 2(x^2 + y^2)^2 = 25(x^2 - y^2) \), \( (3, 1) \) (lemniscate)
30. \( y^2(y^2 - 4) = x^2(x^2 - 5) \), \( (0, -2) \) (devil’s curve)

31. (a) The curve with equation \( y^2 = 5x^4 - x^2 \) is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point \( (1, 2) \).

(b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If
not, you can still graph this curve by graphing its upper
and lower halves separately.)

32. (a) The curve with equation \( y^2 = x^3 + 3x^2 \) is called the
Tschirnhausen cubic. Find an equation of the tangent line to this curve at the point \((1, -2)\).

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the
tangent lines on a common screen.

33–36 Find \( y' \) by implicit differentiation.

33. \( 9x^2 + y^2 = 9 \)  
34. \( \sqrt{x} + \sqrt{y} = 1 \)
35. \( x^3 + y^3 = 1 \)  
36. \( x^4 + y^4 = a^4 \)

37. Fanciful shapes can be created by using the implicit plotting
capabilities of computer algebra systems.
(a) Graph the curve with equation
\[
y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)
\]
At how many points does this curve have horizontal
tangents? Estimate the x-coordinates of these points.

(b) Find equations of the tangent lines at the points \((0, 1)\)
and \((0, 2)\).

(c) Find the exact x-coordinates of the points in part (a).

(d) Create even more fanciful curves by modifying the equation
in part (a).

38. (a) The curve with equation
\[
2y^3 + y^2 - y^3 = x^4 - 2x^3 + x^2
\]
has been likened to a bouncing wagon. Use a computer
algebra system to graph this curve and discover why.

(b) At how many points does this curve have horizontal
tangent lines? Find the x-coordinates of these points.

39. Find the points on the lemniscate in Exercise 29 where the
tangent is horizontal.

40. Show by implicit differentiation that the tangent to the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
at the point \((x_0, y_0)\) is
\[
\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1
\]

41. Find an equation of the tangent line to the hyperbola
\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]
at the point \((x_0, y_0)\).

42. Show that the sum of the \(x\)- and \(y\)-intercepts of any tangent
line to the curve \( \sqrt{x} + \sqrt{y} = \sqrt{c} \) is equal to \(c\).

43. Show, using implicit differentiation, that any tangent line at
a point \(P\) to a circle with center \(O\) is perpendicular to the
radius \(OP\).

44. The Power Rule can be proved using implicit differentiation
for the case where \(n\) is a rational number, \(n = p/q\), and
\( y = f(x) = x^n \) is assumed beforehand to be a differentia-
function. If \( y = x^{p/q} \), then \( y' = x^{p/q - 1} \).
Use implicit differen-
tiation to show that
\[
y' = \frac{p}{q} x^{p/q - 1}
\]

45–54 Find the derivative of the function. Simplify where possible.

45. \( y = \tan^{-1} \sqrt{x} \)
46. \( y = x^2 - 1 \) sec \( x \)
47. \( y = \sin^{-1}(2x + 1) \)
48. \( g(x) = \sqrt{x^2 - 1} \) se
49. \( G(x) = \sqrt{1 - x^2} \) arccos \( x \)
50. \( y = \tan^{-1}(x - \sqrt{1} \) tan \( x \)
51. \( h(t) = \cot^{-1}(t) + \cot^{-1}(1/i) \)
52. \( F(\theta) = \arcsin \sqrt{\sin \theta} \)
53. \( y = \arccos^{-1}(2x^2) \)
54. \( y = \arctan \sqrt{1 - \frac{1}{1 + x^2}} \)

55–56 Find \( f'(x) \). Check that your answer is reasonable by
paring the graphs of \( f \) and \( f' \).

55. \( f(x) = \sqrt{1 - x^2} \) arccos \( x \)
56. \( f(x) = \arctan(x^2 - \)

57. Prove the formula for \((d/dx)(\sec^{-1}x)\) by the same meth-
for \((d/dx)(\sin^{-1}x)\).

58. (a) One way of defining \( \sec^{-1}x \) is to say that \( y = \sec^{-1}x \)
sec \( y = x \) and \( 0 \leq y < \pi/2 \) or \( \pi < y < 2\pi/2 \).
Show that, with this definition,
\[
\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x \sqrt{x^2 - 1}}
\]
(b) Another way of defining \( \sec^{-1}x \) that is sometimes
\( \cos^{-1}x \) to say that \( y = \sec^{-1}x \leftrightarrow \sec y = x \) and \( 0 \leq y \)
\( y \neq 0 \). Show that, with this definition,
\[
\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x| \sqrt{x^2 - 1}}
\]

59–62 Two curves are orthogonal if their tangent lines are
pendicular at each point of intersection. Show that the giver
lies of curves are orthogonal trajectories of each other, the
ey every curve in one family is orthogonal to every curve in the
other family. Sketch both families of curves on the same ax

59. \( x^2 + y^2 = r^2, \quad ax + by = 0 \)
60. \( x^2 + y^2 = ax, \quad x^2 + y^2 = by \)
61. \( y = cx^2, \quad x^2 + 2y^2 = k \)
62. \( y = ax^3, \quad x^2 + 3y^2 = b \)

63. The equation \( x^2 - xy + y^2 = 3 \) represents a "rotated
ellipse," that is, an ellipse whose axes are not parallel to
coordinate axes. Find the points at which this ellipse ce
urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 17 in Section 3.8).

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance \( P(t) \) of someone learning a skill as a function of the training time \( t \). Of particular interest is the rate at which performance improves as time passes, that is, \( dp/dt \).

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If \( p(t) \) denotes the proportion of a population that knows a rumor by time \( t \), then the derivative \( dp/dt \) represents the rate of spread of the rumor (see Exercise 82 in Section 3.4).

**A SINGLE IDEA, MANY INTERPRETATIONS**

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

### 3.7 EXERCISES

1–4 A particle moves according to a law of motion \( s = f(t) \), \( t \geq 0 \), where \( t \) is measured in seconds and \( s \) in feet.

(a) Find the velocity at time \( t \).

(b) What is the velocity after 3 s?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled during the first 8 s.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time \( t \) and after 3 s.

(h) Graph the position, velocity, and acceleration functions for \( 0 \leq t \leq 8 \).

(i) When is the particle speeding up? When is it slowing down?

1. \( f(t) = t^3 - 12t^2 + 36t \)

2. \( f(t) = 0.01t^4 - 0.04t^3 \)

3. \( f(t) = \cos(\pi t/4), \quad t \leq 10 \)

4. \( f(t) = te^{-t/2} \)

5. Graphs of the velocity functions of two particles are shown, where \( t \) is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.

6. Graphs of the position functions of two particles are shown, where \( t \) is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.

7. The position function of a particle is given by \( s = t^3 - 4.5t^2 - 7t, \quad t \geq 0 \).

(a) When does the particle reach a velocity of 5 m/s?
8. If a ball is given a push so that it has an initial velocity of 5 m/s down a certain inclined plane, then the distance it has rolled after $t$ seconds is $s = 5t + 3t^2$.
(a) Find the velocity after 2 s.
(b) How long does it take for the velocity to reach 35 m/s?

9. If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, its height (in meters) after $t$ seconds is $h = 10t - 0.83t^2$.
(a) What is the velocity of the stone after 3 s?
(b) What is the velocity of the stone after it has risen 25 m?

10. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after $t$ seconds is $s = 80t - 16t^2$.
(a) What is the maximum height reached by the ball?
(b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

11. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length $x$ changes. Find $A'(15)$ and explain its meaning in this situation.
(b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length $x$ is increased by an amount $\Delta x$. How can you approximate the resulting change in area $\Delta A$ if $\Delta x$ is small?

12. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If $V$ is the volume of such a cube with side length $x$, calculate $dV/dx$ when $x = 3$ mm and explain its meaning.
(b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).

13. (a) Find the average rate of change of the area of a circle with respect to its radius $r$ as $r$ changes from
(i) 2 to 3  
(ii) 2 to 2.5  
(iii) 2 to 2.1
(b) Find the instantaneous rate of change when $r = 2$.
(c) Show that the rate of change of the area of a circle with respect to its radius (at any $r$) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount $\Delta r$. How can you approximate the resulting change in area $\Delta A$ if $\Delta r$ is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

15. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius $r$ when $r$ is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

16. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius $r$ is measured in micrometers (1 $\mu$m = $10^{-6}$ m). Find the average rate of change of $V$ with respect to $r$ when $r$ changes from (i) 5 to 8 $\mu$m  
(ii) 5 to 6 $\mu$m  
(iii) 5 to 5.1 $\mu$m
(b) Find the instantaneous rate of change of $V$ with respect to $r$ when $r = 5$ $\mu$m.
(c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

17. The mass of the part of a metal rod that lies between its left end and a point $x$ meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when $x$ is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

18. If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli’s Law gives the volume $V$ of water remaining in the tank after $t$ minutes as

$$V = 5000 \left(1 - \frac{t}{40}\right)^2 \quad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

19. The quantity of charge $Q$ in coulombs (C) that has passed through a point in a wire up to time $t$ (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. [See Example 3. The unit of current is an ampere (1 A = 1 C/s).] At what time is the current lowest?

20. Newton’s Law of Gravitation says that the magnitude $F$ of the force exerted by a body of mass $m$ on a body of mass $M$ is

$$F = \frac{GmM}{r^2}$$

where $G$ is the gravitational constant and $r$ is the distance between the bodies.
(a) Find $dF/dr$ and explain its meaning. What does the minus sign indicate?
(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

21. Boyle’s Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.
(a) Find the rate of change of volume with respect to pressure.
(b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.

(c) Prove that the isothermal compressibility (see Example 5) is given by \( \beta = 1/P \).

22. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value \( [A] = [B] = a \) moles/L, then

\[
[C] = a^2k/(akt + 1)
\]

where \( k \) is a constant.

(a) Find the rate of reaction at time \( t \).
(b) Show that if \( x = [C] \), then

\[
\frac{dx}{dt} = k(a - x)^2
\]

(c) What happens to the concentration as \( t \to \infty \)?
(d) What happens to the rate of reaction as \( t \to \infty \)?
(e) What do the results of parts (c) and (d) mean in practical terms?

23. In Example 6 we considered a bacteria population that doubles every hour. Suppose that another population of bacteria triples every hour and starts with 400 bacteria. Find an expression for the number \( n \) of bacteria after \( t \) hours and use it to estimate the rate of growth of the bacteria population after 2.5 hours.

24. The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

\[
n = f(t) = \frac{a}{1 + be^{-0.5}}
\]

where \( t \) is measured in hours. At time \( t = 0 \) the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of \( a \) and \( b \). According to this model, what happens to the yeast population in the long run?

25. The table gives the population of the world in the 20th century.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>1650</td>
<td>1960</td>
<td>3040</td>
</tr>
<tr>
<td>1910</td>
<td>1750</td>
<td>1970</td>
<td>3710</td>
</tr>
<tr>
<td>1920</td>
<td>1860</td>
<td>1980</td>
<td>4450</td>
</tr>
<tr>
<td>1930</td>
<td>2070</td>
<td>1990</td>
<td>5280</td>
</tr>
<tr>
<td>1940</td>
<td>2300</td>
<td>2000</td>
<td>6080</td>
</tr>
<tr>
<td>1950</td>
<td>2560</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
(b) Use a graphing calculator or computer to find a cubic function (a third-degree polynomial) that models the data.

(c) Use your model in part (b) to find a model for the rate of population growth in the 20th century.
(d) Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).
(e) Estimate the rate of growth in 1985.

26. The table shows how the average age of first marriage of Japanese women varied in the last half of the 20th century.

<table>
<thead>
<tr>
<th>Year</th>
<th>Age (t)</th>
<th>Age (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>23.0</td>
<td>1980</td>
</tr>
<tr>
<td>1955</td>
<td>23.8</td>
<td>1985</td>
</tr>
<tr>
<td>1960</td>
<td>24.4</td>
<td>1990</td>
</tr>
<tr>
<td>1965</td>
<td>24.5</td>
<td>1995</td>
</tr>
<tr>
<td>1970</td>
<td>24.2</td>
<td>2000</td>
</tr>
<tr>
<td>1975</td>
<td>24.7</td>
<td></td>
</tr>
</tbody>
</table>

(a) Use a graphing calculator or computer to model these data with a fourth-degree polynomial.
(b) Use part (a) to find a model for \( A'(t) \).
(c) Estimate the rate of change of marriage age for women in 1990.
(d) Graph the data points and the models for \( A \) and \( A' \).

27. Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm², and viscosity \( \eta = 0.027 \).

(a) Find the velocity of the blood along the centerline \( r = 0 \), at radius \( r = 0.005 \) cm, and at the wall \( r = R = 0.01 \) cm.
(b) Find the velocity gradient at \( r = 0 \), \( r = 0.005 \), and \( r = 0.01 \).
(c) Where is the velocity the greatest? Where is the velocity changing most?

28. The frequency of vibrations of a vibrating violin string is given by

\[
f = \frac{1}{2L} \sqrt{\frac{T}{\rho}}
\]

where \( L \) is the length of the string, \( T \) is its tension, and \( \rho \) is its linear density. [See Chapter 11 in D. E. Hall, Musical Acoustics, 3d ed. (Pacific Grove, CA: Brooks/Cole, 2002).]

(a) Find the rate of change of the frequency with respect to
(i) the length (when \( T \) and \( \rho \) are constant),
(ii) the tension (when \( L \) and \( \rho \) are constant), and
(iii) the linear density (when \( L \) and \( T \) are constant).

(b) The pitch of a note (how high or low the note sounds) is determined by the frequency \( f \). (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
(i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
(ii) when the tension is increased by turning a tuning peg,
(iii) when the linear density is increased by switching to another string.
19. The cost, in dollars, of producing $x$ yards of a certain fabric is
\[ C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3 \]
(a) Find the marginal cost function.
(b) Find $C'(200)$ and explain its meaning. What does it predict?
(c) Compare $C'(200)$ with the cost of manufacturing the 201st yard of fabric.

30. The cost function for production of a commodity is
\[ C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3 \]
(a) Find and interpret $C'(100)$.
(b) Compare $C'(100)$ with the cost of producing the 101st item.

31. If $p(x)$ is the total value of the production when there are $x$ workers in a plant, then the average productivity of the workforce at the plant is
\[ A(x) = \frac{p(x)}{x} \]
(a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
(b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

32. If $R$ denotes the reaction of the body to some stimulus of strength $x$, the sensitivity $S$ is defined to be the rate of change of the reaction with respect to $x$. A particular example is that when the brightness $x$ of a light source is increased, the eye reacts by decreasing the area $R$ of the pupil. The experimental formula
\[ R = \frac{40 + 24x^4}{1 + 4x^2} \]
has been used to model the dependence of $R$ on $x$ when $R$ is measured in square millimeters and $x$ is measured in appropriate units of brightness.
(a) Find the sensitivity.
(b) Illustrate part (a) by graphing both $R$ and $S$ as functions of $x$. Comment on the values of $R$ and $S$ at low levels of brightness. Is this what you would expect?

33. The gas law for an ideal gas at absolute temperature $T$ (in kelvins), pressure $P$ (in atmospheres), and volume $V$ (in liters) is $PV = nRT$, where $n$ is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of $0.10$ atm/min and $V = 10$ L and is decreasing at a rate of $0.15$ L/min. Find the rate of change of $T$ with respect to time at that instant if $n = 10$ mol.

34. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation
\[ \frac{dP}{dt} = r_0 \left( 1 - \frac{P(t)}{P_c} \right) P(t) - \beta P(t) \]
where $r_0$ is the birth rate of the fish, $P_c$ is the maximum population that the pond can sustain (called the carrying capacity), and $\beta$ is the percentage of the population that is harvested.
(a) What value of $dP/dt$ corresponds to a stable population?
(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
(c) What happens if $\beta$ is raised to 5%?

35. In the study of ecosystems, predator-prey models are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations
\[ \frac{dC}{dt} = ac - bCW \]
\[ \frac{dW}{dt} = -cW + dCW \]
(a) What values of $dC/dt$ and $dW/dt$ correspond to stable populations?
(b) How would the statement "The caribou go extinct" be represented mathematically?
(c) Suppose that $a = 0.05, b = 0.001, c = 0.05$, and $d = 0.0001$. Find all population pairs $(C, W)$ that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

## 3.8 EXPONENTIAL GROWTH AND DECAY

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time $t$, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant $k$. Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the
You can see that the interest paid increases as the number of compounding periods \((n)\) increases. If we let \(n \to \infty\), then we will be compounding the interest continuously and the value of the investment will be

\[
A(t) = \lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n \to \infty} \left[ \left(1 + \frac{r}{n}\right)^{n/r} \right]^{rt} = A_0 \left[ \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{n/r} \right]^{rt} = A_0 \left[ \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m} \right]^{rt} (\text{where } m = n/r)
\]

But the limit in this expression is equal to the number \(e\). (See Equation 3.6.6). So with continuous compounding of interest at interest rate \(r\), the amount after \(t\) years is

\[A(t) = A_0 e^{rt}\]

If we differentiate this function, we get

\[
\frac{dA}{dt} = rA_0 e^{rt} = rA(t)
\]

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Returning to the example of $1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

\[A(3) = 1000e^{0.06 \times 3} = 1197.22\]

Notice how close this is to the amount we calculated for daily compounding, $1197.20. But the amount is easier to compute if we use continuous compounding.

### 3.8 EXERCISES

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

2. A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.
   (a) Find the relative growth rate.
   (b) Find an expression for the number of cells after \(r\) hours.
   (c) Find the number of cells after 8 hours.
   (d) Find the rate of growth after 8 hours.
   (e) When will the population reach 20,000 cells?

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.
   (a) Find an expression for the number of bacteria after \(r\) hours.
   (b) Find the number of bacteria after 3 hours.
   (c) Find the rate of growth after 3 hours.
   (d) When will the population reach 10,000?

4. A bacteria culture grows with constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.
   (a) Find the initial population.
   (b) Find an expression for the population after \(t\) hours.
(c) Find the number of cells after 5 hours.
(d) Find the rate of growth after 5 hours.
(e) When will the population reach 200,000?

5. The table gives estimates of the world population, in millions, from 1750 to 2000:

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1750</td>
<td>790</td>
<td>1900</td>
<td>1650</td>
</tr>
<tr>
<td>1800</td>
<td>980</td>
<td>1950</td>
<td>2560</td>
</tr>
<tr>
<td>1850</td>
<td>1260</td>
<td>2000</td>
<td>6080</td>
</tr>
</tbody>
</table>

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

6. The table gives the population of the United States, in millions, for the years 1900–2000:

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>76</td>
<td>1960</td>
<td>179</td>
</tr>
<tr>
<td>1910</td>
<td>92</td>
<td>1970</td>
<td>203</td>
</tr>
<tr>
<td>1920</td>
<td>106</td>
<td>1980</td>
<td>227</td>
</tr>
<tr>
<td>1930</td>
<td>123</td>
<td>1990</td>
<td>250</td>
</tr>
<tr>
<td>1940</td>
<td>131</td>
<td>2000</td>
<td>275</td>
</tr>
</tbody>
</table>

(a) Use the exponential model and the census figures for 1900 and 1910 to predict the population in 2000. Compare with the actual figure and try to explain the discrepancy.
(b) Use the exponential model and the census figures for 1980 and 1990 to predict the population in 2000. Compare with the actual population. Then use this model to predict the population in the years 2010 and 2020.
(c) Graph both of the exponential functions in parts (a) and (b) together with a plot of the actual population. Are these models reasonable ones?

7. Experiments show that if the chemical reaction

$$\text{N}_2\text{O}_3 \rightarrow 2\text{NO}_2 + \frac{1}{2}\text{O}_2$$

takes place at $45^\circ\text{C}$, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[\text{N}_2\text{O}_3]}{dt} = 0.0005[\text{N}_2\text{O}_3]$$

(a) Find an expression for the concentration $[\text{N}_2\text{O}_3]$ after $t$ seconds if the initial concentration is $C$.

(b) How long will the reaction take to reduce the concentration of $\text{N}_2\text{O}_3$ to 90% of its original value?

8. Bismuth-210 has a half-life of 5.0 days.
(a) A sample originally has a mass of 800 mg. Find a formula for the mass remaining after $t$ days.
(b) Find the mass remaining after 30 days.
(c) When is the mass reduced to 1 mg?
(d) Sketch the graph of the mass function.

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
(a) Find the mass that remains after $t$ years.
(b) How much of the sample remains after 100 years?
(c) After how long will only 1 mg remain?

10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
(a) What is the half-life of tritium-3?
(b) How long would it take the sample to decay to 20% of its original amount?

11. Scientists can determine the age of ancient objects by the method of radiocarbon dating. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, $^{14}\text{C}$, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates $^{14}\text{C}$ through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of $^{14}\text{C}$ begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 74% as much $^{14}\text{C}$ radioactivity as does plant material on the earthen today. Estimate the age of the parchment.

12. A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point $P$ is twice the $y$-coordinate of $P$. What is the equation of the curve?

13. A roast turkey is taken from an oven when its temperature has reached $185^\circ\text{F}$ and is placed on a table in a room where the temperature is $75^\circ\text{F}$.
(a) If the temperature of the turkey is $150^\circ\text{F}$ after half an hour, what is the temperature after 45 minutes?
(b) When will the turkey have cooled to $100^\circ\text{F}$?

14. A thermometer is taken from a room where the temperature is $20^\circ\text{C}$ to the outdoors, where the temperature is $5^\circ\text{C}$. After one minute the thermometer reads $12^\circ\text{C}$.
(a) What will the reading on the thermometer be after one more minute?
(b) When will the thermometer read $6^\circ\text{C}$?

15. When a cold drink is taken from a refrigerator, its temperature is $5^\circ\text{C}$. After 25 minutes in a $20^\circ\text{C}$ room its temperature has increased to $10^\circ\text{C}$.
(a) What is the temperature of the drink after 50 minutes?
(b) When will its temperature be $15^\circ\text{C}$?