MSRI TALK, APRIL 10, 2014

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1. Isogenies

Formal group G/A:

$$\mathcal{O}_G \approx A[\![x]\!], \qquad x \mapsto F(x_1, x_2)\colon \mathcal{O}_G \to \mathcal{O}_G \widehat{\otimes}_A \mathcal{O}_G.$$

Isogeny: $f\colon G \to G'$ such that $f^*\colon \mathcal{O}_{G'} \to \mathcal{O}_G$ is finite locally free.

$\Longrightarrow K = \mathrm{Ker}(f)$, $\mathcal{O}_K = \mathcal{O}_G \otimes_{\mathcal{O}_{G'}} A$ is finite locally free over A. $\deg(f) = \mathrm{rank}_A \mathcal{O}_K$.

Example? $\widehat{\mathbb{G}}_m/\mathbb{Z}$.

$[p]\colon \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$, $[p]^*(x) = px + \cdots + x^p$.

Not an isogeny over $\mathbb{Z}$. Over $\mathbb{Q}$, isogeny of degree 1 (isomorphism). Over $\mathbb{Z}_p$, isogeny of degree p.

Frobenius isogeny. $\mathbb{F}_p \subseteq A$, any G/A,

$$F^r\colon G \to (\phi^r)^* G, \qquad x \mapsto x^{p^r},$$

degree p^r. ($\phi\colon A \to A$, $\phi(a) = a^p$.)

Over field $k \supseteq \mathbb{F}_p$**.** Unique factorization ($\deg f = p^r$).

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & G' \\ & {\scriptstyle F^r}\searrow \qquad \nearrow{\scriptstyle g}\,{\scriptstyle \sim} & \\ & (\phi^r)^* G & \end{array}$$

2. Deformations

Fix G_0/k: k perfect char p, $[p]_{G_0}$ isogeny of degree p^n. (Height n formal group.)

Deformation structures. Given G/A, $A =$ complete local ring.

$$\mathcal{D}(G/A) = \{\, (i, \alpha) \mid i\colon k \to A/\mathfrak{m},\ \ \alpha\colon i^* G_0 \xrightarrow{\sim} G_{A/\mathfrak{m}} \,\}.$$

Isogeny $f\colon G \to G'$ over $A \Longrightarrow f_*\colon \mathcal{D}(G/A) \to \mathcal{D}(G'/A)$:

$$\begin{array}{ccc} i^* G_0 & \xrightarrow{F^r} & (\phi^r)^* i^* G_0 \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle \exists!\, \alpha'} \\ G_{A/\mathfrak{m}} & \xrightarrow[f_{A/\mathfrak{m}}]{} & G'_{A/\mathfrak{m}} \end{array}$$

$f_*((i, \alpha)) = (i\phi^r, \alpha')$.

Exercise. For $\mathbb{F}_p \subseteq A$:

$$F_* = \phi^*\colon \mathcal{D}(G/A) \to \mathcal{D}(\phi^* G/A).$$

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Pile of deformation structures. Def $=$ Def$_{G_0}$. A complete local ring $\Longrightarrow$

$$\mathrm{Def}(A) := \begin{cases} \textbf{obj}: \ (G/A, \ \ (i,\alpha)\in\mathcal{D}(G/A)), \\ \textbf{mor}: \ f\colon G\to G', \ \ f_*(i,\alpha)=(i',\alpha'). \end{cases}$$

Local homomorphism $g\colon A\to A' \Longrightarrow g^*\colon \mathrm{Def}(A)\to\mathrm{Def}(A')$.

Def: a presheaf of categories on $\{\text{cpt loc rings}\}^{\mathrm{op}}$. "Pile".

Quasi-coherent sheaves on Def. Objects of QCoh(Def) are $(\{M_A\},\{M_g\})$:

$$A \quad\rightsquigarrow\quad M_A\colon \mathrm{Def}(A)^{\mathrm{op}}\to\mathrm{Mod}_A,$$
$$g\colon A\to A' \quad\rightsquigarrow\quad M_g\colon A'\otimes_A M_A \xrightarrow{\sim} M_{A'}\circ g^*.$$

Coherence, etc.

Example. $\omega\in\mathrm{QCoh}(\mathrm{Def})$.

$$\omega_A(G/A) := \{\text{invt 1-forms on } G\},$$

(rank 1 A-module). Forms pullback along homomorphisms.

Example. $\deg\in\mathrm{QCoh}(\mathrm{Def})$.

$$\deg_A(G/A) := A, \qquad f^* = \text{mult. by } \deg(p)\in\mathbb{Z}.$$

3. Digression: elliptic curves and isogenies

Formalism works more generally.

Pile of elliptic curves and isogenies. Ell.

Replace: complete local rings $\to$ schemes, formal groups and def str $\to$ ell curves, isog preserving def str $\to$ all isogenies.

Or just isogenies of pth power degree: Ell^p.

Example. Algebraic de Rham cohomology.

$$C/S\mapsto H^k_{\mathrm{dR}}(C/S), \qquad \text{coh sheaf over } S.$$

This is a functor, so gives object $H^k_{\mathrm{dR}}\in\mathrm{QCoh}(\mathrm{Ell})$.

Hypercohomology ss (algebraic "Hodge to de Rham").

$$H^0_{\mathrm{dR}}(C/S)\approx\mathcal{O}_S,$$
$$0\to H^0(\Omega_{C/S})\to H^1_{\mathrm{dR}}(C/S)\to H^1(\mathcal{O}_{C/S})\to 0,$$

rewrite as

$$0\to\omega\to H^1_{\mathrm{dR}}(C/S)\to\omega^{-1}\otimes\deg\to 0,$$
$$H^2_{\mathrm{dR}}(C/S)\approx\deg.$$

"Hodge class" in $\mathrm{Ext}^1_{\mathrm{Ell}}(\omega^{-1}\otimes\deg,\omega)$.

Remark. For $\mathrm{Ell}^p_{\mathbb{C}}$ (elliptic curves over $\mathbb{C}$ and p-isogenies), have inclusion

$$MF_{\mathrm{weight}=2}(\Gamma_0(p))^{W=-1}\hookrightarrow \mathrm{Ext}^1_{\mathrm{Ell}^p_{\mathbb{C}}}(\omega^{-1}\otimes\deg,\omega).$$

$W=$ Atkin-Lehner involution.

Hodge class corresponds to $E_{2,p}(q)=E_2(q)-pE_2(q^p)$, where $E_2(q)=-\frac{1}{12}+\sum_{n,\,d|n} dq^n$.

Hodge class is non-trivial essentially "because" $E_2(q)$ is not a modular form. (Katz.)

Hope. We will note below that QCoh(Def) has something to do with Morava E-theory (as comm S-algebra).

Dream: QCoh(Ell) has similar relationship to elliptic cohomology, as a globally equivariant ultracommutative ring/scheme.

4. Def is representable; Morava E-theory

Fix G_0/k as before.

$\mathrm{Aut}(G/A)$ acts *freely* on deformation structures $\mathcal{D}(G/A)$.

$\Longrightarrow$ at most one iso between any two objects of $\mathrm{Def}(A)$ ($\mathrm{Def}(A)$ is "0-truncated" in Cat).

Can form $\mathrm{Def}(A)/\sim$: identify isomorphic objects. "Gaunt".

Let $\mathrm{Def}^r(A)/\sim :=$ set of morphisms of degree p^r. (If $r = 0$, these are objects.)

$$\mathrm{Def}^r(A)/\sim \longleftrightarrow \{\, (G, K) \mid K \leq G \text{ subgroup of deg } p^r \,\}.$$

4.1. **Theorem** (Lubin-Tate, Strickland)**.** *There exist complete loc rings A_r, $r \geq 0$, so*

$$\mathrm{Hom}(A_r, B) \approx \mathrm{Def}^r(B)/\sim .$$

(Local homomorphisms.) Isomorphism $A_0 \approx \mathbb{W}_p k[\![u_1, \ldots, u_{n-1}]\!]$.

$\Longrightarrow \coprod \mathrm{Spec} A_r$ is a "graded affine category scheme".

$M \in \mathrm{QCoh}(\mathrm{Def})$ are same as A-comodules:

$$(\psi_r)\colon M \to \textstyle\prod_{r\geq 0} A_r \otimes_{A_0} M \quad \text{such that } \ldots.$$

5. Morava E-theory

5.1. **Theorem** (Morava, Goerss-Hopkins-Miller, Strickland)**.** *There exists essentially unique comm S-algebra $E = E_{G_0/k}$ such that*

$$A_r[u, u^{-1}] \approx E^*(B\Sigma_{p^r})/I, \qquad |u| = 2$$

where $I =$ sum of images of transfers along all $\Sigma_i \times \Sigma_{p^r-i} \subset \Sigma_{p^r}$, $0 < i < p^r$.

In particular, $\pi_* E = A_0[u, u^{-1}]$.

6. Power operations for $K(n)$-local commutative E-algebras

$R =$ comm E-algebra: power operation

$$P_m\colon \pi_0 R \to \pi_0 R^{B\Sigma_m^+} \approx \pi_0 R \otimes_{E_0} E^0 B\Sigma_m.$$

(Iso uses R is $K(n)$-local.)

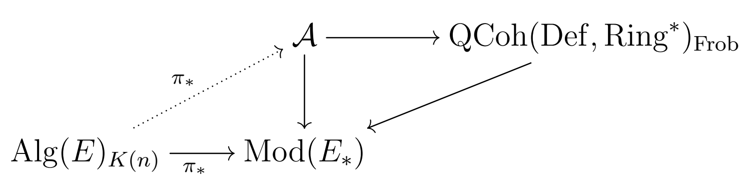
Obtain *ring homomorphims*

$$\psi_r\colon \pi_0 R \to \pi_0 R \otimes_{E_0} E^0 B\Sigma_{p^r} \to \pi_0 R \otimes_{A_0} A_r.$$

This makes $\pi_0 R$ into A-comodule. Hence, we have

$$\pi_0\colon \mathrm{Alg}(E)_{K(n)} \to \mathrm{QCoh}(\mathrm{Def}).$$

6.1. **Proposition.** *Exists $\mathcal{A} = \mathcal{A}_{G_0}$, monadic over complete E_*-modules, and lift*



Forget factors through $\mathcal{A} \to \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring}^*)_{\mathrm{Frob}}$ (graded quasicoherent sheaves of (complete) commutative rings on Def *which satisfy a "Frobenius congruence"). Restricts to equivalence*

$$\mathcal{A}^{\mathrm{tf}} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring}^*)^{\mathrm{tf}}_{\mathrm{Frob}},$$

of full subcategories of p-torsion free objects.

(Ando-Hopkins-Strickland, R., Barthel-Frankland.)

Frobenius congruence. *Skip?* $R \in \mathrm{QCoh}(\mathrm{Def}, \mathrm{Ring})$ such that for $A \supseteq \mathbb{F}_p$,

$$A^{\phi} \otimes_A R_A(G,(i,\alpha)) \xrightarrow{\sim} R_A(\phi^* G, \phi^*(i,\alpha)) = R_A(\phi^* G, F_*(i,\alpha)) \xrightarrow{F^*} R_A(G,(i,\alpha))$$

coincides with relative Frobenius on ring $R_A(G,(i,\alpha))$.

Example. $G_0 = \widehat{\mathbb{G}}_m/\mathbb{F}_p$, $E = KU_p$. All $A_r = \mathbb{Z}_p$.

$\mathcal{A} \approx$ category of p-complete $\mathbb{Z}/2$-graded θ^p-ring (Bousfield).

A θ^p-ring (non-graded) is commutative ring A with function $\theta\colon A \to A$ such that

$$\theta(0) = 0, \qquad \theta(x+y) = \theta(x) + \theta(y) - \tfrac{1}{p}\sum_{k=1}^{p-1} \binom{p}{k} x^k y^{p-k},$$
$$\theta(xy) = x^p\theta(y) + y^p\theta(x) + p\theta(x)\theta(y).$$

The map $\psi(x) := x^p + p\theta(x)$ is ring homomorphism, giving "coaction" $M \to A_1 \otimes_{A_0} M = M$.

7. Quadratic description of QCoh(Def)

Recall that QCoh(Def) are comodules for $\{A_r\}$.

7.1. **Proposition.** *The structure of comodule on is completely determined by an A_0-module M, together with A_0-module map*

$$\psi\colon M \to {}^t{A_1}^s \otimes_{A_0} M$$

such that there exists a dotted arrow A_0-module map in

$$\begin{array}{ccc} M & \xrightarrow{\psi} & {}^t{A_1}^s \otimes_{A_0} M \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\psi} \\ {}^t{A_2}^s \otimes_{A_0} M & \xrightarrow[\nabla\otimes\mathrm{id}]{} & {}^t{A_1}^s \otimes_{A_0} {}^t{A_1}^s \otimes_{A_0} M \end{array}$$

(Note $\nabla \otimes \mathrm{id}$ is always mono.)

Thus, a small amount of data $(A_1, s, t, A_2 \subset A_1 \otimes A_1)$ describes the category QCoh(Def).

7.2. *Remark. Skip?* At height 2, have $w\colon A_1 \to A_1$ ring homomorphism classifying "dual isogeny". Whence isomorphism $(A_1 \otimes_{A_0} A_1)/\nabla(A_2) \approx A_1/s(A_0)$ of A_0-bimodules, using $w \times \mathrm{id}\colon A_1 \otimes_{A_0} A_1 \to A_1$. Condition on ψ is $(w \times \psi)\psi \equiv 0 \mod s(A_0)$.

At height 2, small primes, this has been worked out explicitly (R., Zhu).

7.3. *Remark. Skip?* For a s.s. curve over $\mathbb{F}_2$, have:

$$A_0 = \mathbb{Z}_2[\![a]\!], \quad A_1 = A_0[d]/(d^3 - ad - 2),$$
$$s(a) = a, \qquad t(a) = w(a) = a^2 + 3d - ad^2, \qquad w(d) = a - d^2.$$

At all primes at height 2, can describe everything mod p.

Example: ht 2, any p. *Skip?* $G_0/F_p =$ completion of particular s.s. curve. Then

$$A_0/p \approx \mathbb{F}_p[\![a]\!], \qquad A_1/p \approx \mathbb{F}_p[\![a_0, a_1]\!]/\big((a_0^p - a_1)(a_0 - a_1^p)\big),$$
$$A_2/p \approx \mathbb{F}_p[\![a_0, a_1]\!]/\big((a_0^{p^2} - a_1)(a_0^p - a_1^p)(a_0 - a_1^{p^2})\big).$$
$$s\colon a \mapsto a_0, \quad t\colon a \mapsto a_1, \quad \nabla\colon a_0, a_2 \mapsto 1 \otimes a_0, a_1 \otimes 1.$$

Koszul. QCoh(Def) has finite homological dimension $2n$, and comes with "functorial small resolutions". Assuming we have data as above, we can compute Ext.

Skip? At height 2, $\mathrm{Ext}_{\mathrm{QCoh(Def)}}(M,N)$ for M projective A_0-module is H_* of

$$\mathrm{Hom}_{A_0}(M,N) \to \mathrm{Hom}_{A_0}(M, {}^{t}A_1{}^{s} \otimes_{A_0} N) \to \mathrm{Hom}_{A_0}(M, {}^{w^2 s}(A_1/sA_0)^{s} \otimes_{A_0} N)$$
$$f \mapsto \psi_N f - (\mathrm{id} \otimes f)\psi_M, \qquad g \mapsto (w \times \psi_N)g + (w \times g)\psi_M.$$

8. Spectral sequence for maps in $\mathrm{Alg}(E)_{K(n)}/E$

Let R, F augmented $K(n)$-local E-algebras. $\Longrightarrow$ spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\mathrm{Alg}(E)_{/E}(R,F).$$

For $\pi_* R$ *smooth* as a (complete) $\pi_* E$-algebra, and $\pi_* R$ and $\pi_* F$ concentrated in even degrees,

$$E_2^{s,t} = \begin{cases} \mathcal{A}(\pi_* R, \pi_* F) & (s,t) = (0,0), \\ \mathrm{Ext}^s_{\mathrm{QCoh(Def)}}(\omega^{-1} \otimes \widehat{Q}(\pi_* R), \omega^{t/2-1} \otimes \pi_* \overline{F}) & \text{otherwise.} \end{cases}$$

$\widehat{Q}$ is (completion of) indecomposables; $\pi_*\overline{F} \subset \pi_* F$ is augmentation ideal.

Example. (Special case of conjecture[1] of Hopkins-Lurie.)

Fix $G_0/\overline{\mathbb{F}}_p$ over alg closed field, height 2. (E.g., completion of a supersingular elliptic curve.)

Can show

$$\mathrm{Alg}(S)(\Sigma^\infty_+ \mathbb{Z}, E) \approx \overline{\mathbb{F}}_p^\times \times K(\mathbb{Z}_p, 3).$$

(Same as $\mathrm{Alg}(E)_{/E}\big((E \wedge \Sigma^\infty_+ \mathbb{Z})_{K(n)}, E \times E\big)$.)

This is less exciting than it looks: know $\pi_{*\geq 4} = 0$ by Ravenel-Wilson, and π_3 is known (e.g., Sati-Westerland).

Proof. Have that $\widehat{Q}(E_*^\wedge \mathbb{Z}) \approx \deg$. Calculate explicitly, using explicit height 2 formulas. All $E_2^{s,t}$ vanish *except* $E_2^{0,0} \approx \overline{\mathbb{F}}_p^\times$ and

$$E_2^{1,4} = \mathrm{Ext}^1_{\mathrm{QCoh(Def)}}(\omega^{-1} \otimes \deg, \omega) \approx \mathbb{Z}_p.$$

Remark. Assume G_0 is from s.s. elliptic curve C_0. $E_2^{1,4}$ generated by Hodge class:

$$0 \to \omega \to H^1_{\mathrm{dR}}(C/S) \to \omega^{-1} \otimes \deg \to 0,$$

of universal deformation $C/\mathrm{Spec}(A_0)$.

Remark.

$$\pi_3 \mathrm{Alg}(S)(\Sigma^\infty_+ \mathbb{Z}, \mathrm{TMF}_p) = [\Sigma^3 H\mathbb{Z}, \mathrm{gl}_1(\mathrm{TMF}_p)] \approx \mathbb{Z}_p^{c_p},$$

(p-complete TMF.)

$c_p = \dim \mathrm{MF}_2(\Gamma_0(p))^{W=-1} =$ (s.s. j-invts in $\mathbb{F}_p$) $+ \frac{1}{2}$(s.s. j-invts in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$). (Ogg.)

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[1]Word on the street: theorem.