FIBRATIONS AND HOMOTOPY COLIMITS OF SIMPLICIAL SHEAVES

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Abstract. We show that homotopy pullbacks of sheaves of simplicial sets over a Grothendieck topology distribute over homotopy colimits; this generalizes a result of Puppe about topological spaces. In addition, we show that inverse image functors between categories of simplicial sheaves preserve homotopy pullback squares. The method we use introduces the notion of a sharp map, which is analogous to the notion of a quasi-fibration of spaces, and seems to be of independent interest.

1. Introduction

Dold and Thom [3] introduced a class of maps called quasi-fibrations. A map $f\colon X \to Y$ of topological spaces is called a *quasi-fibration* if for each point $y \in Y$ the fiber $f^{-1}(y)$ is naturally weakly equivalent to the homotopy fiber of f over y. Thus, quasi-fibrations behave for some purposes of homotopy theory very much like other types of fibrations; for example, there is a long exact sequence relating the homotopy groups of X, Y, and $f^{-1}(y)$. A notable feature of quasi-fibrations is that (as shown by Dold and Thom) quasi-fibrations defined over the elements of an open cover of a space Y can sometimes be "patched" together to give a quasi-fibration mapping to all of Y.

In this paper we study a class of maps called sharp maps. In our context, a map $f\colon X \to Y$ will be called *sharp* if for each base-change of f along any map into the base Y the resulting pullback square is homotopy cartesian.

We are particularly interested in sharp maps of sheaves of simplicial sets. We shall show that sharp maps of sheaves of simplicial sets have properties analogous to those of quasi-fibrations of topological spaces. In particular, they can be "patched together", in a sense analogous to the way that quasi-fibrations can be patched together. We give several applications.

1.1. **Applications.** Let $\mathcal{E}$ denote a Grothendieck topos; that is, a category equivalent to a category of sheaves on a small Grothendieck site. The category $s\mathcal{E}$ of *simplicial* objects in $\mathcal{E}$ admits a Quillen closed model category structure, as was shown by Joyal (unpublished), and by Jardine in [5] and [6].

Let $X\colon \mathbf{I} \to s\mathcal{E}$ be a diagram of simplicial sheaves indexed on a small category $\mathbf{I}$. We say that such a diagram is a **homotopy colimit diagram** if the natural map $\operatorname{hocolim}_{\mathbf{I}} X \to \operatorname{colim}_{\mathbf{I}} X$ is a weak equivalence of objects in $s\mathcal{E}$, where hocolim denotes the homotopy colimit functor for simplicial sheaves, generalizing that defined by Bousfield and Kan [1] for simplicial sets.

Date: November 3, 1998.
1991 *Mathematics Subject Classification.* Primary 18G30; Secondary 18B25, 55R99.
Key words and phrases. simplicial sheaves, fibrations, homotopy colimits.

Given a map $f\colon X \to Y$ of $\mathbf{I}$-diagrams of simplicial sheaves, for each object i of $\mathbf{I}$ there exists a commutative square

$$\begin{array}{ccc} Xi & \longrightarrow & \operatorname{colim}_{\mathbf{I}} X \\ {\scriptstyle fi}\downarrow & & \downarrow \\ Yi & \longrightarrow & \operatorname{colim}_{\mathbf{I}} Y \end{array} \tag{1.2}$$

and for each morphism $\alpha\colon i \to j$ of $\mathbf{I}$ there exists a commutative square

$$\begin{array}{ccc} Xi & \xrightarrow{X\alpha} & Xj \\ {\scriptstyle fi}\downarrow & & \downarrow{\scriptstyle fj} \\ Yi & \xrightarrow{Y\alpha} & Yj \end{array} \tag{1.3}$$

The following theorem essentially says that in a category of simplicial sheaves, homotopy pullbacks "distribute" over homotopy colimits.

Theorem 1.4. *Let $f\colon X \to Y$ be a map of $\mathbf{I}$-diagrams of simplicial objects in a topos $\mathcal{E}$, and suppose that Y is a homotopy colimit diagram. Then the following two properties hold.*

1. *If each square of the form (1.2) is homotopy cartesian, then X is a homotopy colimit diagram.*
2. *If X is a homotopy colimit diagram, and each diagram of the form (1.3) is homotopy cartesian, then each diagram of the form (1.2) is also homotopy cartesian.*

The proof of (1.4) is given in Section 7. This result is well-known when $s\mathcal{E}$ is the category of simplicial sets: Puppe [9] formulates and proves a version of the above result for the category of topological spaces, which can be used to derive (1.4) for simplicial sets; see [4, Appendix HL] for more discussion of Puppe's result. Also, Chachólski [2] has proved a result of this type in the category of simplicial sets using purely simplicial methods.

As another application we give the following. Let $p\colon \mathcal{E} \to \mathcal{E}'$ be a geometric morphism of Grothendieck topoi, and let $p^*\colon \mathcal{E}' \to \mathcal{E}$ denote the corresponding inverse image functor. This functor prolongs to a simplicial functor $p^*\colon s\mathcal{E}' \to s\mathcal{E}$.

Theorem 1.5. *The inverse image functor $p^*\colon s\mathcal{E}' \to s\mathcal{E}$ preserves homotopy cartesian squares.*

The proof of (1.5) is given in Section 5. An example of an inverse image functor is the sheafification functor $L^2\colon \mathrm{Psh}\mathbf{C} \to \mathrm{Sh}\mathbf{C}$ associated to a Grothendieck topology on $\mathbf{C}$. Thus, (1.5) shows in particular that sheafification functors preserve homotopy cartesian squares.

1.6. **Organization of the paper.** In Section 2 we define sharp maps and state some of their general properties. In Section 3 we recall facts about sheaf theory and the model category structure on simplicial sheaves. Section 4 gives several useful characterizations of sharp maps of simplicial sheaves, which are used to prove a number of properties in Section 5, as well as the proof of (1.5).

In Section 6 we prove a result about how sharp maps are preserved by taking the diagonal of a simplicial object. This result is used in Section 7 to prove a similar fact about how sharp maps are preserved by homotopy colimits; this result is used in turn to give a proof of (1.4). Section 8 does the hard work of showing that sharp maps which agree "up to homotopy" can be glued together, thus providing lemmas which were needed for Section 6.

Section 9 proves a result about sharp maps in a boolean localization which was needed in Section 4.

In Section 10 we prove that in a boolean localization the local fibrations are the same as the global fibrations, a fact which is used at several places in this paper.

1.7. **Acknowledgements.** I would like to thank Phil Hirschhorn, Mike Hopkins, Mark Johnson, Brooke Shipley, and Carlos Simpson for their comments on various parts of this work.

2. Sharp maps

In this section, we define the notion of a sharp map in a general closed model category, and prove some of its general properties. I learned about the notion of a "sharp" map from Mike Hopkins, who was originally led, for different reasons, to formulate the dual notion of a "flat" map.

Let $\mathbf{M}$ be a closed model category [10], [11]. We say that a map $f\colon X \to Y$ in $\mathbf{M}$ is **sharp** if for each diagram in $\mathbf{M}$ of the form

$$\begin{array}{ccccc} A & \xrightarrow{i} & A' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{j} & B' & \longrightarrow & Y \end{array}$$

in which j is a weak equivalence and each square is a pullback square, the map i is also a weak equivalence. It follows immediately from the definition that the class of sharp maps is closed under base-change.

2.1. **Proper model categories.** A model category $\mathbf{M}$ is said to be **right proper** if for each pullback diagram in $\mathbf{M}$ of the form

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{j} & Y \end{array}$$

such that f is a fibration and j is a weak equivalence, then i is also a weak equivalence. The categories of topological spaces and simplicial sets are two well-known examples of right-proper model categories.

There is an dual notion, in which a model category for which pushouts of weak equivalences along cofibrations are weak equivalences is called **left proper**. A model category is **proper** if it is both left and right proper.

Since the class of fibrations in a model category is closed under base-change, we have the following.

Proposition 2.2. *A model category* $\mathbf{M}$ *is right proper if and only if each fibration is sharp.*

2.3. **Homotopy cartesian squares.** Let

$$(2.4)\qquad \begin{array}{ccc} P & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array}$$

be a commutative square in $\mathbf{M}$. Say such a square is **homotopy cartesian** if for some choice of factorizations $X \to X' \to B$ and $Y \to Y' \to B$ of f and g into weak equivalences followed by fibrations, the natural map $P \to X' \times_B Y'$ is a weak equivalence. It is straightforward to show that the choice of factorizations does not matter.

Clearly, any *pullback* square of the form (2.4) in which f and g are already fibrations is homotopy cartesian. Any square weakly equivalent to a homotopy cartesian square is itself homotopy cartesian.

Lemma 2.5. *In a right proper model category, a pullback square as in (2.4) in which g is a fibration is a homotopy cartesian square.*

Proof. Choose a factorization pi of f into a weak equivalence $i\colon X \to X'$ followed by a fibration $p\colon X' \to B$. Then we obtain pullback squares

$$(2.6)\qquad \begin{array}{ccccc} P & \xrightarrow{j} & P' & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & X' & \xrightarrow{p} & B \end{array}$$

in which j is a weak equivalence by (2.2), since it is obtained by pulling back the weak equivalence i along the fibration h. Thus the square (2.4) is weakly equivalent to the right-hand square of (2.6) which is homotopy cartesian. □

The following proposition gives the characterization of sharp maps which was alluded to in the introduction; it holds only in a right proper model category.

Proposition 2.7. *In a right proper model category, a map $g\colon Y \to B$ is a sharp map if and only if each pullback square (2.4) is a homotopy cartesian square.*

Proof. First suppose that g is sharp. As in the proof of (2.5) choose a factorization $pi\colon X \to X' \to B$ of f into a weak equivalence i followed by a fibration p, obtaining a diagram (2.6). Then the right hand square of this diagram is homotopy cartesian by (2.5), and i and j are weak equivalences, since j is the pullback of the weak equivalence i along the sharp map g.

Conversely, suppose g is a map such that each pullback along g is a homotopy cartesian square. Given a diagram of pullback squares as in (2.6) in which i is a weak equivalence, it follows that j is also a weak equivalence, since both the right-hand square and the outer rectangle are homotopy cartesian squares which are weakly equivalent at the three non-pullback corners. Thus g is sharp. □

Example 2.8. The category of topological spaces is a right proper model category. The class of sharp maps of topological spaces includes all Serre fibrations, as well as all fiber bundles. Every sharp map is clearly a quasi-fibration in the sense of Dold and Thom [3]. It is not the case that all quasi-fibrations are sharp; indeed, the class of quasi-fibrations is not closed under base change, see [3, Bemerkung 2.3]. I do not know of a simple characterization of sharp maps of topological spaces.

3. Facts about topoi

In this section we recall facts about sheaves and simplicial sheaves. Our main reference for sheaf theory is Mac Lane-Moerdijk [8].

3.1. **Grothendieck topoi.** A **Grothendieck topos** $\mathcal{E}$ is a category equivalent to some category Sh$\mathbf{C}$ of sheaves of sets on a small Grothendieck site $\mathbf{C}$. Among the many properties of a Grothendieck topos $\mathcal{E}$, we note that $\mathcal{E}$ has all small limits and colimits, and that $\mathcal{E}$ is cartesian closed. The internal hom object in $\mathcal{E}$ is denoted by Y^X.

Example 3.2.

(1) The category $\mathcal{S}$et is a Grothendieck topos, since it is sheaves on a one-point space.
(2) The presheaf category Psh$\mathbf{C}$, defined to be the category of functors $\mathbf{C}^{\mathrm{op}} \to$ $\mathcal{S}$et, is the category of sheaves of sets in the trivial topology on $\mathbf{C}$, and thus is a Grothendieck topos.
(3) The category Sh(T) of sheaves of sets on a topological space T is a Grothendieck topos.

A **geometric morphism** $f\colon \mathcal{E} \to \mathcal{E}'$ is a pair of adjoint functors

$$f^*\colon \mathcal{E}' \rightleftarrows \mathcal{E} : f_*$$

such that the left adjoint f^* preserves finite limits. The left adjoint f^* is called the **inverse image** functor, and f_* the **direct image** functor.

3.3. **Boolean localizations.** Let $\mathcal{B}$ be a complete Boolean algebra. Then $\mathcal{B}$, viewed as a category via the partial order on $\mathcal{B}$, has a natural Grothendieck topology, and hence gives us a Grothendieck topos Sh$\mathcal{B}$. (This topos is discussed in more detail in Section 10.)

A **Boolean localization** of a topos $\mathcal{E}$ is a geometric morphism $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ such that the inverse image functor $p^*\colon \mathcal{E} \to \mathrm{Sh}\mathcal{B}$ is faithful.

Example 3.4.

(1) The category of sets is its own Boolean localization, since it is equivalent to sheaves on the trivial Boolean algebra.
(2) For a category $\mathbf{C}$, let $\mathbf{C_0} \subset \mathbf{C}$ denote the subcategory consisting of all objects and all identity maps. Then $p\colon \mathrm{Psh}\mathbf{C_0} \to \mathrm{Psh}\mathbf{C}$ is a boolean localization, where $p^*\colon \mathrm{Psh}\mathbf{C} \to \mathrm{Psh}\mathbf{C_0}$ is the obvious restriction functor; this is because Psh$\mathbf{C_0}$ is equivalent to the category of sheaves on the boolean algebra $\mathcal{P}(\mathrm{ob}\mathbf{C})$, the power set of ob$\mathbf{C}$.
(3) For a topological space T, let T^δ denote the underlying set T with the discrete topology. Then $\mathrm{Sh}(T^\delta) \approx \mathrm{Sh}(\mathcal{P}T^\delta)$ is a boolean localization of Sh(T); the inverse image functor $p^*\colon \mathrm{Sh}(T) \to \mathrm{Sh}(T^\delta)$ sends a sheaf X to the collection of all stalks of X over every point of T.

Remark 3.5. In each of the examples above, the Boolean localization turned out to be equivalent to a product of copies of $\mathcal{S}$et. However, there exist topoi $\mathcal{E}$ which do not admit a Boolean localization of this type.

Boolean localizations have the following properties.

(1) Every Grothendieck topos has a Boolean localization.

(2) The inverse image functor p^* associated to a Boolean localization functor $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ reflects isomorphisms, monomorphisms, epimorphisms, colimits, and finite limits.
(3) The topos $\mathrm{Sh}\mathcal{B}$ has a "choice" axiom: every epimorphism in $\mathrm{Sh}\mathcal{B}$ admits a section.
(4) The topos $\mathrm{Sh}\mathcal{B}$ is **boolean**; that is, each subobject $A \subset X$ in $\mathrm{Sh}\mathcal{B}$ admits a "complement", namely a subobject $B \subset X$ such that $A \cup B = X$ and $A \cap B = \varnothing$.

Property (1) is shown in [8, IX.9]. See Jardine [6] for proofs of the other properties.

3.6. **A distributive law.** For our purposes it is important to note the following relationship between colimits and pullbacks in a topos $\mathcal{E}$.

Proposition 3.7. *Let $Y\colon \mathbf{I} \to \mathcal{E}$ be a functor from a small category $\mathbf{I}$ to a topos, and let $A \to B = \mathrm{colim}_{\mathbf{I}}(i \mapsto Yi)$ be a map. Then the natural map*

$$\mathrm{colim}_{\mathbf{I}}(i \mapsto A \underset{B}{\times} Yi) \to A$$

is an isomorphism.

This proposition says that if an object is pulled back along a colimit diagram, then that object can be recovered as the colimit of the pulled-back diagram. It makes sense to think of this as a "distributive law". In fact, in the special case in which $B = X_1 \amalg X_2$, and $A \to B$ is the projection $(X_1 \amalg X_2) \times Y \to X_1 \amalg X_2$ the proposition reduces to the usual distributive law of products over coproducts: $(X_1 \times Y) \amalg (X_2 \times Y) \approx (X_1 \amalg X_2) \times Y$.

To prove (3.7), note that it is true if $\mathcal{E} = \mathcal{S}\mathrm{et}$, and thus is true if $\mathcal{E} = \mathrm{Psh}\mathbf{C}$. The general result now follows from the properties of the sheafification functor $L^2\colon \mathrm{Psh}\mathbf{C} \to \mathrm{Sh}\mathbf{C}$.

Remark 3.8. Consider the diagram X defined by $i \mapsto A \times_B Yi$. It is equipped with a natural transformation $f\colon X \to Y$ with the property that for each $\alpha\colon i \to j$ in $\mathbf{I}$ the map $X\alpha$ is the pullback of $Y\alpha$ along fj.

One can formulate the following "converse" to (3.7) which is false. Namely, given a natural transformation $f\colon X' \to Y$ of $\mathbf{I}$-diagrams such that for each $\alpha\colon i \to j$ in $\mathbf{I}$ the map $X'\alpha$ is the base-change of $Y\alpha$ along fj, one may ask whether the natural maps $X'i \to A \times_B Yi$ are isomorphisms, where $A = \mathrm{colim}_{\mathbf{I}} X'$. A counterexample in $\mathcal{E} = \mathcal{S}\mathrm{et}$ is to take $\mathbf{I}$ to be a group G and $X' \to Y$ to be any map of non-isomorphic G-orbits.

(1.4, part 1) may be viewed as a homotopy theoretic analogue of (3.7). (1.4, part 2) may be viewed as a homotopy theoretic analogue of the "converse" to (3.7).

3.9. **Simplicial sheaves.** We let $s\mathcal{E}$ denote the category of simplicial objects in a Grothendieck topos $\mathcal{E}$. Note that $s\mathcal{E}$ is itself a Grothendieck topos. The full subcategory of **discrete** simplicial objects in $s\mathcal{E}$ is equivalent to $\mathcal{E}$; thus, we regard $\mathcal{E}$ as a subcategory of $s\mathcal{E}$.

For any topos $\mathcal{E}$ there is a natural functor $\mathcal{S}\mathrm{et} \to \mathcal{E}$ sending a set X to the corresponding **constant** sheaf. This prolongs to a functor $s\mathcal{S}\mathrm{et} \to s\mathcal{E}$, and we will thus regard any simplicial set as a constant simplicial sheaf.

3.10. **Model category for simplicial sheaves.** We will make use of the elegant model category structure on simplicial sheaves provided by Jardine in [6]. We summarize here the main properties of this structure which we need. Let $s\mathcal{E}$ denote the category of simplicial objects in a topos $\mathcal{E}$. Let $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ denote a fixed boolean localization of $\mathcal{E}$. A map $f\colon X \to Y$ in $s\mathcal{E}$ is said to be

(1) a **local weak equivalence** (or simply, a **weak equivalence**) if

$$(L^2 \operatorname{Ex}^\infty p^* f)(b)\colon (L^2 \operatorname{Ex}^\infty p^* X)(b) \to (L^2 \operatorname{Ex}^\infty p^* Y)(b)$$

is a weak equivalence for each $b \in \mathcal{B}$. Here $\operatorname{Ex}^\infty\colon s\mathrm{Sh}\mathcal{B} \to s\mathrm{Psh}\mathcal{B}$ denotes the functor obtained by applying Kan's Ex^∞ functor [7] at each $b \in \mathcal{B}$, and L^2 denotes the simplicial prolongation $s\mathrm{Psh}\mathcal{B} \to s\mathrm{Sh}\mathcal{B}$ of the sheafification functor.

(2) a **local fibration** if $p^* f(b)\colon p^* X(b) \to p^* Y(b)$ is a Kan fibration for each $b \in \mathcal{B}$. It should be pointed out that local fibrations are not in general the fibrations in the model category structure on $s\mathcal{E}$; but note (3.14).

(3) a **cofibration** if it is a monomorphism.

(4) a **global fibration** (or simply, a **fibration**) if it has the right lifting property with respect to all maps which are both cofibrations and weak equivalences.

Note that the definition of local weak equivalence simplifies when $\mathcal{E} = \mathrm{Sh}\mathcal{B}$, since $\mathrm{Sh}\mathcal{B}$ is its own boolean localization. Furthermore, a map f in $s\mathcal{E}$ is a local weak equivalence if and only if $p^* f$ in $s\mathrm{Sh}\mathcal{B}$ is a local weak equivalence.

Theorem 3.11. *(Jardine* [6], [5]*) The category $s\mathcal{E}$ with the above classes of cofibrations, global fibrations, and local weak equivalences is a proper simplicial closed model category. Furthermore, the characterizations of local weak equivalences, local fibrations, and global fibrations do not depend on the choice of boolean localization.*

Example 3.12.

(1) When $s\mathcal{E} = s\mathcal{S}\mathrm{et}$ this model category structure coincides with the usual one, and local fibrations coincide with global fibrations.

(2) For $s\mathcal{E} = s\mathrm{Psh}\mathbf{C}$, a map $f\colon X \to Y$ is a local weak equivalence, cofibration, or local fibration if for each $C \in \mathrm{ob}C$, the map $fC\colon X(C) \to Y(C)$ is respectively a weak equivalence, monomorphism, or Kan fibration of simplicial sets.

(3) For $s\mathcal{E} = s\mathrm{Sh}(T)$, a map $f\colon X \to Y$ is a local weak equivalence, cofibration, or local fibration if for each point $p \in T$ the map $f_p\colon X_p \to Y_p$ of stalks is respectively a weak equivalence, monomorphism, or Kan fibration of simplicial sets.

We also need the following property.

Proposition 3.13. [6, Lemma 13(3)] *Let $\mathcal{E}$ be sheaves on a Grothendieck topos. If $f_i\colon X_i \to Y_i$ is a family of local weak equivalences in $s\mathcal{E}$ indexed by a set I, then the induced map $f\colon \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is a local weak equivalence.*

We need one additional fact about fibrations in a boolean localization.

Proposition 3.14. *In the category $s\mathrm{Sh}\mathcal{B}$ of simplicial sheaves on a complete boolean algebra $\mathcal{B}$, the local fibrations are precisely the global fibrations.*

The proof of (3.14) is given in Section 10.

Finally, we note that if $f\colon \mathcal{E} \to \mathcal{E}'$ is a geometric morphism, then the induced inverse image functor $f^*\colon s\mathcal{E}' \to s\mathcal{E}$ preserves cofibrations and weak equivalences; this is because the composite

$$s\mathcal{E}' \xrightarrow{f^*} s\mathcal{E} \xrightarrow{p^*} s\mathrm{Sh}\mathcal{B}$$

must preserve such if $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ is a boolean localization of $\mathcal{E}$.

3.15. **Model category for simplicial presheaves.** Although we will not make much use of it here, we note that if $\mathcal{E} = \mathrm{Sh}\mathbf{C}$ for some Grothendieck site $\mathbf{C}$, then Jardine [6, Thm. 17] constructs a "presheaf" closed model category structure on $s\mathrm{Psh}\mathbf{C}$ related to that on $s\mathcal{E}$ (and not to be confused with the "sheaf" model category structure obtained by applying the remarks of the previous section to $\mathcal{E} = \mathrm{Psh}\mathbf{C}$). In this structure on $s\mathrm{Psh}\mathbf{C}$, the cofibrations are the monomorphisms, and the weak equivalences are the maps in $s\mathrm{Psh}\mathbf{C}$ which sheafify to local weak equivalences in $s\mathcal{E}$. Furthermore, a map in $s\mathrm{Psh}\mathbf{C}$ is called a local fibration if it sheafifies to a local fibration in $s\mathcal{E}$. The natural adjoint pair $s\mathrm{Psh}\mathbf{C} \leftrightarrows s\mathcal{E}$ induces an equivalence of closed model categories in the sense of Quillen; in particular, the homotopy category of $s\mathrm{Psh}\mathbf{C}$ (induced by the presheaf model category structure) is equivalent to the homotopy category of $s\mathcal{E}$. Thus, many results stated for $s\mathcal{E}$ such as (1.4) carry over to the presheaf model category of $s\mathrm{Psh}\mathbf{C}$ without change.

4. Local character of sharp maps of simplicial sheaves

The following theorem provides several equivalent characterizations of sharp maps in $s\mathcal{E}$. There are two types of such statements: (5) and (6) say that sharpness is a "local condition", i.e., sharpness is detected on boolean localizations, while (2), (3), and (4) say that sharpness is detected on "fibers", i.e., by pulling back to the product of a discrete object and a simplex.

Theorem 4.1. *Let $f\colon X \to Y$ be a map of simplicial objects in a Grothendieck topos $\mathcal{E}$. The following are equivalent.*

(1) *f is sharp.*

(2) *For each $n \geq 0$ and each map $S \to Y_n$ in $\mathcal{E}$, the induced pullback square*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ S \times \Delta[n] & \longrightarrow & Y \end{array}$$

is homotopy cartesian.

(3) *For each $n \geq 0$ there exists an epimorphism $S_n \to Y_n$ in $\mathcal{E}$ such that the induced pullback square*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ S_n \times \Delta[n] & \longrightarrow & Y \end{array}$$

is homotopy cartesian.

(4) *For each* $n \geq 0$ *there exists an epimorphism* $S_n \to Y_n$ *in* $\mathcal{E}$ *such that for each map* $\delta\colon \Delta[m] \to \Delta[n]$ *of standard simplices, the induced diagram of pullback squares*

$$\begin{array}{ccccc} P & \xrightarrow{\quad h \quad} & P' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ S_n \times \Delta[m] & \xrightarrow{1\times\delta} & S_n \times \Delta[n] & \longrightarrow & Y \end{array}$$

is such that h *is a weak equivalence of simplicial sheaves.*

(5) *For any boolean localization* $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$*, the inverse image* $p^*f\colon p^*X \to p^*Y$ *of* f *is sharp in* $s\mathrm{Sh}\mathcal{B}$*.*

(6) *There exists a boolean localization* $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ *such that the inverse image* $p^*f\colon p^*X \to p^*Y$ *of* f *is sharp in* $s\mathrm{Sh}\mathcal{B}$*.*

Proof.

(1) implies (2): This follows from (2.7), and the fact that $s\mathcal{E}$ is right proper.

(2) implies (3) and (4): Let $S_n = Y_n$.

either (3) or (4) implies (5): This will follow from (9.1), since $p^*\colon \mathcal{E} \to \mathrm{Sh}\mathcal{B}$ preserves pullbacks and epimorphisms.

(5) implies (6): This is trivial, since every $\mathcal{E}$ has a boolean localization (3.10).

(6) implies (1): If $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ is a boolean localization, and p^*f is sharp, then since $p^*\colon s\mathcal{E} \to s\mathrm{Sh}\mathcal{B}$ preserves pullbacks and reflects weak equivalences (3.10), it follows that f is sharp. □

Remark 4.2. In the case when $s\mathcal{E} = \mathcal{S}$, and $f\colon X \to Y$ a map of simplicial sets, the above theorem implies that the following three statements are equivalent.

(1) f is sharp.

(2) For each map $g\colon \Delta[n] \to Y$ the pullback square of f along g is homotopy cartesian.

(3) For each diagram of pullback squares of the form

$$\begin{array}{ccccc} P & \xrightarrow{\quad h \quad} & P' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \Delta[m] & \xrightarrow{\quad \delta \quad} & \Delta[n] & \longrightarrow & Y \end{array}$$

the map h is a weak equivalence.

Note that characterization (2) is reminiscent of the definition of quasi-fibration of topological spaces.

A sharp map to a simplicial set Y induces a "good diagram" indexed by the simplices of Y, in the sense of Chachólski [2].

Remark 4.3. Recall from (3.4) that if $\mathcal{E} = \mathrm{Psh}\mathbf{C}$ is a category of presheaves on $\mathbf{C}$, then a suitable boolean localization for $\mathcal{E}$ is $\mathrm{Psh}\mathbf{C_0}$. This implies using part (6) of (4.1) that a map $f\colon X \to Y$ of presheaves on $\mathbf{C}$ is sharp if and only if for each object $C \in \mathbf{C}$ the map $f(C)\colon X(C) \to Y(C)$ is a sharp map of simplicial sets.

Remark 4.4. Recall from (3.4) that if $\mathcal{E} = \mathrm{Sh}(T)$ where T is a topological space, then a boolean localization for $\mathcal{E}$ is $\mathrm{Sh}T^{\delta}$. This implies using part (6) of (4.1) that a map $f\colon X \to Y$ of sheaves over T is sharp if and only if for each point $p \in T$ the induced map $f_p\colon X_p \to Y_p$ on stalks is a sharp map of simplical sets.

Remark 4.5. The statement of (4.1) remains true if we replace $s\mathcal{E}$ with $s\mathrm{Psh}\mathbf{C}$ equipped with the presheaf model category structure of (3.15), and replace boolean localizations $\mathrm{Sh}\mathcal{B} \to \mathcal{E}$ with composite maps $\mathrm{Sh}\mathcal{B} \to \mathrm{Sh}\mathbf{C} \to \mathrm{Psh}\mathbf{C}$. That this is the case follows easily from the observation that $f\colon X \to Y \in s\mathrm{Psh}\mathbf{C}$ is sharp if and only if $L^2 f\colon L^2 X \to L^2 Y \in s\mathrm{Sh}\mathbf{C}$ is sharp, the proof of which is straightforward.

5. Basic properties of sharp maps of simplicial sheaves

In this section we give some basic properties of sharp maps in a simplicial topos.

Theorem 5.1. *The following hold for simplicial objects in a topos $\mathcal{E}$.*

- P1 *Local fibrations are sharp.*
- P2 *For any object $X \in s\mathcal{E}$ the projection map $X \to 1$ is sharp.*
- P3 *Sharp maps are closed under base-change.*
- P4 *If f is a map such that the base-change of f along some epimorphism is sharp, then f is sharp.*
- P5 *If maps f_α are sharp for each $\alpha \in A$ for some set A, then the coproduct $\amalg f_\alpha$ is sharp.*
- P6 *If $p\colon \mathcal{E} \to \mathcal{E}'$ is a geometric morphism of topoi, the inverse image functor $p^*\colon s\mathcal{E}' \to s\mathcal{E}$ preserves sharp maps.*

Proof. Property P1 follows from part (6) of (4.1), the fact that global fibrations are sharp (2.2), and the fact that local fibrations are global fibrations in a Boolean localization (3.14).

Property P2 follows immediately from the fact weak equivalences in $s\mathcal{E}$ are precisely those maps f such that $(L^2 \operatorname{Ex}^\infty p^* f)(b)$ is a weak equivalence for each $b \in \mathcal{B}$ (where $p\colon \mathrm{Sh}\mathcal{B} \to \mathcal{E}$ is a boolean localization), together with the fact that the functor $L^2 \operatorname{Ex}^\infty p^*$ preserves products.

Property P3 has already been noted in Section 2.

To prove property P4, consider the pull-back squares

$$\begin{array}{ccccccc}
Q & \xrightarrow{\;\;q\;\;} & Q' & \longrightarrow & P & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
C_n \times \Delta[m] & \xrightarrow{1\times\delta} & C_n \times \Delta[n] & \longrightarrow & C & \xrightarrow{\;p\;} & Y
\end{array}$$

where g is sharp and p is an epimorphism. Then q is a weak equivalence since $1 \times \delta$ is, whence f is sharp by part (4) of (4.1), since the map $C_n \to Y_n$ is an epimorphism in $\mathcal{E}$.

To prove property P5, let $f_\alpha\colon X_\alpha \to Y_\alpha$ be a collection of sharp maps, and let $f = \amalg_{\alpha\in I} f_\alpha$. Then P5 follows from the fact that a coproduct of weak equivalences is a weak equivalence (3.13) and using part (4) of (4.1).

Property P6 follows easily from part (4) of (4.1), and the fact that inverse image functors preserve pullbacks, epimorphisms, and weak equivalences. □

We can now easily prove (1.5).

Proof of (1.5). Recall (3.10) that any homotopy cartesian square is weakly equivalent to a pullback square in which all the maps are fibrations. Since fibrations are sharp by (2.2), the square obtained by applying the inverse image functor $p^*\colon \mathcal{E}' \to \mathcal{E}$ is a pullback square in which the maps are sharp by P6, and hence is a homotopy cartesian square by (2.7). Since p^* preserves weak equivalences the conclusion follows. □

Remark 5.2. Parts P1–P5 of (5.1) remain true if we replace $s\mathcal{E}$ with $s\mathrm{Psh}\mathbf{C}$ equipped with the presheaf model category of (3.15), for the reasons discussed in (4.5).

6. Diagonal of a simplicial object

Let $X\colon \Delta^{\mathrm{op}} \to s\mathcal{E}$ be a simplicial object in $s\mathcal{E}$; we write $[n] \mapsto X(n)$ where $X(n) \in s\mathcal{E}$. The **diagonal** $|X|$ of X is an object in $s\mathcal{E}$ defined by $[n] \mapsto X(n)_n$.

Theorem 6.1. *Let $p\colon X \to Y$ be a map of simplicial objects in $s\mathcal{E}$ such that each $p(n)\colon X(n) \to Y(n)$ is sharp, and each square*

$$\begin{array}{ccc} X(n) & \longrightarrow & X(m) \\ \downarrow & & \downarrow \\ Y(n) & \longrightarrow & Y(m) \end{array}$$

is homotopy cartesian. Then $|p|\colon |X| \to |Y|$ is sharp.

We prove this theorem using the following well-known inductive construction of the diagonal of a simplicial object. Namely, $|X| \approx \mathrm{colim}_n\, F_n|X|$, where $F_0|X| = X(0)$ and each $F_n|X|$ is obtained from $F_{n-1}|X|$ by a pushout diagram of the form

$$\begin{array}{ccc} X(n) \times \partial\Delta[n] \displaystyle\bigcup_{L_{n-1}X \times \partial\Delta[n]} L_{n-1}X \times \Delta[n] & \longrightarrow & X(n) \times \Delta[n] \\ \downarrow & & \downarrow \\ F_{n-1}|X| & \longrightarrow & F_n|X| \end{array} \tag{6.2}$$

where $L_{n-1}X$ denotes the subobject of $X(n)$ which is the union of the images of all degeneracy maps $s_i\colon X_{n-1} \to X_n$ for $0 \le i \le n$.

6.3. **Colimits on posets of proper subsets.** Before going to the proof of (6.1) we collect some facts about colimits of diagrams indexed by the subsets of a finite set. These facts will also be needed in Sections 8 and 9.

If S is a finite set, let $\mathcal{P}S$ denote the poset of subsets of S, and let $\bar{\mathcal{P}}S$ denote the poset of *proper* subsets of S; we regard $\mathcal{P}S$ and $\bar{\mathcal{P}}S$ as categories with $T \to T'$ if $T \subseteq T' \subseteq S$.

Given a functor $X\colon \mathcal{P}S \to s\mathcal{E}$ and a subset $S' \subset S$, we define $X|_{S'}\colon \mathcal{P}S' \to s\mathcal{E}$ to be the restriction of X to $\mathcal{P}S'$ via the formula $X|_{S'}(T) = X(T)$ for $T \subset S'$. We also speak of the restriction $X|_{S'}\colon \bar{\mathcal{P}}S' \to s\mathcal{E}$ to $\bar{\mathcal{P}}S'$. We say that a functor $X\colon \mathcal{P}S \to s\mathcal{E}$ (resp. a functor $X\colon \bar{\mathcal{P}}S \to s\mathcal{E}$) is **cofibrant** if for each subset (resp. proper subset) $T \subset S$ the induced map $\mathrm{colim}_{\bar{\mathcal{P}}T}\, X|_T \to X(T)$ is a monomorphism. (The cofibrant functors are in fact the cofibrant objects in a model category structure on the categories of functors $\mathcal{P}S \to s\mathcal{E}$ and $\bar{\mathcal{P}}S \to s\mathcal{E}$.)

Say that $S = \{1, \dots, n\}$, and let $S' = \{1, \dots, n-1\}$. Define $X'\colon \bar{\mathcal{P}}S' \to s\mathcal{E}$ by the formula $X'(T) = X(T \cup \{n\})$ for $T \subset S'$. There is a natural map $X|_{S'} \to X'$ of diagrams indexed by $\bar{\mathcal{P}}S'$.

Proposition 6.4. *Suppose $X\colon \bar{\mathcal{P}}S \to s\mathcal{E}$ is a functor. There is a natural pushout square*

$$\begin{array}{ccc} \operatorname{colim}_{\bar{\mathcal{P}}S'} X|_{S'} & \longrightarrow & \operatorname{colim}_{\bar{\mathcal{P}}S'} X' \\ \big\downarrow & & \big\downarrow \\ X(S') & \longrightarrow & \operatorname{colim}_{\bar{\mathcal{P}}S} X \end{array}$$

and if f is a cofibrant functor, then both vertical maps in the above square are monomorphisms.

Proof. This is a straightforward induction argument on the size of S, using the fact that in a topos, pushouts of monomorphisms are again monomorphisms. □

Corollary 6.5. *Given a cofibrant functor $X\colon \mathcal{P}S \to s\mathcal{E}$ such that for all $T \subset T'$ the map $X(T) \to X(T')$ is a weak equivalence, the induced map $\operatorname{colim}_{\bar{\mathcal{P}}S} X \to X(S)$ is a weak equivalence.*

Proof. This is proved by a straightforward induction argument on the size of S, using (6.4). □

Corollary 6.6. *Given cofibrant functors $X, Y\colon \bar{\mathcal{P}}S \to s\mathcal{E}$ and a map $f\colon X \to Y$ such that each map $X(T) \to Y(T)$ is a weak equivalence, then the induced map $\operatorname{colim}_{\bar{\mathcal{P}}S} X \to \operatorname{colim}_{\bar{\mathcal{P}}T} Y$ is a weak equivalence.*

Proof. This is proved by induction on the size of S, using (6.4) and the fact that $s\mathcal{E}$ is a left proper model category. □

6.7. **The proof of the theorem.** The object $L_{n-1}X$ has an alternate description using the above notation. Let $S = \{1, \dots, n\}$. Define a functor $F\colon \mathcal{P}S \to s\mathcal{E}$ sending $T \subset S$ to $X(\#T)$, and sending $i\colon T \to T'$ to the map induced by the simplicial operator $\sigma\colon [\#T'] \to [\#T]$ defined by $\sigma(0) = 0$ and $\sigma(k) = \max(\ell \mid i(k) \leq \ell)$ for $0 < k \leq \#T'$. Then $L_{n-1}X = \operatorname{colim}_{\bar{\mathcal{P}}S} F$. Furthermore, F is a cofibrant functor

Proof of (6.1). Each map $\partial\Delta[n] \to \Delta[n]$ is mono, as are the maps $L_{n-1}X \to X(n)$, and the top horizontal arrow in (6.2). The proof is a straightforward induction following the inductive construction of diagonal given above and using (8.1) together with (6.4).

That is, suppose by induction that $F_{n-1}|X| \to F_{n-1}|Y|$ is sharp. Using (8.1, 3) one shows that $L_{n-1}X \to L_{n-1}Y$ is sharp. Then using (8.1, 2) one shows that the induced map from the upper left-hand corner of (6.2) to the upper left-hand corner of the corresponding square for Y is sharp. Applying (8.1, 2) to the whole square (6.2) gives that $F_n|X| \to F_n|Y|$ is sharp. Finally, (8.1, 1) shows that $|X| \to |Y|$ is sharp, as desired. □

Remark 6.8. If $f\colon X \to Y$ is a map of simplicial objects in $s\mathcal{E}$ such that in each degree n the map $f(n)\colon X(n) \to Y(n)$ is a weak equivalence, then one may show by using the above inductive scheme together with (6.6) that $|f|\colon |X| \to |Y|$ is a weak equivalence, since $s\mathcal{E}$ is a proper model category and the cofibrations are precisely the monomorphisms.

7. Homotopy colimits

Let $X\colon \mathbf{I} \to s\mathcal{E}$ be a diagram of simplicial sheaves. As in [1] the **homotopy colimit** of X, denoted $\operatorname{hocolim}_{\mathbf{I}} X$, is defined to be the diagonal of the simplicial object in $s\mathcal{E}$ given in each degree $n \geq 0$ by

$$[n] \mapsto \coprod_{i_0 \to \cdots \to i_n} X i_0,$$

where the coproduct is taken over all composable strings of arrows in $\mathbf{I}$ of length n. From (3.13) and (6.8) it follows that $\operatorname{hocolim}_{\mathbf{I}} X$ is a weak homotopy equivalence invariant of X.

Let $(\mathbf{I} \downarrow i)$ denote the category of objects over a fixed object i in $\mathbf{I}$. Given an $\mathbf{I}$-diagram X, one can define an $\mathbf{I}$-diagram $\widetilde{X}$ by $\widetilde{X} i = \operatorname{hocolim}_{(\mathbf{I}\downarrow i)} X$. Thus, $\widetilde{X} i$ is the diagonal of the simplicial object in $s\mathcal{E}$ given by

$$[n] \mapsto \coprod_{i_0 \to \cdots \to i_n \to i} X i_0.$$

There is a natural map $\widetilde{X} \to X$ of $\mathbf{I}$-diagrams, and an isomorphism of simplicial sheaves $\operatorname{hocolim}_{\mathbf{I}} X \approx \operatorname{colim}_{\mathbf{I}} \widetilde{X}$. (This is the construction of [1].)

Theorem 7.1. *Let $f\colon X \to Y$ be a map of $\mathbf{I}$-diagrams of simplicial sheaves such that*

(1) *each map $fi\colon Xi \to Yi$ is sharp for $i \in \operatorname{ob}\mathbf{I}$, and*

(2) *each square* $\begin{array}{ccc} Xi & \longrightarrow & Xj \\ \downarrow & & \downarrow \\ Yi & \longrightarrow & Yj \end{array}$ *for $\alpha\colon i \to j \in \mathbf{I}$ is homotopy cartesian.*

Then

(a) *the induced map $\operatorname{hocolim}_{\mathbf{I}} f\colon \operatorname{hocolim}_{\mathbf{I}} X \to \operatorname{hocolim}_{\mathbf{I}} Y$ is sharp, and*

(b) *for each object i in $\mathbf{I}$ the square* $\begin{array}{ccc} \widetilde{X} i & \longrightarrow & \operatorname{hocolim} X \\ \downarrow & & \downarrow \\ \widetilde{Y} i & \longrightarrow & \operatorname{hocolim} Y \end{array}$ *is a pull-back square.*

Proof. First, we note that (b) follows without need of the hypotheses (1) and (2). This is because for each $n \geq 0$, the square

$$\begin{array}{ccc} \coprod_{i_0 \to \cdots \to i_n \to i} X i_0 & \longrightarrow & \coprod_{i_0 \to \cdots \to i_n} X i_0 \\ \downarrow & & \downarrow \\ \coprod_{i_0 \to \cdots \to i_n \to i} Y i_0 & \longrightarrow & \coprod_{i_0 \to \cdots \to i_n} Y i_0 \end{array}$$

is a pullback square by the distributive law (3.7), and taking diagonals of bisimplicial objects commutes with limits.

To show (a), we consider the square

$$\begin{array}{ccc}\coprod_{i_0\to\cdots\to i_n} Xi_0 & \longrightarrow & \coprod_{j_0\to\cdots\to j_m} Xj_0 \\ \downarrow & & \downarrow \\ \coprod_{i_0\to\cdots\to i_n} Yi_0 & \longrightarrow & \coprod_{j_0\to\cdots\to j_m} Yj_0\end{array}$$

where the horizontal arrows are induced by a map $\delta\colon [m] \to [n] \in \Delta$. The vertical arrows are sharp by (5.1, P5), and the square is homotopy cartesian using (7.2) . (In fact, the square is a *pullback* square except when δ is a simplicial operator for which $\delta(0) \neq 0$, in which case the square is only homotopy cartesian.) The result then follows from (6.1). □

Lemma 7.2. *In $s\mathcal{E}$, an arbitrary coproduct of homotopy cartesian squares is homotopy cartesian.*

Proof. A coproduct of weak equivalences is a weak equivalence by (3.13), and a coproduct of pullback squares is a pullback square by the distributive law (3.7). Thus it suffices to factor the sides of each square into a weak equivalence followed by a fibration and demonstrate the result for the resulting pullback squares; since fibrations are sharp (2.2), the coproduct of sharp maps is sharp (5.1, P5), and pullbacks along sharp maps are homotopy cartesian, the result follows. □

Proof of (1.4). To prove (1), choose a factorization

$$\operatorname{colim}_{\mathbf{I}} X \xrightarrow{j} W' \xrightarrow{p} \operatorname{colim}_{\mathbf{I}} Y$$

such that j is a weak equivalence and p is sharp (e.g., a fibration). Define an $\mathbf{I}$-diagram X' by $X'i = W' \underset{\operatorname{colim} Y}{\times} Yi$; by the distributive law (3.7) we see that $\operatorname{colim}_{\mathbf{I}} X' \approx W'$. Note also that the induced map $Xi \to X'i$ is a weak equivalence, since p is sharp and by the hypothesis that each square (1.2) is homotopy cartesian. In the diagram

$$\begin{array}{ccccc}\operatorname{colim} \widetilde{X} & \xrightarrow[\sim]{i} & \operatorname{colim} \widetilde{X'} & \longrightarrow & \operatorname{colim} \widetilde{Y} \\ \downarrow{\scriptstyle k} & & \downarrow & & {\scriptstyle \ell}\downarrow{\scriptstyle \sim} \\ \operatorname{colim} X & \xrightarrow[\sim]{j} & \operatorname{colim} X' & \xrightarrow{p} & \operatorname{colim} Y\end{array}$$

the map p is sharp and the indicated maps are weak equivalences; that i and ℓ are weak equivalences follows from the homotopy invariance of homotopy colimits and the hypothesis that Y is a homotopy colimit diagram. Thus to show that k is a weak equivalence, and hence that X is a homotopy colimit diagram, it suffices to show that the right-hand square is a pull-back square.

Since each $X'i$ is defined to be the pullback of $\operatorname{colim} X' \to \operatorname{colim} Y$ along a map $Yi \to \operatorname{colim} Y$, we see that $\widetilde{X}i$ is the pullback of $\operatorname{colim} X'$ along the composite map $\widetilde{Y}i \to Yi \to \operatorname{colim} Y$. The assertion that the right-hand square is a pullback square now follows using the distributive law (3.7).

To prove (2), choose a factorization of $f\colon X \to Y$ into $X \xrightarrow{j} X' \xrightarrow{p} Y$, in which j is an object-wise weak equivalence and p is an object-wise fibration, and hence object-wise sharp. Then the square

$$\begin{array}{ccc} \widetilde{X'}i & \longrightarrow & \operatorname{colim}_{\mathbf{I}} \widetilde{X'} \\ \big\downarrow & & \big\downarrow \\ \widetilde{Y}i & \longrightarrow & \operatorname{colim}_{\mathbf{I}} \widetilde{Y} \end{array}$$

is homotopy cartesian by (7.1), and since X is by hypothesis a homotopy colimit diagram it follows that this square is weakly equivalent to (1.2), and we get the desired result. □

8. Lemmas on sharp maps of special diagrams

In this section we show that for special kinds of maps of diagrams, the induced map of colimits is sharp. These results were the key element of the proof of (6.1).

Proposition 8.1. *Let* $\mathbf{I}$ *denote a small category, and* $p\colon X \to Y$ *a map of* $\mathbf{I}$-*diagrams in* $s\mathcal{E}$. *Suppose that* $pi\colon Xi \to Yi$ *is sharp for each* $i \in \operatorname{ob}\mathbf{I}$, *and that*

$$\begin{array}{ccc} Xi & \xrightarrow{X\alpha} & Xj \\ {\scriptstyle pi}\big\downarrow & & \big\downarrow{\scriptstyle pj} \\ Yi & \xrightarrow{Y\alpha} & Yj \end{array}$$

is a homotopy cartesian square for each $\alpha\colon i \to j$ *in* $\mathbf{I}$. *Then under each of the following cases (1)–(3), the induced map* $\operatorname{colim}_{\mathbf{I}} X \to \operatorname{colim}_{\mathbf{I}} Y$ *is sharp, and for each* $i \in \operatorname{ob}\mathbf{I}$, *the square*

$$\begin{array}{ccc} Xi & \longrightarrow & \operatorname{colim}_{\mathbf{I}} X \\ \big\downarrow & & \big\downarrow \\ Yi & \longrightarrow & \operatorname{colim}_{\mathbf{I}} Y \end{array}$$

is homotopy cartesian.

(1) $\mathbf{I}$ *is the category obtained from the poset* $\mathbb{N}$ *of non-negative integers, and each map* $X(n) \to X(n+1)$ *and* $Y(n) \to Y(n+1)$ *is a monomorphism.*
(2) $\mathbf{I} = (i_1 \xleftarrow{\alpha} i_0 \xrightarrow{\beta} i_2)$ *and* $X\beta$ *and* $Y\beta$ *are monomorphisms.*
(3) $\mathbf{I}$ *is the category obtained from the poset* $\bar{\mathcal{P}}S$ *of* proper *subsets of a finite set* S, *and* X *and* Y *are cofibrant diagrams in the sense of (6.3).*

Lemma 8.2. *Consider a countable sequence of maps over* B

$$Y(0) \xrightarrow{i_0} Y(1) \xrightarrow{i_1} Y(2) \xrightarrow{i_2} \ldots \to B$$

such that each i_n *is a trivial cofibration, and each map* $q_n\colon Y(n) \to B$ *is sharp. Then the induced map* $q\colon \operatorname{colim}_n Y(n) \to B$ *is sharp.*

Proof. Given a map $f\colon A \to B$, consider the pullbacks $X(n) = Y(n) \times_B A$. By the distributive law (3.7), $\operatorname{colim}_n X(n) = \operatorname{colim}_n Y(n) \times_B A$. Since each map $Y(n) \to B$ is sharp and each i_n is a weak equivalence, it follows that each $X(n) \to X(n+1)$ is a weak equivalence and thus a trivial cofibration. Thus the composite $X(0) \to \operatorname{colim}_n X(n)$ is a trivial cofibration, and so base-change of q along f yields a homotopy cartesian square. □

Proof of part 1 of (8.1). Let $X'(n) = Y(n) \times_{\operatorname{colim} Y(n)} \operatorname{colim} X(n)$, whence we have that $\operatorname{colim} X'(n) \approx \operatorname{colim} X(n)$ by the distributive law (3.7). It suffices to show

(1) that each map $X'(n) \to Y(n)$ is sharp, and
(2) that each map $X(n) \to X'(n)$ is a weak equivalence.

This is because (1), together with (4.1, P4) and the fact that $\coprod_n Y(n) \to \operatorname{colim} Y(n)$ is epi, implies that $\operatorname{colim} X(n) \to \operatorname{colim} Y(n)$ is sharp, and (2) then demonstrates that the appropriate squares are homotopy cartesian.

Let $X(n,m) = Y(n) \times_{Y(m)} X(m)$ for $m \geq n$. Then $X'(n) \approx \operatorname{colim}_m X(n,m)$ by the distributive law. We have that $X(n,n) \approx X(n)$, and each map $X(n,m) \to X(n,m+1)$ is a weak equivalence since $X(m+1) \to Y(m+1)$ is sharp. Thus $X(n) \to \operatorname{colim}_m X(n,m) \approx X'(n)$ is a weak equivalence, proving (2). Claim (1) follows from (8.2) applied to the sequence $X(n,m)$ over $Y(n)$. □

The following lemma describes conditions under which one may "glue" an object onto a sharp map and still obtain a sharp map.

Lemma 8.3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \longrightarrow & B \end{array}$$

be a commutative diagram such that the top square is a push-out square, p, pf, and qg are sharp, f is a weak equivalence, and either i or f is a monomorphism. Then q is also sharp.

Proof. It suffices by (2.7) to show that every base-change of q along a map $U \to B$ produces a homotopy cartesian square. Since qg is sharp it suffices to show that

$$U \underset{B}{\times} g \colon U \underset{B}{\times} Y \to U \underset{B}{\times} Y'$$

is a weak equivalence. Via the pushout square

$$\begin{array}{ccc} U \underset{B}{\times} X & \xrightarrow[U \times_B i]{} & U \underset{B}{\times} Y \\ {\scriptstyle U \times_B f}\downarrow & & \downarrow \\ U \underset{B}{\times} X' & \longrightarrow & U \underset{B}{\times} Y' \end{array}$$

in which either the top or the left arrow is a cofibration, we see that it suffices to show that $U \times_B f$ is a weak equivalence, since $s\mathcal{E}$ is a left-proper model category.

In fact, $U \times_B f \approx (U \times_B A) \times_A f$; that is, $U \times_B f$ is a base-change of f along a map into A. Thus since p and pf are sharp, this base-change of f is a weak equivalence, as desired. □

We have need of the following peculiar lemma.

Lemma 8.4. *In a Grothendieck topos $\mathcal{E}$ consider a diagram of the form*

$$\begin{array}{ccc} A & \rightarrowtail & X \\ {\scriptstyle p}\downarrow & & \downarrow \\ A' & \rightarrowtail & X' \\ \downarrow & & \downarrow{\scriptstyle q} \\ B & \rightarrowtail & Y \end{array}$$

in which the horizontal arrows are mono, the top square is a pushout square, and the large rectangle is a pullback rectangle. Then the bottom square is also a pullback square.

Proof. It suffices to show that the lemma holds in a Boolean localization of $\mathcal{E}$. In this case every subobject has a complement, so we may write $X = A \amalg C$, $X' = A' \amalg C'$, and $Y = B \amalg D$. To show that the lower square is a pullback, it suffices to show that $q(C') \subset D$. Since the top square is a pushout, $p(C) = C'$, and since the big rectangle is a pullback, $qp(C) \subset D$, producing the desired result. □

Proof of part 2 of (8.1). We have a diagram of the form

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_0 & \overset{i}{\rightarrowtail} & X_2 \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_2} \\ Y_1 & \longleftarrow & Y_0 & \overset{j}{\rightarrowtail} & Y_2 \end{array} \tag{8.5}$$

where p_n is sharp for $n = 0, 1, 2$, each square is homotopy cartesian, and i and j are mono. We must show that the induced map $X_{12} \to Y_{12}$ of pushouts is sharp, and that each square

$$\begin{array}{ccc} X_n & \longrightarrow & X_{12} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_{12}} \\ Y_n & \longrightarrow & Y_{12} \end{array} \tag{8.6}$$

is homotopy cartesian for $n = 0, 1, 2$.

We prove the claim by proving it for the following cases:

(a) under the additional hypothesis that both of the squares in (8.5) are pullback squares,
(b) under the additional hypothesis that the right-hand square in (8.5) is a pull back square, and
(c) under no additional hypotheses.

In case (a), each square of the form (8.6) is necessarily a pullback square since i and j are mono; this can be seen by passing to a boolean localization, in which case $X_2 \approx X_0 \amalg X_2'$ and $Y_2 \approx Y_0 \amalg Y_2'$ so that $X_{12} \approx X_1 \amalg X_2'$ and $Y_{12} \approx Y_1 \amalg Y_2'$. Thus $p_{12} \colon X_{12} \to Y_{12}$ must be sharp using (4.1, P4), since the pullback of p_{12} along the epimorphism $Y_1 \amalg Y_2 \to Y_{12}$ is sharp.

In case (b), we let $X_0' = Y_0 \underset{Y_1}{\times} X_1$ and $X_2' = X_0' \underset{X_0}{\cup} X_2$, obtaining a diagram of the form

$$\begin{array}{ccccc}
 & & X_0 & \overset{i}{\rightarrowtail} & X_2 \\
 & & \sim\downarrow & & \sim\downarrow \\
X_1 & \longleftarrow & X_0' & \rightarrowtail & X_2' \\
p_1\downarrow & & p_0'\downarrow & & \downarrow p_2' \\
Y_1 & \longleftarrow & Y_0 & \overset{j}{\rightarrowtail} & Y_2
\end{array}$$

The map p_0' is sharp since it is a base-change of the sharp map p_1. The map p_2' is sharp by (8.3) since i is mono. The lower right-hand square is a pullback square by (8.4). Then the claim reduces to case (a), since $X_1 \cup_{X_0'} X_2' \approx X_{12}$.

In case (c), let $X_0' = Y_0 \underset{Y_2}{\times} X_2$ and $X_1' = X_1 \underset{X_0}{\cup} X_0'$, obtaining a diagram of the form

$$\begin{array}{ccccc}
X_1 & \longleftarrow & X_0 & & \\
\sim\downarrow & & \downarrow\sim & & \\
X_1' & \longleftarrow & X_0' & \overset{i'}{\rightarrowtail} & X_2 \\
p_1'\downarrow & & p_0'\downarrow & & \downarrow p_2 \\
Y_1 & \longleftarrow & Y_0 & \overset{j}{\rightarrowtail} & Y_2
\end{array}$$

The map p_0' is the base-change of a sharp map p_2 and hence is sharp; the map p_1' is sharp by (8.3) (note that $X_0 \to X_0'$ is mono). Thus the claim reduces to case (b), since $X_1' \cup_{X_0'} X_2 \approx X_{12}$. □

Proof of part 3 of (8.1). Let $S = \{1, \dots, n\}$; we prove the result by induction on n. The cases $n = 0, 1$ are trivial, and case $n = 2$ follows from (8.1, part 2).

For a set T, as in (6.3) let $\bar{P}T$ denote the poset of *proper* subsets of T as in (6.3). Then (6.4) provides a pushout square

$$\begin{array}{ccc}
\operatorname{colim}_{\bar{P}S'} X|_{S'} & \longrightarrow & \operatorname{colim}_{\bar{P}S'} X' \\
\downarrow & & \downarrow \\
X(S') & \longrightarrow & \operatorname{colim}_{\bar{P}S} X
\end{array}$$

in which the vertical arrows are mono; here $S' = \{1, \dots, n-1\}$. There is a similar diagram for Y. One now deduces the result by induction on the size of S, applying (8.1, part 2) to the above square to carry out the induction step.

Note that in order to apply the induction step, we need to know that the square

$$\begin{array}{ccc}
\operatorname{colim}_{\bar{P}S'} X|_{S'} & \longrightarrow & \operatorname{colim}_{\bar{P}S'} X' \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\bar{P}S'} Y|_{S'} & \longrightarrow & \operatorname{colim}_{\bar{P}S'} Y'
\end{array}$$

is homotopy cartesian. This follows by induction from the fact that each of the squares in

$$\begin{array}{ccccccc}
\operatorname{colim}_{\bar{\mathcal{P}}S'} X|_{S'} & \longrightarrow & X(S') & \longrightarrow & X(S) & \longleftarrow & \operatorname{colim}_{\bar{\mathcal{P}}S'} X' \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\operatorname{colim}_{\bar{\mathcal{P}}S'} Y|_{S'} & \longrightarrow & Y(S') & \longrightarrow & Y(S) & \longleftarrow & \operatorname{colim}_{\bar{\mathcal{P}}S'} Y'
\end{array}$$

are homotopy cartesian. □

9. Sharp maps in a boolean localization

In this section we go back to prove the results needed in the proof of (4.1) on sharp maps in a boolean localization. In the following $\mathcal{B}$ denotes a complete boolean algebra.

Proposition 9.1. *Let $f\colon X \to Y$ be a map in $s\mathrm{Sh}\mathcal{B}$. The following are equivalent.*

(1) *f is sharp.*

(2) *For all $n \geq 0$ and all $S_n \to Y_n$ in $\mathrm{Sh}\mathcal{B}$ the induced pullback square*

$$\begin{array}{ccc}
P & \longrightarrow & X \\
\big\downarrow & & \big\downarrow{\scriptstyle f} \\
S_n \times \Delta[n] & \longrightarrow & Y
\end{array}$$

is homotopy cartesian.

(3) *For each $n \geq 0$ there exists an epimorphism $S_n \to Y_n$ in $\mathrm{Sh}\mathcal{B}$ such that for each map $\delta\colon \Delta[m] \to \Delta[n]$ of standard simplices, the induced diagram of pullback squares*

$$\begin{array}{ccccc}
P & \xrightarrow{\quad h \quad} & P' & \longrightarrow & X \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\
S_n \times \Delta[m] & \xrightarrow{1\times\delta} & S_n \times \Delta[n] & \longrightarrow & Y
\end{array}$$

is such that h is a weak equivalence.

Let γ be an ordinal, viewed as a category. Given a functor $X\colon \gamma \to s\mathcal{E}$ such that $\operatorname{colim}_{\alpha<\beta} X(\alpha) \approx X(\beta)$ for each limit ordinal $\beta < \gamma$, we call the induced map $X(0) \to \operatorname{colim}_{\alpha<\gamma} X(\alpha)$ a **transfinite composition** of the maps in X.

Proof. The implications (1) implies (2) implies (3) are straightforward, so it suffices to prove (3) implies (1).

Consider a diagram of pullback squares

$$\begin{array}{ccccc}
A & \xrightarrow{g} & A' & \longrightarrow & X \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\
B & \xrightarrow{h} & B' & \longrightarrow & Y
\end{array} \tag{9.2}$$

with h a weak equivalence and f as in (3). We want to show that g is a weak equivalence. By factoring h into a trivial cofibration followed by a trivial fibration, we see that we can reduce to the case when h is a trivial cofibration.

Let $\mathcal{C}$ denote the class of trivial cofibrations $h\colon B \to B'$ such that for all diagrams of the form (9.2) in which the squares are pullbacks, the map g is a weak equivalence. In order to show that $\mathcal{C}$ contains *all* trivial cofibrations, it will suffice by (10.16) to show that $\mathcal{C}$

(1) is closed under retracts,
(2) is closed under cobase-change,
(3) is closed under transfinite composition, and
(4) contains all maps of the form $U \times \Lambda^k[n] \to U \times \Delta[n]$ where $U \in \mathcal{E}$ is a discrete object.

Part (1) is straightforward. Part (2) follows from the fact that pullbacks of monomorphisms are monomorphisms, and the fact that the cobase-change of a trivial cofibration is again a trivial cofibration. Part (3) follows from the fact that a transfinite composite of trivial cofibrations is a trivial cofibration, and from the fact that if $\{Y_\alpha\}$ is some sequence and $f\colon X \to Y = \operatorname{colim}_\alpha Y_\alpha$ a map, then $\operatorname{colim}_\alpha Y_\alpha \times_Y X \approx X$ by the distributive law (3.7).

Part (4) is (9.3). $\square$

Let $\Lambda^k[n] \subset \Delta[n]$ denote the "k-th horn" of the standard n-simplex; that is, $\Lambda^k[n]$ is the largest subcomplex of $\Delta[n]$ not containing the k-th face. The following lemma, though simple, is crucial to proving anything about sharp maps. It is essentially Lemma 7.4 of Chachólski [2], at least in the case when $\mathrm{Sh}\mathcal{B} = \mathcal{S}\mathrm{et}$.

Lemma 9.3. *Let $f\colon X \to Y$ be a map in $s\mathrm{Sh}\mathcal{B}$, and let $U_n \to Y_n$ be some map in $\mathrm{Sh}\mathcal{B}$. Suppose that for each map $\delta\colon \Delta[m] \to \Delta[n]$ of standard simplices the map i in the diagram*

$$\begin{array}{ccccc} P_\delta & \xrightarrow{\quad i \quad} & P & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ U_n \times \Delta[m] & \xrightarrow{1\times\delta} & U_n \times \Delta[n] & \longrightarrow & Y \end{array}$$

of pullback squares is a weak equivalence. Then for any inclusion $\Lambda^k[n] \to \Delta[n]$ of a simplicial horn into a standard simplex the map j in the diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\quad j \quad} & P & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ U_n \times \Lambda^k[n] & \xrightarrow{1\times\delta} & U_n \times \Delta[n] & \longrightarrow & Y \end{array}$$

of pullback squares is a weak equivalence.

Proof. Let $S = \{0, \ldots, n\}$ be a set, and identify S with the set of vertices of $\Delta[n]$. There is a functor $F\colon \mathcal{P}S \to \mathcal{S}$ sending $T \subseteq S$ to the smallest subobject of $\Delta[n]$ containing the vertices T; thus $F(T) \approx \Delta[|T|]$. Note that F is a cofibrant functor in the sense of (6.3). Let $S' = S \setminus \{k\}$, and define $\widetilde{F}\colon \mathcal{P}S' \to \mathcal{S}$ by $\widetilde{F}(T) = F(T \cup \{k\})$. Then $\operatorname{colim}_{\overline{\mathcal{P}}S'} \widetilde{F} \approx \Lambda^k[n]$, and $\widetilde{F}(S') \approx \Delta[n]$, and $\widetilde{F}$ is a cofibrant functor.

Now define $G\colon \mathcal{P}S' \to s\mathrm{Sh}\mathcal{B}$ by $G(T) = X \times_Y (U_n \times \widetilde{F}(T))$. Then $\operatorname{colim}_{\overline{\mathcal{P}}S'} G \approx Q$ by the distributive law (3.7) and $G(S') \approx P$, and the lemma follows immediately from (6.5), since G is a cofibrant functor. $\square$

10. Local fibrations are global fibrations in a boolean localization

The purpose of this section is to prove (3.14), as well as (10.16), which was used in Section 9. It is possible with some work to derive these facts from Jardine's construction of the model category on $s\mathrm{Sh}\mathcal{B}$. However, it seems more enlightening (and no more difficult) to proceed by constructing the model category structure on $s\mathrm{Sh}\mathcal{B}$ from scratch, and showing that it has the desired properties while coinciding with Jardine's structure; it turns out that the construction of the model category structure on $s\mathrm{Sh}\mathcal{B}$ is somewhat simpler than the more general case of simplicial sheaves on an arbitrary Grothendieck site.

10.1. **Sheaves on a complete boolean algebra.** Let $\mathcal{B}$ be a **complete boolean algebra**. Thus $\mathcal{B}$ is a complete distributive lattice with minimal and maximal elements 0 and 1, such that every $b \in \mathcal{B}$ has a complement $\bar{b}$; that is, if $\vee$ denotes meet and $\wedge$ denotes join, then $b \vee \bar{b} = 1$ and $b \wedge \bar{b} = 0$. We view $\mathcal{B}$ as a category, with a map $b \to b'$ whenever $b \leq b'$.

A **presheaf** on $\mathcal{B}$ is a functor $X \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. A **sheaf** on $\mathcal{B}$ is a presheaf X such that for a collection of elements $\{b_i \in \mathcal{B}\}_{i\in I}$, the diagram

$$X(b) \longrightarrow \prod_{i\in I} X(b_i) \rightrightarrows \prod_{i,j\in I} X(b_i \wedge b_j) \tag{10.2}$$

is an equalizer whenever $\bigvee_{i\in I} b_i = b$.

Say a collection of elements $\{b_i \in \mathcal{B}\}_{i\in I}$ is a **decomposition** of $b \in \mathcal{B}$ if $\bigvee_{i\in I} b_i = b$ and $b_i \wedge b_j = 0$ if $i \neq j$. We write $\coprod_{i\in I} b_i = b$ to denote a decomposition of b. The collection of decompositions of $b \in \mathcal{B}$ forms a directed set under refinement, with the trivial decomposition $\{b\}$ as minimal element.

Proposition 10.3. *A presheaf $X \in \mathrm{Psh}\mathcal{B}$ is a sheaf if and only if for each decomposition $b = \coprod_{i\in I} b_i$ of b, the induced map $X(b) \to \prod_{i\in I} X(b_i)$ is an isomorphism.*

Proof. The only if statement follows from the definition of a sheaf. To prove the if statement, suppose $\bigvee_{i\in I} b_i = b$. For each $S \subset I$, let

$$b_S = b \wedge \left(\bigwedge_{i\in S} b_i\right) \wedge \left(\bigwedge_{j\in S\setminus I} \bar{b}_j\right).$$

Note that for $T \subset I$ we have that $\bigvee_{i\in T} b_i = \coprod_{S\subset T} b_S$ and $\bigwedge_{i\in T} b_i = \coprod_{S\supset T} b_S$. In particular, $b_\varnothing = 0$ and $b = \coprod_{S\subset I} b_S$.

To show that X is a sheaf, we need to show that every sequence of the form (10.2) is exact. By hypothesis this sequence is isomorphic to the diagram

$$\prod_{S\subset I} X(b_S) \to \prod_{i\in I}\prod_{T\ni i} X(b_T) \rightrightarrows \prod_{i,j\in I}\prod_{U\ni i,j} X(b_U).$$

But the above sequence is manifestly exact, since it is a product of sequences of the form

$$X(b_S) \to \prod_{i\in S} X(b_S) \rightrightarrows \prod_{i,j\in S} X(b_S)$$

for each $S \subset I$. □

Define a functor $L \colon \mathrm{Psh}\mathcal{B} \to \mathrm{Psh}\mathcal{B}$ by

$$(LX)(b) = \operatorname*{colim}_{b=\amalg b_i} \prod_i X(b_i),$$

the colimit being taken over the directed set of decompositions of b. There is a natural transformation $\eta\colon X \to LX$ corresponding to the trivial decompositions of $b \in \mathcal{B}$. Typically, one proves that the composite functor $L^2 = L \circ L$ is a sheafification functor for sheaves on $\mathcal{B}$. The following shows that L *is itself a sheafification functor.*

Proposition 10.4. *Given $X \in \mathrm{Psh}\mathcal{B}$, the object LX is a sheaf; furthermore, $\eta_X\colon X \to LX$ is an isomorphism if X is a sheaf. Thus L induces a sheafification functor $L\colon \mathrm{Psh}\mathcal{B} \to \mathrm{Sh}\mathcal{B}$ left adjoint to inclusion $\mathrm{Sh}\mathcal{B} \to \mathrm{Psh}\mathcal{B}$.*

Proof. The proof is a straightforward element chase, using (10.3) and the fact that decompositions of an element $b \in \mathcal{B}$ form a directed set. □

10.5. **A model category structure for $s\mathrm{Sh}\mathcal{B}$.** Let $f\colon X \to Y$ be a map in $s\mathrm{Sh}\mathcal{B}$. Say that f is

(1) a **cofibration** if it is a monomorphism,
(2) a **fibration** if each $X(b) \to Y(b)$ is a Kan fibration for all $b \in \mathcal{B}$, and
(3) a **weak equivalence** if $(L\,\mathrm{Ex}^\infty X)(b) \to (L\,\mathrm{Ex}^\infty Y)(b)$ is a weak equivalence of simplicial sets.

Here $\mathrm{Ex}^\infty\colon \mathrm{Sh}\mathcal{B} \to \mathrm{Psh}\mathcal{B}$ denotes the functor obtained by applying Kan's Ex^∞ functor at each $b \in \mathcal{B}$. This functor commutes with finite limits and preserves fibrations; the same is true of the composite $L\,\mathrm{Ex}^\infty\colon \mathrm{Sh}\mathcal{B} \to \mathrm{Sh}\mathcal{B}$.

We say a map is a **trivial cofibration** if it is both a weak equivalence and a cofibration, and we say a map is a **trivial fibration** if it is both a weak equivalence and a fibration. We say an object X is **fibrant** if the map $X \to 1$ to the terminal object is a fibration.

Theorem 10.6. *With the above structure, $s\mathrm{Sh}\mathcal{B}$ is a closed model category in the sense of Quillen.*

Proof of (3.14) using (10.6). The cofibrations and weak equivalences in a closed model category determine the fibrations. Since $\mathrm{Sh}\mathcal{B}$ serves as its own boolean localization, the weak equivalences (resp. fibrations) of (10.6) are precisely the local weak equivalences (resp. local fibrations) of (3.10). Thus the two model category structures coincide, and thus local fibrations are model category theoretic fibrations. □

10.7. **Characterization of trivial fibrations.**

Lemma 10.8. *If $X \in s\mathrm{Sh}\mathcal{B}$ is fibrant, then $X(b) \to (L\,\mathrm{Ex}^\infty X)(b)$ is a weak equivalence for each $b \in \mathcal{B}$.*

Proof. Since X is fibrant, each map $X(b) \approx \prod_i X(b_i) \to \prod_i \mathrm{Ex}^\infty X(b_i)$ is a product of weak equivalences between fibrant simplicial sets, and thus is a weak equivalence. Thus the map

$$X(b) \to (L\,\mathrm{Ex}^\infty X)(b) \approx \operatorname*{colim}_{b=\amalg b_i} \prod_i \mathrm{Ex}^\infty X(b_i)$$

is a weak equivalence, since the colimit is taken over a directed set. □

Corollary 10.9. *If $X, Y \in s\mathrm{Sh}\mathcal{B}$ are fibrant, then $f\colon X \to Y$ is a weak equivalence if and only if $f(b)\colon X(b) \to Y(b)$ is a weak equivalence for each $b \in \mathcal{B}$*

The following lemma implies that if $f\colon X \to Y \in s\mathrm{Sh}\mathcal{B}$ is a map such that each $f(b)$ is a weak equivalence of simplicial sets, then f is a weak equivalence.

Lemma 10.10. *If $f\colon X \to Y \in s\mathrm{Psh}\mathcal{B}$ is a map of simplicial presheaves such that each $f(b)\colon X(b) \to Y(b)$ is a weak equivalence, then $Lf\colon LX \to LY$ is a weak equivalence of simplicial sheaves.*

Proof. First, note that if W is a simplicial presheaf, then $L\,\mathrm{Ex}^\infty LW \approx L\,\mathrm{Ex}^\infty W$. Now if each $f(b)$ is a weak equivalence, then so is each $\mathrm{Ex}^\infty f(b)$, and thus we conclude that $L\,\mathrm{Ex}^\infty f$ is a weak equivalence using (10.4). □

Let $y\colon \mathcal{B} \to \mathrm{Sh}\mathcal{B}$ denote the canonical functor sending b to the representable sheaf $\hom_{s\mathrm{Sh}\mathcal{B}}(-, b)$. Note that y identifies $\mathcal{B}$ with the category of subobjects of the terminal object 1 in $\mathrm{Sh}\mathcal{B}$.

Proposition 10.11. *A map $f\colon X \to Y \in s\mathrm{Sh}\mathcal{B}$ is a trivial fibration if and only if each $f(b)\colon X(b) \to Y(b)$ is a trivial fibration of simplicial sets.*

Proof. The "if" part follows immediately from (10.10) and the definition of fibrations.

To prove the "only if" part, note that since f is a fibration it suffices to show that for each vertex $v \in Y_0(b)$ that the fiber of $f(b)$ over v is a contractible Kan complex. Let $u\colon yb \to Y$ be the map representing v, and form the pullback square

$$\begin{array}{ccc} P & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ yb & \xrightarrow{u} & Y \end{array}$$

Note that g is a fibration, and that the inclusion $yb \to 1$ of discrete objects is easily seen to be a fibration. Thus P and yb are fibrant. Thus, to show that $P(b)$ is contractible it suffices by (10.9) to show that g is a weak equivalence, since $yb(b)$ is a point.

The functor $L\,\mathrm{Ex}^\infty$ preserves pullbacks, fibrations, and weak equivalences, and furthermore $L\,\mathrm{Ex}^\infty yb = yb$. Thus $(L\,\mathrm{Ex}^\infty f)(b')$ is a trivial fibration for each $b' \in \mathcal{B}$, whence so is each $(L\,\mathrm{Ex}^\infty g)(b')$, and thus g is a weak equivalence as desired. □

10.12. **Factorizations.** We produce factorizations of maps in $s\mathrm{Sh}\mathcal{B}$ by use of the "small object argument".

Choose an infinite cardinal $c > 2^{|\mathcal{B}|}$ and let γ be the smallest ordinal of cardinality c. Then for each $b \in \mathcal{B}$ the object $yb \in \mathrm{Sh}\mathcal{B}$ is **small** with respect to γ. That is, given a functor $X\colon \gamma \to \mathrm{Sh}\mathcal{B}$, any map $yb \to \mathrm{colim}_{\alpha<\gamma} X_\alpha$ factors through some X_β with $\beta < \gamma$.

Lemma 10.13. *Given $f\colon X \to Y$, there exists a factorization $f = pi$ as a cofibration i followed by a trivial fibration p.*

Proof. We inductively define a functor $X\colon \gamma \to s\mathrm{Sh}\mathcal{B}$ as follows. Let $X(0) = X$. Let $X(\alpha) = \mathrm{colim}_{\beta<\alpha} X(\beta)$ if $\alpha < \gamma$ is a limit ordinal. Otherwise, define $X(\alpha+1)$ by the pushout square

$$\begin{array}{ccc} \coprod yb \times \partial\Delta[n] & \longrightarrow & \coprod yb \times \Delta[n] \\ \downarrow & & \downarrow \\ X(\alpha) & \longrightarrow & X(\alpha+1) \end{array}$$

where the coproducts are taken over the set of all diagrams of the form

$$\begin{array}{ccc} yb \times \partial\Delta[n] & \longrightarrow & yb \times \Delta[n] \\ \downarrow & & \downarrow \\ X(\alpha) & \longrightarrow & Y \end{array}$$

The desired factorization is $X \to \operatorname{colim}_{\alpha<\gamma} X(\alpha) \to Y$. □

Lemma 10.14. *Trivial fibrations have the right lifting property with respect to cofibrations.*

Proof. Note that (10.11) and the "choice" axiom for $\mathrm{Sh}\mathcal{B}$ (3.3) implies that trivial fibrations are precisely the maps which have the right lifting property with respect to all maps of the form $S \times \partial\Delta[n] \to S \times \Delta[n]$, where S is any discrete object in $\mathrm{Sh}\mathcal{B}$. Thus the result follows when we note that if $i\colon A \to B$ is a monomorphism in $s\mathrm{Sh}\mathcal{B}$, then we can write $B_n \approx A_n \amalg S_n$ for some $S_n \in s\mathrm{Sh}\mathcal{B}$ since $\mathrm{Sh}\mathcal{B}$ is boolean, and in this way construct an ascending filtration $F_nB \subset B$ for $-1 \leq n < \infty$ such that

$$F_nB \approx F_{n-1}B \bigcup_{S_n \times \partial\Delta[n]} S_n \times \Delta[n],$$

$F_{-1}B \approx A$, and $\operatorname{colim}_n F_nB \approx B$. □

Let $\mathcal{C}$ denote the class of maps in $s\mathrm{Sh}\mathcal{B}$ which are retracts of transfinite compositions of pushouts along maps of the form $yb \times \Lambda^k[n] \to yb \times \Delta[n]$, where $b \in \mathcal{B}$ and $n \geq k \geq 0$.

Lemma 10.15. *Given $f\colon X \to Y$, there exists a factorization $f = qj$ as a map $j \in \mathcal{C}$ followed by a fibration q.*

Proof. We perform the small object argument by taking pushouts along coproducts of maps $yb \times \Lambda^k[n] \to yb \times \Delta[n]$, indexed by diagrams of the form

$$\begin{array}{ccc} yb \times \Lambda^k[n] & \longrightarrow & yb \times \Delta[n] \\ \downarrow & & \downarrow \\ X(\alpha) & \longrightarrow & Y \end{array}$$

Otherwise, the proof is similar to that of (10.13). □

Lemma 10.16. *The class $\mathcal{C}$ is precisely the class of trivial cofibrations.*

Proof. First we show that $\mathcal{C}$ consists of trivial cofibrations. It is already clear that every map in $\mathcal{C}$ is a cofibration.

Suppose that X is a simplicial sheaf. Consider the following pushout square in the category of simplicial *presheaves.*

$$\begin{array}{ccc} \coprod yb \times \Lambda^k[n] & \longrightarrow & \coprod yb \times \Delta[n] \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

That is, the products, coproducts, and pushouts are to be taken in the category of presheaves. It is then clear that $X(b') \to Y(b')$ is a weak equivalence for each $b' \in \mathcal{B}$, and thus by (10.10) the map $X \to LY$ is a weak equivalence in $s\mathrm{Sh}\mathcal{B}$; the sheaf LY is the pushout of the corresponding square of simplicial *sheaves*. Since the functor $L\,\mathrm{Ex}^{\infty}$ commutes with directed colimits in $s\mathrm{Sh}\mathcal{B}$, and since weak equivalences are closed under retracts, we may conclude that every map in $\mathcal{C}$ is a weak equivalence.

We show that any trivial cofibration $f\colon X \to Y$ is in $\mathcal{C}$ by a standard retract trick. Namely, by (10.15) we can factor $f = qj$ into a map $j \in \mathcal{C}$ followed by a fibration q. But q must also be a weak equivalence since f and j are, and hence q is a trivial fibration. Thus we can show using (10.14) that f is a retract of j and thus $f \in \mathcal{C}$. □

Proof of (10.6). Quillen's [11] axioms CM1, CM2, and CM3 are clear: $s\mathrm{Sh}\mathcal{B}$ has all small limits and colimits, the classes of fibrations, cofibrations, and weak equivalences are closed under retracts, and if any two of f, g, and gf are weak equivalences, then so is the third. The factorization axiom CM5 follows from (10.13), (10.15), and (10.16).

One half of the lifting axiom CM4 follows from (10.16), since fibrations are clearly characterized by having the right lifting property with respect to all maps of the form $yb \times \Lambda^k[n] \to yb \times \Delta[n]$. The other half of CM4 is (10.14). □

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