Isogenies, power operations, and homotopy theory

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Overview

Plan.

- Context: “power operations” in cohomology theories.
- Recent advances: Morava E-theories. Formal groups and isogenies.
- Applications and vistas.

K-theory

Motivating example.

K-theory (Grothendieck; Atiyah-Hirzebruch; 1950s)

$$K(X) = K^0(X) := \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{vector bundles } /X \end{array} \right\} / \sim$$

$$V \sim V_1 + V_2 \qquad \text{if} \qquad 0 \to V_1 \to V \to V_2 \to 0.$$

Functors on vector bundles give operations on $K(X)$, e.g.,:

$$V, W \mapsto V \otimes W, \qquad V \mapsto \Lambda^n V, \qquad V \mapsto \mathrm{Sym}^n V.$$

$K(X)$ is a Λ-ring (Grothendieck)

Functions $\lambda^n \colon K(X) \to K(X)$ satisfying axioms

$$\lambda^n(x+y) = \cdots, \qquad \lambda^n(xy) = \cdots, \qquad \lambda^m\lambda^n(x) = \cdots$$

$\ldots =$ explicit polynomials in $\lambda^i(x), \lambda^j(y)$.

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Equivariant K-theory

Compact Lie $G \curvearrowright X$.

Equivariant K-theory

$$K_G(X) = K(X /\!\!/ G) := \{G \text{ equivariant vb } /X\}/\sim .$$

(Atiyah, 1966) tensor power is an operation

$$V \mapsto V^{\otimes n}: \qquad K_G(X) \to K_{G \times \Sigma_n}(X) \approx K_G(X) \otimes R\Sigma_n.$$

$K_G(\text{point}) = K(\text{point} /\!\!/ G) = RG =$ representation ring of G.
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Λ-rings $\iff$ representation theory of symmetric groups

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Λ-ring structure is complicated to describe, but is easy for "nice" rings.

(Wilkerson, 1982)

R torsion free comm. ring:

$$\left\{ \begin{array}{c} \Lambda\text{-ring} \\ \text{structures on } R \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \{\psi^p \colon R \to R\}_{p \text{ prime}} \text{ lifts of Frobenius,} \\ \psi^p\psi^q = \psi^q\psi^p. \end{array} \right\}$$

- Adams operations ψ^n, $n \geq 1$; $\psi^m\psi^n = \psi^{mn}$, ring homomorphisms
  Adams congruence $\psi^p(x) \equiv x^p \mod p$, p prime
- any Λ-ring has $\psi^p, \theta^p \colon R \to R$ satisfying

$$\psi^p \text{ is a ring homomorphism,} \qquad \psi^p(x) = x^p + p\,\theta^p(x)$$

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$$\text{Line bundle } L \to X \quad \Longrightarrow \quad \psi^n(L) = L^{\otimes n} \text{ in } K(X)$$

Multiplicative group scheme

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}]) \approx \mathrm{Spec}(K(\mathrm{pt}/\!\!/ U(1)))$$

Adams operation $\Longrightarrow$ *isogeny* of $\mathbb{G}_m$:

$$(K(\mathrm{pt}/\!\!/ U(1)) \xrightarrow{\psi^n} K(\mathrm{pt}/\!\!/ U(1))) \quad \Longleftrightarrow \quad \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m$$

Isogeny: finite flat homomorphism of group schemes

Remarks.

- $\widehat{\mathbb{G}}_m = \mathrm{Spf}\, K(BU(1))$, multiplicative *formal group*
- These properties useful in classical applications (e.g., Adams work on vector fields on spheres, image of J, . . .)

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Other examples of power operations

$h^*(-)$ = generalized cohomology theory, commutative ring valued

Would like to have

$$h^*(X) \xrightarrow{P^n} h^*_{\Sigma_n}(X) = h^*(X \times B\Sigma_n) \qquad \text{refines of } n\text{th power } x \mapsto x^n$$

Do these exist? **Yes** if $h^*(-)$ represented by a *structured commutative ring spectrum* (= commutative S-algebra = E_∞-ring spectrum = ...)

Examples.

- **(Steenrod, 1953)** reduced power operations in $H^*(-, \mathbb{F}_p)$
  (Sq^i for $p = 2$, P^i for p odd)
- **(Voevodsky, 2001)** motivic reduced power operations
- **(Quillen, 1971)** power operations in bordism theories
  based on $M \mapsto M^{\times n} \curvearrowleft \Sigma_n$
  used to prove $\pi_* MU$ classifies formal group laws

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What is elliptic cohomology?

Theory	K-theory	elliptic cohomology
Group scheme	$\mathbb{G}_m$	elliptic curve
Cycles	vector bundles	???

??? = 2-dim conformal field theories? **(Segal, ...)**

Examples:

- **(Goerss-Hopkins-Miller)** tmf = "topological modular forms"
  associated to *universal* elliptic curve over $\mathcal{M}_{\mathrm{Ell}}$
  structured comm ring spectrum $\Longrightarrow$ power operations!
- **(Lurie)** *Equivariant* elliptic cohomology theories

  Open question: Which equivariant elliptic cohomology theories admit power operations?

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A nice example: Elliptic cohomology at the Tate curve

Tate curve $T[\![q]\!] = \text{“}\mathbb{C}^\times/q^{\mathbb{Z}}\text{”}$, defined over $\operatorname{Spec}\mathbb{Z}[\![q]\!]$.

Equivariant elliptic cohomology at Tate curve

$$\mathrm{Ell}_{\mathrm{Tate}}(X/\!\!/G) \underset{\text{approx}}{:=} K\Big(\mathcal{L}^{\mathrm{ghost}}(X/\!\!/G)/\!\!/U(1)\Big)$$

“ghost loops” = contstant loops; RHS is K of “twisted sectors” (see e.g., Ruan 2000, Lupercio-Uribe 2002)

(Ganter, 2007, 2013) Power operations for $\mathrm{Ell}_{\mathrm{Tate}}$

$\mathrm{Ell}_{\mathrm{Tate}}(X/\!\!/G)$ is an **elliptic Λ-ring**: *two* families of operations

$$\lambda^n\colon \mathrm{Ell}_{\mathrm{Tate}} \to \mathrm{Ell}_{\mathrm{Tate}}, \qquad \mu^m\colon \mathrm{Ell}_{\mathrm{Tate}} \to \mathrm{Ell}_{\mathrm{Tate}} \otimes_{\mathbb{Z}[\![q]\!]} \mathbb{Z}[\![q^{1/m}]\!]$$

$\{\lambda^n\}$ are Λ-ring structure, $\{\mu^m\}$ are Λ-ring homomorphisms

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A nice example: Elliptic cohomology at the Tate curve

Tate curve $T[\![q]\!] = \text{“}\mathbb{C}^\times/q^{\mathbb{Z}}\text{”}$, defined over $\operatorname{Spec}\mathbb{Z}[\![q]\!]$.

Equivariant elliptic cohomology at Tate curve

$$\mathrm{Ell}_{\mathrm{Tate}}(X/\!\!/G) \underset{\text{approx}}{:=} K\Big(\mathcal{L}^{\mathrm{ghost}}(X/\!\!/G)/\!\!/U(1)\Big)$$

“ghost loops” = contstant loops; RHS is K of “twisted sectors” (see e.g., Ruan 2000, Lupercio-Uribe 2002)

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Morava E-theory: introduction

- *Morava E-theories* are "designer cohomology theories" — manufactured using homotopy theory, not coming from "nature"
- some arise as completions of "natural" theories, e.g.

$$K_p^\wedge, \qquad \mathrm{Ell}^\wedge_{\text{s.-s. point}}$$

- have rich theory of power operations (Ando, Hopkins, Strickland, R.)

Goal: describe what we know about this theory (a lot)
Recall: Power operations for K-theory are "controlled" by isogenies of $\mathbb{G}_m$

Slogan

Power operations for Morava E-theories are "controlled" by "deformations" of Frobenius isogenies of 1-dimensional formal groups

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Let $G_0/\mathbb{F}_p$ = one dimensional commutative formal group of height $n \in \{1, 2, \ldots\}$.

(Morava, 1978; Goerss-Hopkins-Miller 1993–2004)

There exists a cohomology theory E_{G_0} (*Morava E-theory*) which

- is represented by a structured commutative ring spectrum
- is complex orientable; formal group $\mathrm{Spf}(E^0\mathbb{CP}^\infty)$ = *universal deformation* of G_0 (in sense of Lubin-Tate)

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Formal groups and complex oriented theories

Formal group is object locally described by a formal group law.

Formal group law (commutative, 1-dimensional)

$S(x,y) \in R[\![x,y]\!]$ satisfying axioms for abelian group:

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For future reference, we note the *p-series* of G_0:

$$[p](x) = \underbrace{S(x,S(x,\ldots S(x,x)))}_{x \text{ appears } p \text{ times}}$$

Complex oriented cohomology theory

Ring-valued cohomology theory E such that $E^*(\mathbb{CP}^\infty) = E^*[\![x]\!]$, and x restricts to fundamental class of $\mathbb{CP}^1 = S^2$.

Examples: $H^*(-,\mathbb{Z})$, K-theory, Ell, **Morava E-theories**, ...

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$G_0/\mathbb{F}_p$ formal group of *height* n (i.e., $[p]_{G_0}(x) = c\,x^{p^n} + O(x^{p^n+1})$, $c \neq 0$)
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Groupoid $\mathrm{Def}^0_{G_0}(R)$ of deformations of $G_0/\mathbb{F}_p$ to R

Deformation (G, α):

- G is a formal group over R,
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Isomorphism $(G, \alpha) \to (G', \alpha')$ of deformations:

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Classified up to canonical iso by Lubin and Tate:

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$\exists$ *universal deformation* $(G_{\mathrm{univ}}, \alpha_{\mathrm{univ}})$ over $A \approx \mathbb{Z}_p[\![a_1, \dots, a_{n-1}]\!]$

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Isogenies

Isogeny of formal groups over R

Homomorphism $f\colon G \to G'$ given locally over R by $f(x) = cx^n +$ higher degree terms, $c \in R^\times$. ($n = \deg f$)

$G_0/\mathbb{F}_p$ has a distinguished family of *Frobenius isogenies*

$$\mathrm{Frob}^r\colon G_0 \to G_0, \qquad r \geq 0,$$

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Category $\mathrm{Def}_{G_0}(R)$ of deformations of Frobenius

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The "pile" $\mathrm{Def} = \mathrm{Def}_{G_0}$

We have assignments

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If $\mathrm{Def}(R)$ were a groupoid, we would call it a (pre-)stack

Def is the "pile" of deformations of powers of Frob

Sheaves on Def

A *sheaf of modules* on Def is a collection of functors

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with compatibility wrt base change along local homomorphisms $R \to R'$

Likewise, a *sheaf of commutative rings* on Def is . . .

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Def is the "pile" of deformations of powers of Frob

Sheaves on Def

A *sheaf of modules* on Def is a collection of functors

$$A_R \colon \mathrm{Def}(R) \to (R\text{-modules})$$

with compatibility wrt base change along local homomorphisms $R \to R'$
Likewise, a *sheaf of commutative rings* on Def is ...

Notation: $\mathrm{Mod}(\mathrm{Def})$, $\mathrm{Com}(\mathrm{Def})$.

$\mathrm{Mod}(\mathrm{Def}) = \mathrm{Mod}(\Gamma)$ for a certain ring Γ

(Ando-Hopkins-Strickland 2004; see R. 2009)

Power operations make Morava E-cohomology E_{G_0} a functor

$$E^*(-)\colon \mathrm{Spaces} \to \mathrm{Com}^*(\mathrm{Def})$$

Key step **(Strickland 1997, 1998)**:
$E^0 B\Sigma_{p^r}/I$ classifies subgroups of rank p^r of deformations

Broader context: We have $E^*(X) = \pi_*(E^{X_+})$ where $A = E^{X_+}$ is

(i) a structured commutative E-algebra spectrum,

(ii) $K(n)$-local ($\Leftrightarrow \pi_* A$ complete wrt $(a_1, \dots, a_{n-1})$ in a suitable sense)

The real theorem is

(ibid)

π_* lifts to a functor

$$\pi_*\colon h\mathrm{Com}(E)_{K(n)} \to \mathrm{Com}^*(\mathrm{Def})$$

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Examples

$\mathrm{Mod}(\mathrm{Def})$ = modules for a certain ring Γ

Height 1

G_0 = multiplicative formal group; $E_{G_0} = K_p^\wedge$

$$\Gamma = \mathbb{Z}_p[\psi^p] \qquad \text{gen. by Adams operation } \psi^p$$

Height 2 **(R., arXiv:0812.1320)**

$G_0/\mathbb{F}_2$ = completion of s.-s. elliptic curve $y^2 + y = x^3$ over $\mathbb{F}_2$

$$\Gamma = \mathbb{Z}_2[\![a]\!]\langle Q_0, Q_1, Q_2\rangle \Big/ \left(\begin{array}{rcl} Q_0\,a &=& a^2\,Q_0 - 2a\,Q_1 + 6\,Q_2 \\ Q_1\,a &=& 3\,Q_0 + a\,Q_2 \\ Q_2\,a &=& -a\,Q_0 + 3\,Q_1 \\ \\ Q_1\,Q_0 &=& 2\,Q_2 Q_1 - 2\,Q_0 Q_2 \\ Q_2\,Q_0 &=& Q_0 Q_1 + a\,Q_0 Q_2 - 2\,Q_1 Q_2 \end{array} \right)$$

(Y. Zhu, 2014) gives similar description at height 2, $p = 3$

There is a uniform description of Γ/p at height 2, all primes p

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Properties of Γ

$n =$ height of $G_0/\mathbb{F}_p$

(Ando 1995)

$\mathrm{Center}(\Gamma) = \mathbb{Z}_p[\tilde{T}_1, \ldots, \tilde{T}_n]$, (*Hecke algebra*)

(R. arXiv:1204.4831)

Γ is *quadratic*, i.e.,

$$\Gamma \approx \text{Tensor alg.}(C_1) \,/\, (\text{ideal gen. by } C_2)$$

where C_1 and $C_2 \subseteq C_1 \otimes_{E_0} C_1$ are $E_0 = \mathbb{Z}_p[\![a_1, \ldots, a_{n-1}]\!]$ bimodules

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Γ is *Koszul*: have Γ-bimodule resolution

$$0 \leftarrow \Gamma \leftarrow \Gamma \otimes_{E_0} C_0 \otimes_{E_0} \Gamma \leftarrow \cdots \leftarrow \Gamma \otimes_{E_0} C_n \otimes_{E_0} \Gamma \leftarrow 0,$$

each C_k is E_0-bimod, free and f.g. as right E_0-mod; $C_0 = E_0$

$\implies gl.\dim(\Gamma) = 2n$

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Remark about "Koszul" property

(R. ibid)

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- This was conjectured by Ando-Hopkins-Strickland
- It is purely a theorem about formal algebraic geometry
- Only general proof is a purely "topological" proof, using ingredients:
  (1) $\Gamma =$ "primitives" of the Hopf algebra $\bigoplus_{m\geq 0} E_0(B\Sigma_m)$ **(Strickland)**
  (2) bar complex of Γ in degree k is "primitives" in $\bigoplus_{m_1,\dots,m_k} E_0 B(\Sigma_{m_1} \wr \cdots \wr \Sigma_{m_k})$
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  Proof inspired by role of partition complexes as "derivatives of identity functor" in Goodwillie's functor calculus

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Homotopy groups of $K(n)$-local E-algebras have some more structure:

$$\pi_*\colon h\mathrm{Com}(E)_{K(n)} \to \big(T\text{-algebras}\big)$$

"T-algebras" $=$ a complicated algebraic catgeory (like Λ-rings)

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R p-torsion free commutative E_*-algebra:

$$\left\{ \begin{array}{c} T\text{-algebra} \\ \text{structures on } R \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} A \in \mathrm{Com}^*(\mathrm{Def}) \text{ with } A(G_{\mathrm{univ}}) = R \\ \text{satisfying "Frobenius congruence"} \end{array} \right\}$$

Frobenius congruence: $Qx \equiv x^p \mod pR$ for a certain $Q \in \Gamma$

There is a (non-additive) *witness* to the Frobenius congruence:

$$\theta\colon R \to R \qquad \text{satisfying} \quad Qx = x^p + p\,\theta(x)$$

where R is a T-algebra

Congruence criterion

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Application 1: nilpotence

Easy consequence of existence of "witness" θ such that $Q(x) = x^p + p\,\theta(x), \quad Q(x+y) = Q(x) + Q(y)$:

If $A \in \mathrm{Com}(E)_{K(n)}$, then

$$x \in \pi_* A, \quad p^r x = 0 \quad \Longrightarrow \quad x^{(p+1)^r} = 0.$$

Idea: deduce relation $\theta(px) = p^{p-1}x - Q(x) = (p^{p-1} - 1)x^p - p\,\theta(x)$.
If $px = 0$, then $0 = x\,\theta(px) = -x^{p+1}$.
Mathew-Noel-Naumann observe this, and use it (with Nilpotence Theorem of Devinatz-Hopkins-Smith) to give an easy proof of a conjecture of May:

(Mathhew-Noel-Naumann 2014)

If $R =$ structured commutative ring spectrum, then the kernel of the Hurewicz map

$$\pi_* R \to H_*(R, \mathbb{Z})$$

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Application 2: units and orientations

$A =$ structured commutative ring $\implies$ *units spectrum* $\mathrm{gl}_1 A$

$$(\mathrm{gl}_1 A)^0(X) = (A^0(X))^\times$$

Question. Does there exist structured commutative ring map $MG \to A$, where $MG =$ spectrum representing bordism ($G \in \{U, SU, O, SO, \mathrm{Spin}, \dots\}$)?

Answer (May-Quinn-Ray-Tornehave 1977). Yes iff the composite

$$g \to o \xrightarrow{J} \mathrm{gl}_1 S \to \mathrm{gl}_1 A$$

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Logarithmic operations

Logarithmic operation: spectrum map $\ell\colon \mathrm{gl}_1 A \to A$

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$A = K_p^\wedge$: exists $\ell\colon \mathrm{gl}_1 K_p^\wedge \to K_p^\wedge$, giving $\ell\colon K_p^\wedge(X)^\times \to K_p^\wedge(X)$ by

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Must construct null-homotopy of $\alpha\colon \mathrm{string} \to \mathrm{gl}_1\mathrm{tmf}$

Proof idea:

- Above techniques give "locally defined" logarithms $\ell_n\colon \mathrm{gl}_1\mathrm{tmf}_p^\wedge \to \mathrm{tmf}_{K(n)}$, $n = 1, 2$, all primes p
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Application 3: derived indecomposables

Commutative ring k; augmented comm. k-algebra $\pi\colon R \to k$

Indecomposables

$$Q_k(R) := I/I^2, \qquad I = \mathrm{Ker}\big(\pi\colon R \to k\big)$$

("cotangent space at π")

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(Basterra 1999, Basterra-Mandell 2005) Derived version

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Also called *reduced topological André-Quillen homology/cohomology*

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HA = Eilenberg-MacLane spectrum, representing $H^*(-, A)$

(Sullivan 1977)

X = simply connected f. type space; $H\mathbb{Q}^{X_+}$ = rational cochains spectrum

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Yes: $K(n)$-local R, such as Morava E-theories

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E = Morava E-theory at height n; $X = S^{2d-1}$ odd dimensional sphere

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- Known power operations for $\mathrm{Ell}_{\mathrm{Tate}}$ and $\mathrm{Ell}^{\wedge}_{\text{s.-s.}}$ are consistent with this picture

Conjecturally, equivariant elliptic cohomologies which are etale over $\mathcal{M}_{\mathrm{Ell}}$ should be "classified" by the etale site of Isog

Vista: power operations in equivariant elliptic cohomology

Q: Does equivariant elliptic cohomology admit power operations?

- Analogue of Def:
  Isog = "pile" of all elliptic curves and isogenies between them
  $\Longrightarrow \mathrm{Mod}(\mathrm{Isog})$, $\mathrm{Com}(\mathrm{Isog})$
- $\mathrm{Mod}(\mathrm{Isog})$ has analog of Koszul property
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