COMPLEX ANALYTIC ELLIPTIC COHOMOLOGY AND DOUBLE LOOP GROUPS (TALK: REGENSBURG, JUNE 2017)

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Conjecturally, certain CFTs (in dimensions 0,1,2) give rise to classes in certain (equivariant) cohomology theories. The following picture is meant to be vague and imprecise (though some parts can be made absolutely precise).

(2,1,0) elliptic cohomology	$\Longrightarrow$	(2,1) Tate K-theory loop group repr	$\Longrightarrow$	(2) ?
		$\Uparrow$		$\Uparrow$
		(1,0) K-theory	$\Longrightarrow$	(1) 2-periodic de Rham character sheaves
				$\Uparrow$
				(0) de Rham $/\mathbb{C}$

?=Grojnowski's complex analytic elliptic cohomology. (You could also put Devoto's equivariant elliptic cohomology for finite groups here.)

The horizontal arrows are "restriction", and correspond to character maps in the cohomology theories. The vertical arrows are "dimensional reduction". There is an emerging picture of the vertical arrow in the middle column, at least on the level of cohomology (Ganter, Kitchloo, Z. Huan., others). I want to suggest a similar picture for (0) $\Longrightarrow$ (2).[1]

The double loop construction

Fix

- Σ = smooth orientable genus 1 surface, with diffeomorphism group $\mathrm{Diff}(\Sigma)$;
- G = topological group, usually a compact connected abelian Lie group (i.e., $U(1)^d$).

The **wreath product** of G by $\mathrm{Diff}(\Sigma)$ is the semidirect product

$$\mathcal{W}(G) = \mathcal{W}^{\Sigma}(G) := \mathrm{Diff}(\Sigma) \ltimes \mathrm{Map}(\Sigma, G)$$

with group law

$$(\phi_1, g_1) \cdot (\phi_2, g_2) := (\phi_1 \circ \phi_2, (g_1 \circ \phi_2) \cdot g_2), \qquad g_1, g_2 \in \mathrm{Map}(\Sigma, G), \quad \phi_1, \phi_2 \in \mathrm{Diff}(\Sigma).$$

(Replace Σ with a finite set to get the conventional wreath product construction.)

The wreath product is the *extended gauge group* of the trivial G-bundle over Σ:

$$(\phi, g) \in \mathcal{W}(G) \qquad \rightsquigarrow \qquad \begin{array}{ccc} \Sigma \times G & \xrightarrow{(x,y)\mapsto(\phi(x),\, y\cdot g(x)^{-1})} & \Sigma \times G \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow[\phi]{} & \Sigma \end{array}$$

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[1]Inspired by ideas of Ganter about the Tate K-theory case, see arXiv:1301.2754.

I.e., elements of $\mathcal{W}(G)$ are bundle maps which cover diffeomorphisms.

Given $G \curvearrowright X$ space with a G-action, define

$$\mathcal{P}(G \curvearrowright X) := \mathrm{Map}_G(\Sigma \times G, X)_{h\mathcal{W}(G)} = \mathrm{Map}(\Sigma, X)_{h\mathcal{W}(G)},$$

where $\mathcal{W}(G) \curvearrowright \mathrm{Map}(\Sigma, X)$ by $(\phi, g) \cdot f := (g \cdot f) \circ \phi^{-1}$.

Side remark. This construction is a kind of classifying space:

$$\left\{ T \dashrightarrow \mathrm{Map}_G(\Sigma \times G, X)_{h\mathcal{W}(G)} \right\} \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{ccc} & E & \\ {\scriptstyle \mathcal{W}(G) \curvearrowright \Sigma \times G \text{ bun}} \swarrow & & \searrow {\scriptstyle G \text{ eq}} \\ T & & X \end{array} \right\}$$

Note that $E \to T$ can be decomposed as

$$E \xrightarrow{p} S \xrightarrow{q} T,$$

where q is a Σ-bundle, and p is a principal G-bundle which is trivializable on each fiber of q.

Cohomology of the double loop construction

This is not quite what I want: the group $\mathcal{W}(G)$ has a connected part and a discrete quotient:

$$1 \to \mathcal{W}_0(G) \to \mathcal{W}(G) \to \overline{\mathcal{W}}(G) \to 1.$$

We need to separate out the discrete part of the action from the connected part. Set

$$\mathcal{P}_0(G \curvearrowright X) := \mathrm{Map}(\Sigma, X)_{h\mathcal{W}_0(G)}$$

which carries a resuidual $\overline{\mathcal{W}}(G)$-action.

We calculate $H^*(\mathcal{P}_0(G \curvearrowright *); \mathbb{C})$ together with its $\mathcal{W}(G)$-action. To do this, fix models:

- $\Sigma := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,
- $G := U(1)^d$.

We have weak equivalences

$$\mathrm{Map}(\mathbb{T}^2, U(1)^d) \xleftarrow{\sim} \mathrm{Hom}(\mathbb{T}^2, U(1)^d) \times U(1)^d = \mathbb{Z}^{d\times 2} \times U(1)^d,$$

and

$$\begin{aligned} \mathcal{W}(U(1)^d) &= \mathrm{Diff}(\mathbb{T}^2) \ltimes \mathrm{Map}(\mathbb{T}^2, U(1)^d) \\ &\approx \big(GL_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big) \ltimes \big(\mathbb{Z}^{d\times 2} \times U(1)^d\big) \\ &\approx \underbrace{\big(GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{d\times 2}\big)}_{\overline{\mathcal{W}}(G)} \ltimes \underbrace{\big(\mathbb{T}^2 \times U(1)^d\big)}_{\mathcal{W}_0(G)}. \end{aligned}$$

Thus

$$\begin{aligned} H^*(\mathcal{P}(U(1)^d \curvearrowright *)) &\approx H^* B\mathcal{W}_0(G) \\ &\approx H^* B(\mathbb{T}^2 \times U(1)^d) \\ &\approx \mathbb{C}[t_1, t_2, y_1, \dots, y_d], \end{aligned}$$

with action by $\overline{\mathcal{W}}(G) = GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{d\times 2}$:

$$\begin{aligned} A \cdot (t_1, t_2, y_1, \dots, y_d) &= (at_1 + bt_2,\ ct_1 + dt_2,\ y_1, \dots, y_d), \\ m \cdot (t_1, t_2, y_1, \dots, y_d) &= (t_1, t_2,\ y_1 + m_{11}t_1 + m_{12}t_2,\ \dots,\ y_d + m_{d1}t_1 + m_{d2}t_2), \end{aligned}$$

$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})$, $m \in \mathbb{Z}^{d\times 2}$.

Magically, this data presents interesting geometry.

$d = 0$: We get the evident action of $GL_2(\mathbb{Z})$ on $\mathcal{O}^{\mathrm{alg}}_{\mathbb{C}^2} = \mathbb{C}[t_1, t_2]$, i.e., the action on algebraic functions induced by $GL_2(\mathbb{Z}) \curvearrowright \mathbb{C}^2$. There is an additional $\mathbb{C}^\times$ by $\lambda \cdot (t_1, t_2) = (\lambda t_1, \lambda t_2)$ corresponding to grading in cohomology.

The action restricts to the analytic open subset

$$\mathcal{X} := \{\, (t_1, t_2) \mid \mathbb{C} = \mathbb{Z}t_1 + \mathbb{Z}t_2 \,\} \subset \mathbb{C}^2,$$

corresponding to pairs which generate a lattice. Thus we get

$$\mathbb{C}^\times \times GL_2(\mathbb{Z}) \quad \curvearrowright \quad \mathcal{X},$$

which presents $\mathcal{M}$ = the moduli stack of complex analytic elliptic curves.

$d = 1$: We an action of $\mathbb{C}^\times \times GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C}[t_1, t_2, y]$, induced by an action on $\mathbb{C}^2 \times \mathbb{C}$, which restricts to

$$\mathbb{C}^\times \times GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \quad \curvearrowright \quad \mathcal{X} \times \mathbb{C}.$$

This presents $\mathcal{E}$ = the universal elliptic curve: the new $\mathbb{Z}^2$ factor acts in each fiber of $\mathcal{X} \times \mathbb{C} \to \mathcal{X}$ as translations by the lattice.

$d \geq 2$: We get

$$\mathbb{C}^\times \times GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{d\times 2} \quad \curvearrowright \quad \mathcal{X} \times \mathbb{C}^d$$

which presents $\mathcal{E}^{\times d} = \mathcal{E} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{E}$.

Note: cohomology produces affine varieties acted on by discrete groups. The final step to produce an object in complex analytic geometry is a bit ad hoc.

Aside: Double loop groups (Etingof & Frenkel, 1993)

We can think about this action in terms of double loop groups.

Fix $\Sigma = \mathbb{T}^2$ and G a connected compact group. The "extended double loop group" $\mathbb{T}^2 \ltimes \mathrm{Map}(\mathbb{T}^2, G) \subset \mathcal{W}(G)$ contains a maximal torus

$$\mathbb{T}^2 \times T \subset \mathbb{T}^2 \ltimes \mathrm{Map}(\mathbb{T}^2, G)$$

where $\mathbb{T}^2$ is the rotation subgroup, and $T \subset \mathrm{Map}(\mathbb{T}^2, G)$ corresponds to constant maps $\Sigma \to T \subset G$ to the maximal torus.

The torus is acted on by the "elliptic Weyl group"

$$W_{\mathrm{ell}} = W \rtimes \mathrm{Hom}(\mathbb{Z}^2, \check{T}),$$

the 2-dimensional analogue of the affine Weyl group. The action $GL_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ extends to a further action

$$\widetilde{W}_{\mathrm{ell}} := GL_2(\mathbb{Z}) \ltimes W_{\mathrm{ell}} \curvearrowright \mathbb{T}^2 \times T.$$

Write $\mathfrak{t}^{\mathrm{ext}}_G = \mathrm{Lie}(\mathbb{T}^2 \times T)$ for this torus which inherits the action. In the case of $G = e$ (trivial group), we have

$$\mathfrak{t}^{\mathrm{ext}}_e \otimes \mathbb{C} = \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C} \approx \mathbb{C}^2.$$

For general G we have a pullback square

$$\begin{array}{ccc} \mathcal{X}_G & \rightarrowtail & \mathfrak{t}^{\mathrm{ext}}_G \otimes \mathbb{C} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \mathcal{X}_e & \rightarrowtail & \mathfrak{t}^{\mathrm{ext}}_e \otimes \mathbb{C} \end{array}$$

where $\mathcal{X}_e \subset \mathfrak{t}^{\mathrm{ext}}_e \otimes \mathbb{C}$ is the subset of pairs which generate a lattice, and we get $\mathcal{W}_\ell \curvearrowright \mathcal{X}_G$.

Picking a complex structure on $\mathbb{T}^2$ (compatible with the group strucure) plus a holomorphic 1-form determines a point $t \in \mathcal{X}_e$. The induced action

$$W_{\mathrm{ell}} \;\curvearrowright\; \pi^{-1}(t) \approx \mathfrak{t}_G \otimes \mathbb{C}$$

is the same as one described by E&F: they observe that for simply connected G, $(\mathfrak{t}_G \otimes \mathbb{C})/W_{\text{ell}}$ is a space of equivalence classes of flat and unitary holomorphic G-bundles on $(\mathbb{T}^2, t)$.

If $G = T = U(1)^d$, we recover the story described above via the Chern-Weil isomorphism. We have

$$\widetilde{W}_G = GL_2(\mathbb{Z}) \ltimes W_{\text{ell}} = GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{d\times 2} \approx \overline{W}(G)$$

acting on

$$H^*(B\mathcal{W}_0(G); \mathbb{C}) \approx H^*(B(\mathbb{T}^2 \times T); \mathbb{C}) \approx \operatorname{Sym}(\mathfrak{t}_T^{\text{ext}} \otimes \mathbb{C})^*.$$

Side remark. In E&F the role of the choice of complex structure on $\mathbb{T}^2$ in describing the action is a little obscure. The above picture is a little cleaner than what they actually say. The moral is that incorporating the *extended* torus somehow takes into account the choice of the complex structure + holomorphic 1-form: i.e., $\mathcal{X}_e/GL_2(\mathbb{Z})$ is a coarse moduli space of such structures.

Variation: $G_\phi = G$ extended by $K(\mathbb{Z}, 2)$

Let $G = U(1)^d$, and choose

$$\phi \in H^4(BG; \mathbb{Z}) \quad \text{represented by} \quad BG \to K(\mathbb{Z}, 4).$$

The cohomology class ϕ can be represened by a quadratic function $\mathbb{Z}^d = H_2BG \to \mathbb{Z}$, which we also write as "ϕ"; in coordinates $\phi(y) = \frac{1}{2}\sum c_{ij}y_iy_j$ with $c_{ij} = c_{ji} \in \mathbb{Z}$ and c_{ii} even.

This gives an extension of topological groups

$$1 \to K(\mathbb{Z}, 2) \to G_\phi \to G \to 1.$$

Carrying out the program for $\mathcal{W}(G_\phi) = \operatorname{Diff}(\Sigma) \ltimes \operatorname{Map}(\Sigma, G_\phi)$ we get:

$$\mathcal{W}_0(G_\phi) \approx \big(\mathbb{T}^2 \times U(1)^d \times K(\mathbb{Z}, 1)^2\big) \ltimes K(\mathbb{Z}, 2),$$
$$\overline{W}(G_\phi) \approx GL_2(\mathbb{Z}) \ltimes \big(\mathbb{Z}^{d\times 2} \rtimes \mathbb{Z}\big),$$

where $\mathbb{Z}^{d\times 2} \rtimes \mathbb{Z}$ is a central extension of $\mathbb{Z}^{d\times 2}$ by $\mathbb{Z}$ depending on ϕ. We obtain

$$H^*\mathcal{P}_0(G_\phi \curvearrowright *) = H^*B\mathcal{W}_0(G_\phi) \approx \mathbb{C}[t_1, t_2, y_1, \dots, y_d, x_1, x_2]/\big(\phi(y) + t_1x_1 + t_2x_2\big).$$

The resulting action

$$\mathbb{C}^\times \times \overline{\mathcal{W}}(G_\phi) \quad \curvearrowright \quad \mathcal{Q}_\phi = \{\, (t, y, x) \mid \mathbb{C} = \mathbb{Z}t_1 + \mathbb{Z}t_2,\ \phi(y) = -t_1x_1 - t_2x_2 \,\} \subset \mathcal{X} \times \mathbb{C}^d \times \mathbb{C}^2$$

presents the total space of a $\mathbb{C}^\times$-torsor $\mathcal{L}_\phi \to \mathcal{E}^{\times d}$. These are Looijenga's line bundles, whose sections are certain kinds of theta functions which appear, e.g., as characters of representations of loop groups. (See arXiv:1608.03548.)

Side remark. For the record, the action can be described by

$$\begin{aligned} A \cdot (t, y, x) &= (At, y, xA^{-1}), && A \in GL_2(\mathbb{Z}), \\ m \cdot (t, y, x) &= (t, y + mt, x - \beta(y, m) - \omega(mt, m)), && m = (m_1, m_2) \in \mathbb{Z}^{d\times 2}, \\ n \cdot (t, y, x_1, x_2) &= (t, y, x_1 - nt_2, x_2 + nt_1), && n \in \mathbb{Z}, \end{aligned}$$

where t, y, and x represent the evident vectors, ω is a choice of bilinear function $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}$ such that $\omega(y, y) = \phi(y)$, and $\beta(y, y') := \omega(y, y') + \omega(y', y) = \phi(y + y') - \phi(y) - \phi(y')$. The group law for the central extension $\mathbb{Z}^{d\times 2} \rtimes \mathbb{Z}$ is

$$(m, n) \cdot (m', n') = (m + m', n + n' + \omega(m_1, m_2') - \omega(m_2, m_1')),$$

while $GL_2(\mathbb{Z})$ acts on $\mathbb{Z}^{d\times 2} \times \rtimes\ \mathbb{Z}$ (from the right), by

$$(m, n) \cdot A = (mA, n(\det A)).$$

(Note that $\omega((mA)_1, (m'A)_2) - \omega((mA)_2, (m'A)_1) = \big(\omega(m_1, m_2') - \omega(m_2, m_1')\big)(\det A)$.)

Variation: covering maps and isogenies

Fix a finite covering map $\psi\colon \Sigma \to \Sigma'$ of smooth tori. This has a symmetry group

$$\mathrm{Diff}(\psi) = \{ (\phi, \phi') \mid \psi\phi = \phi'\psi \} \subset \mathrm{Diff}(\Sigma) \times \mathrm{Diff}(\Sigma').$$

For G define

$$\mathcal{W}^{\psi,\mathrm{src}}(G) := \mathrm{Diff}(\psi) \ltimes \mathrm{Map}(\Sigma, G), \qquad \mathcal{W}^{\psi,\mathrm{tar}}(G) := \mathrm{Diff}(\psi) \ltimes \mathrm{Map}(\Sigma', G).$$

We get a diagram of classifying spaces

$$\begin{array}{ccccccc}
B\mathcal{W}^{\Sigma'}(G) & \longleftarrow & B\mathcal{W}^{\psi,\mathrm{tar}}(G) & \xrightarrow{\psi^*} & B\mathcal{W}^{\psi,\mathrm{src}}(G) & \longrightarrow & B\mathcal{W}^{\Sigma}(G) \\
\searrow & & \searrow & & \swarrow & & \swarrow \\
B\,\mathrm{Diff}(\Sigma') & & \xleftarrow{\mathrm{tar}} & B\,\mathrm{Diff}(\psi) & \xrightarrow{\mathrm{src}} & & B\,\mathrm{Diff}(\Sigma)
\end{array}$$

We can plug this into cohomology (e.g., $\mathbb{C}^\times \times \overline{W}^?(G) \curvearrowright H^*(B\mathcal{W}^?_0(G); \mathbb{C}))$ as above.

Take $\Sigma = \Sigma' = \mathbb{T}^2$, $\psi = \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $G = U(1)$. We get

$$\begin{array}{ccccccc}
\mathcal{E} & \longleftarrow & \mathrm{tar}^*\mathcal{E} & \xrightarrow{\psi^*} & \mathrm{src}^*\mathcal{E} & \longrightarrow & \mathcal{E} \\
\searrow & & \searrow & & \swarrow & & \swarrow \\
\mathcal{M} & & \xleftarrow{\mathrm{tar}} & \mathcal{M}_{\Gamma_0(N)} & \xrightarrow{\mathrm{src}} & & \mathcal{M}
\end{array}$$

where $\mathcal{M}_{\Gamma_0(N)}$ is presented by $\mathbb{C}^\times \times \Gamma_0(N) \curvearrowright \mathcal{X}$ (because $\pi_0\,\mathrm{Diff}(\psi) = \Gamma_0(N) = \{A \equiv_N \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$). The map ψ^* is the universal example of an isogeny of degree N with cyclic kernel: over $t \in \mathcal{X}$ it looks like the projection $\mathbb{C}/(\mathbb{Z}(Nt_1) + \mathbb{Z}t_2) \to \mathbb{C}/(\mathbb{Z}t_1 + \mathbb{Z}t_2)$.

Side remark. Fix $\psi \in M_{2\times 2}(\mathbb{Z})$ with $\det\psi \neq 0$, giving $\psi\colon \mathbb{T}^2 \to \mathbb{T}^2$, and let $\Gamma(\psi) = GL_2(\mathbb{Z}) \cap \psi^{-1} GL_2(\mathbb{Z})\psi$. We have isomorphisms

$$H^*(B\mathcal{W}^{\psi,\mathrm{tar}}_0(G); \mathbb{C}) \approx H^*(B\mathcal{W}^{\psi,\mathrm{src}}_0(G); \mathbb{C}) \approx \mathbb{C}[t_1, t_2, y_1, \dots, y_d].$$

with ψ^* acting as the identity map. We get

$$\overline{W}^{\psi,\mathrm{src}}(G) = \Gamma(\psi) \ltimes \mathrm{Hom}(\mathbb{T}^2, G), \qquad (A, m)\cdot(A', m') = (AA', mA' + m')$$

acting on $\mathcal{X} \times \mathbb{C}^d$ by

$$(A, m)\cdot(t, y) = (At, y + mt),$$

and

$$\overline{W}^{\psi,\mathrm{tar}}(G) = \Gamma(\psi) \ltimes \mathrm{Hom}(\mathbb{T}^2, G), \qquad (A, m)\cdot(A', m') = (A\psi A'\psi^{-1}, mA' + m')$$

acting on $\mathcal{X} \times \mathbb{C}^d$ by

$$(A, m)\cdot(t, y) = (At, y + m\psi t).$$

That is, at a point $t \in \mathcal{X}$ this models the isogeny $\mathbb{C}/(\mathbb{Z}\psi(t_1) + \mathbb{Z}\psi(t_2)) \to \mathbb{C}/(\mathbb{Z}t_1 + \mathbb{Z}t_2)$.

Side remark. If we use the group G_ϕ instead of $G = U(1)^d$, we get a map

$$(\mathrm{tar}^*\mathcal{L}_\phi)^{\otimes(\det\psi)} \to \psi^*\mathrm{src}^*\mathcal{L}_\phi$$

of $\mathbb{C}^\times$-torsors over $\mathrm{tar}^*\mathcal{E}^{\times d}_{\mathcal{M}}$.

Orbits

Now we consider $\mathcal{P}(G \curvearrowright X)$ where $X \approx G/H$ is a G-orbit.

Return to $G = U(1)^d$, let $H \leq G$ be a closed subgroup. Using $\Sigma = \mathbb{T}^2$ we find that

$$\begin{aligned}\mathcal{P}_0(G \curvearrowright G/H) &\approx \mathrm{Map}(\mathbb{T}^2, G/H)_{h(\mathbb{T}^2\times G)} \\ &\approx \big(\mathrm{Hom}(\mathbb{T}^2, G/H) \times G/H\big)_{h(\mathbb{T}^2\times G)} \\ &\approx \coprod_{f\colon \mathbb{T}^2\to G/H} BK_f, \qquad K_f = \{\, (t,g) \mid f(t) \equiv_H g \,\} \subseteq \mathbb{T}^2 \times G.\end{aligned}$$

Thus $K_f \subseteq \mathbb{T}^2 \times G$ is the preimage of the graph of f in $\mathbb{T}^2 \times G/H$, and K_f is abstractly isomorphic to $\mathbb{T}^2 \times H$. (Because $\mathbb{T}^2 \times G$ acts on $\mathrm{Hom}(\mathbb{T}^2, G/H) \times G/H \subseteq \mathrm{Map}(\mathbb{T}^2, G/H)$ by $(t,g)\cdot(f, xH) = (f, gf(t)^{-1}xH)$.)

The projection $G/H \to *$ induces $\mathcal{P}_0(G \curvearrowright G/H) \to \mathcal{P}_0(G \curvearrowright *)$, which corresponds to the evident map $\coprod_{f\colon \mathbb{T}^2\to G/H} BK_f \to B(\mathbb{T}^2 \times G)$ induced by inclusion of subgroups. As before, we take cohomology and consider the corresponding analytic spaces schemes equipped with $\overline{\mathcal{W}}(G)$-actions.

Example. Let $G = U(1)$ and $G/H = U(1)/e$. For $f\colon \mathbb{T}^2 \to U(1)$ defined by $[t_1, t_2] \mapsto e^{2\pi i\,(n_1t_1+n_2t_2)}$, the group $K_f \subseteq \mathbb{T}^2 \times U(1)$ is the graph subgroup of f, which is isomorphic to $\mathbb{T}^2$. We get an induced map of cohomology rings

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{C}[t_1,t_2,y]/(y - n_1t_1 - n_2t_2) \leftarrow \mathbb{C}[t_1,t_2,y].$$

I.e., the subobject of $\mathcal{E}$ which is the image of the identity section.

Example. More generally, let $G = U(1)$ and $G/H = U(1)/\mu_N$. Then for $f\colon \mathbb{T}^2 \to U(1)/\mu_N$ defined by $[t_1, t_2] \mapsto [e^{2\pi i\,(n_1t_1+n_2t_2)/N}]$, the group $K_f \approx \mathbb{T}^2 \times \mu_N$. We get

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{C}[t_1,t_2,y]/(y - n_1t_1/N - n_2t_2/N) \leftarrow \mathbb{C}[t_1,t_2,y].$$

I.e., the n-torsion points in $\mathcal{E}$.

There is a general recipe for computing this for any orbit. We note one feature: if G/H has dimension r, then the corresponding subspace of $\mathcal{X} \times \mathbb{C}^d$ is a disjoint union of hyperplanes of (complex) codimension r, with components indexed by $\mathrm{Hom}(\mathbb{T}^2, G/H) \approx \mathbb{Z}^{r\times 2}$.

Side remark. In general, if $X = G/H$ is a G-orbit, then the "space" associated to $H^*\mathcal{P}_0(G \curvearrowright G/H)$ is the pullback of

$$\mathcal{X} \times (\mathbb{C}\otimes H_1G) \to \mathcal{X} \times (\mathbb{C}\otimes H_1X) \xleftarrow{(t,ft)\leftarrow\!\shortmid(t,f)} \mathcal{X} \times \mathrm{Hom}(\mathbb{Z}^2, H_1X)$$

where $H_1 = H_1(-;\mathbb{Z})$.

Cohomology theory and ghost maps

We have a functor

$$(G \curvearrowright X) \quad\longmapsto\quad \big(\mathbb{C}^\times \times \overline{\mathcal{W}}(G) \ \curvearrowright\ H^*(\mathcal{P}_0(G \curvearrowright X); \mathbb{C}[u^\pm])\big)$$

from G-spaces to $H^*(B\mathcal{W}_0(G); \mathbb{C}[u^\pm])$-algebras with $\mathbb{C}^\times \times \overline{W}(G)$-action. This functor even lifts to spectra, by replacing the target with the function spectrum $\mathcal{F}(\Sigma^\infty_+\mathcal{P}_0(G \curvearrowright X), H\mathbb{C}[u^\pm])$. However it is not a cohomology theory.

Note: Here $H^*(-;\mathbb{C}[u^\pm])$ denotes 2-periodic ordinary cohomology. The $\mathbb{C}^\times$ acts on the spectrum $H\mathbb{C}[u^\pm]$. It is more correct to say that there is an action by $\underline{\mathrm{Aut}}(H\mathbb{C}[u^\pm]) \approx \mathrm{Spec}\, H\mathbb{C}[u^\pm]$, a derived group scheme over $\mathrm{Spec}\, H\mathbb{C}$.

There is an approximation which is a cohomology theory: restricting to G-orbits gives a functor $\mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Spectra}$, and thus a naive G-spectrum. This cohomology theory can be given a geometric intepretation in terms of *ghost maps*.

For $G \curvearrowright X$, a **ghost map** $f\colon \Sigma \to X$ is a map such that the image $f(\Sigma)$ is contained in a single G-orbit. The subspace

$$\mathrm{Map}^{\mathrm{gh}}(\Sigma, X) \subseteq \mathrm{Map}(\Sigma, X)$$

of ghost maps is invariant under the $\mathcal{W}(G) = \mathrm{Diff}(\Sigma) \ltimes \mathrm{Map}(\Sigma, G)$ action, and the construction $\mathrm{Map}^{\mathrm{gh}}(\Sigma, -)$ preserves homotopy colimits of G-CW complexes.

Set

$$\mathcal{P}_0^{\mathrm{gh}}(G \curvearrowright X) := \mathrm{Map}^{\mathrm{gh}}(\Sigma, X)_{h\mathcal{W}_0(G)}.$$

We obtain a functor

$$(G \curvearrowright X) \quad \longmapsto \quad \mathcal{E}^*(G \curvearrowright X) := \big(\mathbb{C}^\times \times \overline{\mathcal{W}}(G) \ \curvearrowright\ H^*(\mathcal{P}_0^{\mathrm{gh}}(G \curvearrowright X); \mathbb{C}[u^\pm])\big)$$

which is a cohomology theory on finite G-CW complexes.

This is not what we want; it's just the best we can do with these tools. Problems:

(1) It does not take values in coherent sheaves on the complex manifold $\mathcal{X} \times \mathbb{C}^d$ as we would like it to. Ideally, we should "analytify" to turn algebraic sheaves into analytic sheaves, but it is not clear that this is possible functorially (because the algebraic sheaves aren't coherent).

    An additional complication is that we should do everything derived, with sheaves on derived complex analytic spaces.

(2) It is not a "genuine" equivariant theory: for instance, we expect compactifications of complex representations to give invertible sheaves, but they don't.

These two issues are connected. For instance, consider S^L where L is the standard 1-dimensional $U(1)$-representation. Then $S^L = \mathrm{Cof}(U(1) \to *)$, so $\widetilde{\mathcal{E}}^0(U(1) \curvearrowright S^L)$ should be the "fiber" of $\mathcal{E}^0(U(1) \curvearrowright U(1)) \leftarrow \mathcal{E}^0(U(1) \curvearrowright *)$, i.e.,

$$\prod_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{C}[t_1, t_2, y]/(y - n_1 t_1 - n_2 t_2) \leftarrow \mathbb{C}[t_1, t_2, y].$$

If these are somehow turned into analytic sheaves in a sensible way, this would look like

$$\mathcal{O}/\mathcal{I} \leftarrow \mathcal{O},$$

a surjective map whose fiber=kernel is the ideal sheaf $\mathcal{I}$ cutting out a codimimension one subobject, so invertible.

Side remark. Pick $\phi \in H^4(BG; \mathbb{Z})$ and $G \curvearrowright X$ represented by a map $\phi\colon BG \to K(\mathbb{Z}, 4)$. The resulting group $G_\phi = G \ltimes K(\mathbb{Z}, 2)$ acts on X via the projection $G_\phi \to G$, and so we get

$$(G \curvearrowright X),\ \phi \quad \longmapsto \quad \mathcal{E}^{*,\phi} := (\mathbb{C}^\times \times \overline{\mathcal{W}}(G_\phi) \ \curvearrowright\ H^*(\mathcal{P}_0^{\mathrm{gh}}(G_\phi \curvearrowright X); \mathbb{C}[u^\pm]),$$

taking values in "sheaves" on $\mathcal{L}_\phi$. This is a twisted version of $\mathcal{E}^*$.

Side remark. These theories will have "Hecke operators", associated to the isogenies given by finite covers $\psi\colon \Sigma \to \Sigma'$.

Conjectures

Instead of trying to manufacture an analytic cohomology theory from this, it may be better to think of this a "shadow" of a more geometric picture.

Guess.[2] There should be a derived stack $\mathcal{X}^{\mathrm{der}}$ over $H\mathbb{C}[u^\pm]$, which is the moduli space of certain "torus-like objects" (e.g., derived supermanifolds whose underlying space is a smooth torus Σ, equipped with some additional structures). The underlying space of $\mathcal{X}^{\mathrm{der}}$ should be $\mathcal{X} \subset \mathbb{C}^2$. The object $\mathcal{X}^{\mathrm{der}}$ should have an inherent derived complex analytic structure, so that $\pi_0\mathcal{O}^{\mathrm{der}}$ is the

[2]This is all vague, and should not be taken too seriously. It is inspired by discussions with Dan Berwick-Evans.

sheaf of holomorphic functions on $\mathcal{X}$. The forgetful map from geometry to topology should give a map of stacks over $H\mathbb{C}[u^{\pm}]$:

$$\begin{array}{ccc} \mathcal{X}^{\mathrm{der}} & \longrightarrow & \underline{B\,\mathrm{Diff}_0(\Sigma)}_{H\mathbb{C}[u^{\pm}]} \\ & \searrow \qquad \swarrow & \\ & \operatorname{Spec} H\mathbb{C}[u^{\pm}] & \end{array}$$

inducing a map of sections

$$H^0(\mathcal{X}^{\mathrm{der}}, \mathcal{O}^{\mathrm{der}}) \leftarrow H^0(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}[u^{\pm}])$$

which should look like the inclusion of algebraic functions on $\mathcal{X} \subset \mathbb{C}^2$ into analytic functions. We can vary this by adding a principal G-bundle or G_ϕ-bundle to the data of a point in $\mathcal{X}^{\mathrm{der}}$. For $G = U(1)$ you should get a derived (and "oriented") elliptic curve over $\mathcal{X}^{\mathrm{der}}$, giving equivariant elliptic cohomology theories by Lurie's machine.

What I have described in this talk would merely be the "homotopical shadow" of the geometric picture I'm guessing at.

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