STUFF ABOUT QUASICATEGORIES

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1. Introduction to ∞-categories

I’ll give a brief discussion to motivate the notion of ∞-categories.

1.1. Groupoids. Modern mathematics is based on sets. The most basic way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring $R$, we can consider the set $X(R)$ of tuples $(x,y,z) \in R^3$ which satisfy the equation $x^5 + y^5 = z^5$. We can express such sets as limits; for instance, $X(R)$ is the pullback of the diagram of sets

$$
R \times R \xrightarrow{(x,y) \mapsto x^5+y^5} R \xleftarrow{z^5} R.
$$

Another way to construct new sets is by taking “quotients”; e.g., as sets of equivalence classes of an equivalence relation. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that; but some have “higher” structure standing behind them, which is often encoded in the form of a groupoid. Here are some examples.

- Given a topological space $X$, we can define an equivalence relation on the set of points, so $x \sim x'$ if and only if there is a continuous path connecting them. The set of equivalence classes is the set $\pi_0 X$ of path components. Standing behind this equivalence relation is the fundamental groupoid $\Pi_1 X$, whose objects are points of $X$, and whose morphisms are path-homotopy classes of paths between two points.

- Given any category $C$, there is an equivalence relation on the collection of objects, so that $X \sim Y$ if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the core of $C$ (also called the maximal subgroupoid), which is a groupoid having the same objects as $C$, but having as morphisms only the isomorphisms in $C$.

- As a special case of the above, let $C = \text{Vect}_F$ be the category of finite dimensional vector spaces and linear maps over some field $F$. Then isomorphism classes of objects correspond to non-negative integers, via the notion of dimension. The core $\text{Vect}_F^{\text{core}}$ is a groupoid whose objects are finite dimensional vector spaces, and whose morphisms are invertible linear maps.

Note that many interesting problems are about describing isomorphism classes; e.g., classifying finite groups of a given order, or principal $G$-bundles on a space. In practice, one learns that when you try to classify some type of objects up to isomorphism, you will need to have a good handle on the isomorphisms between such objects, including the groups of automorphisms of such objects. So you will likely need to know about the groupoid, even if it is not the primary object of interest.

For instance, a problem such as: “describe the groupoid $\text{Bun}_G(M)$ of principal $G$-bundles on a space $M$” as a more sophisticated analogue of: “find the set $X(R)$ of solutions to $x^5 + y^5 = z^5$ in the ring $R$”. (In fact, the theory of “moduli stacks” exactly develops this analogy between the two problems.) To do this, you can imagine having a “groupoid-based mathematics”, generalizing the usual set-based one. Here are some observations about this.

- We regard two sets as “essentially the same” if they are isomorphic, i.e., if there is a bijection $f: X \to X'$ between them. Any such bijection has a unique inverse bijection $f^{-1}: X' \to X$.

- On the other hand, we regard two categories as “essentially the same” if they are merely equivalent, i.e., if there is a functor $f: C \to C'$ which admits an inverse up to natural

---

1I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].
isomorphism. It is not the case that such an inverse up to natural isomorphism is itself unique. These same remarks apply in particular to equivalences of groupoids.

Although any equivalence of categories admits some kind of inverse, the failure to be unique leads to complications. For example, one goal of every course in abstract linear algebra is to demonstrate and exploit an equivalence of categories

\[ f : \text{Mat}_F \to \text{Vect}_F. \]

Here Mat\(_F\) is the matrix category, whose objects are non-negative integers, and whose morphisms \( n \to m \) are \( m \times n \)-matrices with entries in \( F \). The functor \( f \) is defined by an explicit construction; e.g., it sends the object \( n \) to the vector space \( F^n \). However, there is no completely “natural” way to construct an inverse functor \( f^{-1} : \text{Vect}_F \to \text{Mat}_F \): producing such an inverse functor requires making an arbitrary choice, for each abstract vector space \( V \), of a basis for \( V \).

- We can consider “solutions to equations” in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids; thus, we need to consider “weak” or “homotopy” limits.

For example, suppose \( M \) is a space which is a union of two open sets \( U \) and \( V \). The weak pullback of

\[ \text{Bun}_G(U) \to \text{Bun}_G(U \cap V) \leftarrow \text{Bun}_G(V) \]

is a groupoid, whose objects are triples \((P, Q, \alpha)\), where \( P \to U \) and \( Q \to U \) are \( G \)-bundles, and \( \alpha : P|_{U \cap V} \xrightarrow{\sim} Q|_{U \cap V} \) is an isomorphism of \( G \)-bundles over \( U \cap V \); the morphisms \((P, Q, \alpha) \to (P', Q', \alpha')\) are pairs \((f : P \to P', g : Q \to Q')\) are pairs of bundle maps which are compatible over \( U \cap V \) with the isomorphisms \( \alpha, \alpha' \). Compare this with the strict pullback, which consists of \((P, Q)\) such that \( P|_{U \cap V} = Q|_{U \cap V} \) as bundles; in particular, \( P|_{U \cap V} \) and \( Q|_{U \cap V} \) must be the identical sets.

A basic result about bundles is that \( \text{Bun}_G(M) \) is equivalent to this weak pullback. The strict limit may fail to be equivalent to this; in fact, it is impossible to describe the strict pullback without knowing precisely what definition of \( G \)-bundle we are using, whereas the identification of weak pullback is insensitive to the precise definition of \( G \)-bundle. (The point being, there can exist many non-identical “precise definitions of \( G \)-bundle”, because what we really care about in the end is understanding \( \text{Bun}_G(M) \) up to equivalence, rather than up to isomorphism.)

These kinds of issues persist when dealing with higher groupoids and categories.

1.2. Higher groupoids. There is a category \( \text{Gpd} \) of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here; there are natural transformations between functors \( f, f' : G \to G' \) of groupoids. That is, \( \text{Fun}(G, G') \) forms not merely a set, but a category. We can consider the collection consisting of (0) groupoids, (1) equivalences between groupoids, and (2) natural isomorphisms between equivalences; this is an example of a 2-groupoid\(^2\). There is no reason to stop at 2-groupoids: there are \( n \)-groupoids, the totality of which are an example of an \((n + 1)\)-groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider \( \infty \)-groupoids.

It turns out to be difficult to construct an “algebraic” definition of \( n \)-groupoid. The approach which in seems to work best in practice is to use homotopy theory. We start with the observation that every groupoid \( G \) has a classifying space \( BG \). This is defined explicitly as a quotient space

\[ G \mapsto BG := \left( \coprod_{x_0 \to x_1 \to \cdots \to x_n} \Delta^n \right) / (\sim), \]

\(^2\)More precisely, a “quasistrict 2-groupoid”.
where we glue in a topological $n$-simplex $\Delta^n$ for each $n$-fold sequence of composable arrows in $G$, modulo certain identifications. It turns out (i) the fundamental groupoid of $BG$ is equivalent to $G$, and (ii) the higher homotopy groups $\pi_k$ of $BG$ are trivial, for $k \geq 2$. A space like this is called a 1-type. Furthermore, (iii) there is a bijection between equivalence classes of groupoids up to equivalence and CW-complexes which are 1-types, up to homotopy equivalence. (More is true, but I’ll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should define $n$-groupoids as $n$-types (CW complexes with trivial homotopy groups in dimensions $> n$), with equivalences being homotopy equivalences. Removing the restriction on homotopy groups leads to modelling $\infty$-groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classifying space construction factors through a “combinatorial” construction, called the “nerve”. That is, we have

$$(G \in \text{Gpd}) \mapsto (NG \in \text{sSet}) \mapsto (|NG| = BG \in \text{Top}),$$

where $NG$ is the nerve of the groupoid, and is an example of a simplicial set; $|X|$ denotes the geometric realization of a simplicial set $X$. In fact, the nerve is a particular kind of simplicial set called a Kan complex. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that $\infty$-groupoids are precisely the Kan complexes.

1.3. $\infty$-categories. An $\infty$-category is a generalization of $\infty$-groupoid in which morphisms are no longer required to be invertible in any sense.

There are a number of approaches to defining $\infty$-categories. Here are two which build on top of the identification of $\infty$-groupoids with Kan complexes.

- A category $C$ consists of a set $\text{ob} C$ of objects, and for each pair of objects a set $\text{hom}_C(x, y)$ of maps from $x$ to $y$. If we replace the set $\text{hom}_C(x, y)$ with a Kan complex (or more generally a simplicial set) $\text{map}_C(x, y)$, we obtain a category enriched over Kan complexes (or simplicial sets). This leads to one model for $\infty$-categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category $C$, we have a simplicial set $NC$. In general, $NC$ is not a Kan complex; however, it does land in a special class of simplicial sets, which are called quasicategories. This leads to another model for $\infty$-categories: quasicategories.

In this paper we focus on the second case: the quasicategory model for $\infty$-categories.

1.4. Prerequisites. I assume only familiarity with basic concepts of category theory, such as those discussed in the first few chapters of [Rie16]. It is helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher’s textbook).

Some categorical prerequisites: you should be at least aware of the following notions (or know where to turn to in order to learn them):

- categories, functors, and natural transformations;
- full subcategories;
- groupoids;
- products and coproducts;
- pushouts and pullbacks;
- general colimits and limits.
- adjoint functors.
1.5. **Historical remarks.** Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of $\infty$-categories. This development began with the work of Joyal, first published in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls “$\infty$-categories”), recasting and extending Joyal’s work significantly.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a “categorical equivalence” between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I follow here is essentially that of Joyal. However, I have tried to follow Lurie’s terminology and notation in most places.

1.6. **Goal of this book.** The goal of this book is to give a reasonably approachable introduction to the subject of higher category theory. In particular, I am writing with the following ideas in mind.

- The prerequisites are merely some basic notions of category theory, as seen in a first year algebraic topology or algebraic geometry course. In particular, no advanced training in homotopy theory is assumed.
- The book is written in “lecture notes” style rather than “textbook” style. That is, I will try to avoid introducing a lot of theory in section 3 which is only to be used in section 42, even if that is the “natural” place for it. The goal is to introduce new ideas near where they are first used, so that motivations are clear.
- The structure of the exposition is organized around the following type of question: Here is a [definition we can make/theorem we can prove] for ordinary categories; how do we generalize it to quasicategories? In some cases the answer is easy. In others, it can require a significant detour.
- The exposition is largely from the bottom up, rather than from the top down. Thus, I attempt to give complete details about everything I prove, so that nothing is relegated to references. (The current document does not achieve this yet, but that is the plan; in some cases, such details will be put into appendices.)

1.7. **Things to add.** This is a place for me to remind myself of things I might add.

- A discussion of $n$-truncation and $n$-groupoids, including the equivalence of ordinary groupoids to 1-groupoids (so connecting with the introduction).
- Pointwise criterion for limits/colimits: Show that $S^\triangledown \to \text{Fun}(D,C)$ is a colimit cone if each projection to $S^\triangledown \to \text{Fun}(\{d\},C) \approx C$ is one.

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**Part 1. Basic notions**

2. **Simplicial sets**

In the subsequent sections, we will define quasicategories as a generalization of the notion of a category. To accomplish this, we will recharacterize categories as a particular kind of *simplicial set*; relaxing this characterization will lead us to the definition of quasicategories.

Simplicial sets were introduced as a combinatorial framework for the homotopy theory of spaces. There are a number of treatments of simplicial sets from this point of view. We recommend Greg Friedman’s survey [Fri12] as a starting place for learning about this viewpoint, and we will
discuss this point of view later on in §???. Here we will focus on what we need in order to develop quasicategories.

2.1. The simplicial indexing category $\Delta$. We write $\Delta$ for the category whose

- **objects** are the non-empty totally ordered sets $[n] := \{0 < 1 < \ldots < n\}$ for $n \geq 0$, and
- **morphisms** $f : [n] \to [m]$ are weakly monotone functions, i.e., such that $x \leq y$ implies $f(x) \leq f(y)$.

Note that we exclude the empty set from $\Delta$. Morphisms in $\Delta$ are often called simplicial operators.

Because $[n]$ is an ordered set, you can also think of it as a category: the objects are the elements of $[n]$, and there is a morphism (necessarily unique) $i \to j$ if and only if $i \leq j$. Thus, morphisms in the category $\Delta$ are precisely the functors between the categories $[n]$. We can, and will, also think of $[n]$ as the category “freely generated” by the picture

$$
\begin{array}{cccc}
0 & \to & 1 & \to \cdots & \to n-1 & \to n.
\end{array}
$$

Arbitrary non-identity morphisms in $[n]$ can be expressed uniquely as iterated composites of the arrows which are displayed in the picture.

We will often use the following notation for morphisms in $\Delta$:

$$f = \langle f_0 \cdots f_n \rangle : [n] \to [m] \quad \text{with } f_0 \leq \cdots \leq f_n \quad \text{represents the function } \quad k \mapsto f_k.$$

2.2. Remark. There are distinguished simplicial operators called face and degeneracy operators:

- $d^i := \langle 0, \ldots, \widehat{i}, \ldots, n \rangle : [n-1] \to [n], \quad 0 \leq i \leq n,$
- $s^i := \langle 0, \ldots, i, \ldots, n \rangle : [n+1] \to [n], \quad 0 \leq i \leq n.$

All maps in $\Delta$ can be obtained as a composition of face and degeneracy operators, and in fact $\Delta$ can be described as the category generated by the above symbols, subject to a set of relations called the “simplicial identities”, which can be found in various places, e.g., [Fri12, Def. 3.2].

2.3. Simplicial sets. A simplicial set is a functor $X : \Delta^{op} \to \text{Set}$, i.e., a contravariant functor (or “presheaf”) from $\Delta$ to sets.

It is typical to write $X_n$ for $X([n])$, and call it the set of $n$-simplices in $X$. Sometimes people speak informally of the set of all simplices of $X$, i.e., of the disjoint union of the $X_n$s. Furthermore, given a simplex $a \in X_n$ and a simplicial operator $f : [m] \to [n]$, I will write $af \in X_m$ as shorthand for $X(f)(a)$. That is, I’ll think of simplicial operators as acting on simplices from the right; this is a convenient choice given that $X$ is a contravariant functor. In this language, a simplicial set consists of

- a sequence of sets $X_0, X_1, X_2, \ldots,$
- functions $a \mapsto af : X_n \to X_m$ for each simplicial operator $f : [m] \to [n]$, such that
- $a \text{id} = a$, and $(af)g = a(fg)$ for any simplex $a$ and simplicial operators $f$ and $g$ whenever this makes sense.

If I need to have the simplicial operator act from the left, I’ll write $f^*(a) = af$.

Sometimes I’ll use a subscript notation when speaking of the action of particular simplicial operators. So, given a simplicial operator of the form $f = \langle f_0 \cdots f_m \rangle : [m] \to [n]$, we can indicate the action of $f$ on a simplex using subscripts:

$$af_0 \cdots f_m := af = a(\langle f_0 \cdots f_m \rangle).$$

In particular, applying simplicial operators of the form $\langle i \rangle : [0] \to [n]$ gives the vertices $a_0, \ldots, a_n \in X_0$ of an $n$-simplex $a \in X_n$, while applying simplicial operators of the form $\langle ij \rangle : [1] \to [n]$ for $i \leq j$ gives the edges $a_{ij} \in X_1$ of $a$. 
2.4. **The category of simplicial sets.** A simplicial set is a functor; therefore a map of simplicial sets is a natural transformation of functors. Explicitly, a map \( f : X \to Y \) between simplicial sets is a collection of functions \( \phi : X_n \to Y_n, n \geq 0 \), which commute with simplicial operators:

\[
(\phi a)f = \phi(af) \quad \text{for all simplicial operators } f \text{ and simplices } a \text{ in } X, \text{ when this makes sense.}
\]

I’ll write \( sSet \) for the category of simplicial sets and maps between them.

2.5. **Discrete simplicial sets.** A simplicial set \( X \) is discrete if every simplicial operator \( f \) induces a bijection \( f^* : X_n \to X_m \).

Every set \( S \) gives us a discrete simplicial set \( S^{\text{disc}} \), defined so that \((S^{\text{disc}})_n = S\), and so that each simplicial operator acts according to the identity map of \( S \). This construction defines a functor \( S \mapsto S^{\text{disc}} : \text{Set} \to sSet \).

2.6. **Exercise.** Show that (i) every discrete simplicial set \( X \) is isomorphic to \( S^{\text{disc}} \) for some set \( S \), (and that in fact you can take \( S = X_0 \)), and (ii) for every pair of sets \( S \) and \( T \), the evident function \( \text{Hom}_{\text{Set}}(S, T) \to \text{Hom}_{sSet}(S^{\text{disc}}, T^{\text{disc}}) \) is a bijection.

Let \( sSet^{\text{disc}} \) denote the full subcategory of \( sSet \) spanned by discrete simplicial sets. That is, objects of \( sSet^{\text{disc}} \) are discrete simplicial sets, and morphisms of \( sSet^{\text{disc}} \) are all simplicial maps between them. Then the above exercise amounts to saying that the full subcategory of discrete simplicial sets is equivalent to the category of sets.

For this reason, it is often convenient to (at least informally) “identify” sets with their corresponding discrete simplicial sets (i.e., for a set \( S \) we also write \( S \) for the discrete simplicial set \( S^{\text{disc}} \) defined above).

2.7. **Standard \( n \)-simplex.** The standard \( n \)-simplex \( \Delta^n \) is the simplicial set defined by

\[
\Delta^n := \text{Hom}_\Delta(-,[n]).
\]

That is, the standard \( n \)-simplex is exactly the functor represented by the object \([n]\). Explicitly, this means that

\[
(\Delta^n)_m = \text{Hom}_\Delta([m],[n]) = \{\text{simplicial operators } a : [m] \to [n]\},
\]

while the action of simplicial operators on simplices in \( \Delta^n \) is given by composition: \( f : [m'] \to [m] \) sends \( a \in (\Delta^n)_m \) to \((af) : [m'] \to [n] \in (\Delta^n)_{m'}\).

The **generator** of \( \Delta^n \) is the simplex

\[
t_n := (01 \ldots n) = \text{id}_{[n]} \in (\Delta^n)_n
\]

corresponding to the identity map of \([n]\).

The **Yoneda lemma** (applied to the category \( \Delta \)) asserts that the function

\[
\text{Hom}_{sSet}(\Delta^n, X) \to X_n,
\]

\[
g \mapsto g(t_n),
\]

is a bijection for every simplicial set \( X \). (**Exercise:** if this fact is not familiar to you, prove it.)

The Yoneda lemma can be stated this way: for each simplex \( a \in X_n \) there exists a unique map \( f_a : \Delta^n \to X \) of simplicial sets which sends the generator to it, i.e., such that \( f_a(t_n) = a \). We call the map \( f_a \) the **representing map** of the simplex \( a \).

We will often use the bijection provided by the Yoneda lemma implicitly. In particular, instead of using notation such as \( f_a \), we will typically abuse notation and write \( a : \Delta^n \to X \) for the representing map of the simplex \( a \in X_n \). We reiterate that the map \( a : \Delta^n \to X \) is characterized as the unique map sending the generator \( t_n \) of \( \Delta^n \) to \( a \). Thus with our notation we have \( a = a(t_n) \), where the two appearances of “\( a \)” denote respectively the element of \( X_n \) and the representing morphism \( \Delta^n \to X \).

\Lurie\footnote{Lurie [Lur09] uses \( \text{Set}_\Delta \) to denote the category of simplicial sets. **Perhaps I should try to be consistent with this?} uses \( \text{Set}_\Delta \) to denote the category of simplicial sets.
2.8. Exercise. Show that the representing map \( f : \Delta^n \to X \) of \( a \in X_n \) sends \( \langle f_0 \cdots f_k \rangle \in (\Delta^n)_k \) to \( a\langle f_0 \cdots f_k \rangle \in X_k \).

Note that if \( X = \Delta^m \) is also a standard simplex, then the Yoneda lemma gives a bijection

\[
\text{Hom}_{s\text{Set}}(\Delta^n, \Delta^m) \cong (\Delta^m)_n = \text{Hom}_\Delta([n], [m]).
\]

The inverse of this bijection sends a simplicial operator \( f : [n] \to [m] \) to the map \( \Delta^f : \Delta^n \to \Delta^m \) of simplicial sets defined on simplices \( g \in (\Delta^n)_k = \text{Hom}_\Delta([k], [n]) \) by \( g \mapsto fg \). (Exercise: prove this.)

I will commonly abuse notation, and write \( f : \Delta^n \to \Delta^m \) instead of \( \Delta^f \) for the map induced by the simplicial operator \( f \), as it is also the representing map of the corresponding simplex \( f \in (\Delta^m)_n \).

2.9. The standard 0-simplex and the empty simplicial set. The standard 0-simplex \( \Delta^0 \) is the terminal object in \( s\text{Set} \). Sometimes I write \( \ast \) instead of \( \Delta^0 \) for this object. Note that it is the only standard \( n \)-simplex which is discrete.

The empty simplicial set \( \emptyset \) is the functor \( \Delta^\text{op} \to \text{Set} \) sending each \( [n] \) to the empty set. It is the initial object in \( \text{Set} \).

2.10. Exercise. Show that a simplicial set \( X \) is isomorphic to the empty simplicial set if and only if \( X_0 \) is isomorphic to the empty set.

2.11. Standard simplices on totally ordered sets. The definition of the standard simplices \( \Delta^n \) can be extended to simplicial sets “generated” by arbitrary totally ordered sets.

For instance, for any non-empty finite totally ordered set \( S = \{ s_0 < s_1 < \cdots < s_n \} \), there is a unique order preserving bijection \( S \approx [n] \) for a unique \( n \geq 0 \). We write \( \Delta^S \) for the simplicial set with \( (\Delta^S)_k = \{ \text{order preserving } [k] \to S \} \). There is a unique isomorphism \( \Delta^S \approx \Delta^n \) of simplicial sets (Exercise: prove this). We can also apply this idea to the empty ordered set \( S = \emptyset \), in which case \( (\Delta^S)_k = \emptyset \) for all \( k \), i.e., \( \Delta^S \) is the empty simplicial set.

This notation is especially convenient for subsets \( S \subseteq [n] \) with induced ordering, as the simplicial set \( \Delta^S \) is in a natural way a subcomplex of \( \Delta^n \) (i.e., a collection of subsets of the \( (\Delta^n)_k \) closed under action of simplicial operators; we will return to the notion of subcomplex below §4.9).

Furthermore, any simplicial operator \( f : [m] \to [n] \) factors through its image \( S = f([m]) \subseteq [n] \), giving a factorization

\[
[m] \xrightarrow{f_{\text{sur}}} S \xrightarrow{f_{\text{in}}} [n]
\]

of maps between ordered sets, and thus a factorization \( \Delta^n \xrightarrow{\Delta^f_{\text{sur}}} \Delta^S \xrightarrow{\Delta^f_{\text{in}}} \Delta^n \) of the induced map \( \Delta^f \) of simplicial sets.

2.12. Exercise. Show that \( \Delta^f_{\text{in}} \) and \( \Delta^f_{\text{sur}} \) respectively induce maps between simplicial sets which are (respectively) injective and surjective on sets of \( k \)-simplices for all \( k \). (The case of \( \Delta^f_{\text{in}} \) is formal, but the case of \( \Delta^f_{\text{sur}} \) is not completely formal.)

2.13. Pictures of standard simplices. When we draw a “picture” of \( \Delta^n \), we draw a geometric \( n \)-simplex: the convex hull of \( n + 1 \) points in general position, with vertices labelled by \( 0, \ldots, n \). The faces of the geometric simplex correspond exactly to injective simplicial operators into \([n]\): these simplices are called non-degenerate simplices of \( \Delta^n \). For each non-degenerate simplex \( f \) in \( \Delta^n \), there is an infinite collection of degenerate simplices with the same “image” as \( f \) (when viewed as a simplicial operator with target \([n]\)).

Here are some “pictures” of standard simplices, which show their non-degenerate simplices. Note that we draw the 1-simplices in \( \Delta^n \) as arrows; this lets us easily see the total ordering on the vertices
We'll extend the terminology of “degenerate” and “non-degenerate” simplices to arbitrary simplicial sets in §15.5.

3. The nerve of a category

The nerve of a category is a simplicial set which retains all the information of the original category. In fact, the nerve construction provides a full embedding of Cat, the category of categories, into sSet, which means that we are able to think of categories as just a special kind of simplicial set.

3.1. Construction of the nerve. Given a category $C$, the nerve of $C$ is the simplicial set $NC$ defined so that

$$(NC)_n := \text{Hom}_{\text{Cat}}([n], C),$$

the set of functors from $[n]$ to $C$, and so that simplicial operators $f : [m] \to [n]$ act by precomposition: $a \mapsto af$ for an element $a : [n] \to C$ in $(NC)_n$.

3.2. Example. There is an evident isomorphism $N[n] \cong \Delta^n$.

Given a functor $F : C \to D$ between categories, we obtain a map $NF : NC \to ND$ of simplicial sets, sending $(a : [n] \to C) \in (NC)_n$ to $(Fa : [n] \to D) \in (ND)_n$. Thus the nerve construction defines a functor $N : \text{Cat} \to \text{sSet}$.

3.3. Structure of the nerve. We observe the following, whose verification we leave to the reader.

- $(NC)_0$ is canonically identified with the set of objects of $C$.
- $(NC)_1$ is canonically identified with the set of morphisms of $C$.
- The operators $(0)^* : (NC)_0 \to (NC)_1$ assign to a morphism its source and target respectively.
- The operator $(00)^* : (NC)_0 \to (NC)_1$ assigns to an object its identity map.
- $(NC)_2$ is in bijective correspondence with the set of pairs $(f, g)$ of morphisms such that $gf$ is defined, i.e., such that the target of $f$ is the source of $g$. This bijection is given by sending $a \in (NC)_2$ to $(a_{01}, a_{12}) \in (NC)_1 \times (NC)_1$.
- The operator $(02)^* : (NC)_2 \to (NC)_1$ assigns, to a simplex corresponding to a pair $(f, g)$ of morphisms, the composite morphism $gf$.

We have the following general description of $n$-simplices in the nerve.

3.4. Proposition. Let $C$ be a category.

1. There is a bijective correspondence

$$(NC)_n \cong \{ (g_1, \ldots, g_n) \in (\text{mor} \ C)^{\times n} \mid \text{target}(g_{i-1}) = \text{source}(g_i) \},$$

which sends $(a : [n] \to C) \in (NC)_n$ to the sequence $(a(0, 1), \ldots, a(n-1, n))$.

2. With respect to the correspondence of (1), the map $f^* : (NC)_n \to (NC)_m$ induced by a simplicial operator $f : [m] \to [n]$ coincides with the function

$$(g_1, \ldots, g_n) \mapsto (h_1, \ldots, h_m), \quad h_k = \begin{cases} 
\text{id} & \text{if } f(k-1) = f(k) \\
g_{j}g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k).
\end{cases}$$
Proposition. 3.8. Suppose there exists a category $C$ and an isomorphism $NC \approx X$. 3.5. Remark. It is clear from the above remarks that most of the information in the nerve of $C$ is redundant: we only needed $(NC)_k$ for $k = 0, 1, 2$ and certain simplicial operators between them to recover $C$.

3.6. Exercise. Show that for any discrete simplicial set $X$ there exists a category $C$ and an isomorphism $NC \approx X$.

3.7. Characterization of nerves. This leads to the question: given a simplicial set $X$, how can we detect that it is isomorphic to the nerve of some category?

3.8. Proposition. A simplicial set $X$ is isomorphic to the nerve of some category if and only if for all $n \geq 2$ the function

$$\phi_n: X_n \to \{ (g_1, \ldots, g_n) \in (X_1)^n \mid g_{i-1}(1) = g_i(0), 1 \leq i \leq n \}$$

which sends $a \in X_n$ to $(a_{01}, \ldots, a_{n-1,n})$ is a bijection.

Proof. First, suppose $X = NC$ for some category $C$. Then the function $\phi_n$ is precisely the bijection of (3.4)(1). Thus, if $X$ is isomorphic to the nerve of some category then its $\phi_n$ are bijections.

Now suppose $X$ is a simplicial set such that the $\phi_n$ are bijections. We define a category $C$, with

(objects of $C$) = $X_0$, (morphisms of $C$) = $X_1$,

following the discussion in (3.3). Thus, the source and target of $g \in X_1$ are $g_0$ and $g_1$ in $X_0$ respectively, the identity map of $x \in X_0$ is $x_{00} \in X_1$, while the composite of $(g, h)$ such that $g_1 = h_0$ is $a_{02}$, where $a \in X_2$ is the unique 2-simplex with $a_{01} = g$ and $a_{12} = h$. We leave the remaining details (e.g., unit and associativity properties) to the reader, though we note that proving associativity requires consideration of elements of $X_3$. (Or look ahead to (5.10), where we carry out the argument explicitly in a slightly different context.)

Next, we claim that for $a \in X_n$, and for $0 \leq i \leq j \leq k \leq n$, we have that

$$a_{i,k} = a_{j,k}a_{i,j},$$

where $a_{i,k}, a_{i,j}, a_{j,k} \in X_1$ are images of $a$ under face operators $[1] \to [n]$, and right-hand side represents composition of two morphisms in $C$. To see this, note first that for $b \in X_2$, we have $b_{02} = b_{12}b_{01}$ by construction of $C$. The general case follows from this by setting $b = a_{i,j,k}$.

Now we can define maps $\psi_n: X_n \to (NC)_n$ by sending $a \in X_n$ to $\psi(a) = a_{i,j} = a_{i,j}$, which is a functor by the above remarks. These maps $\psi_n$ are seen to be bijections using the bijections $\phi_n$ and (3.4), since $\psi_n(a)(i - 1) = a_{i-1,i}$. If $f: [m] \to [n]$ is a simplicial operator, then we compute

$$\psi_n(af)(i \leq j) = (af)_{i,j} = af(i)af(j) = (\psi_n(a))(f(i) \leq f(j)) = (\psi_n(a)f)(i \leq j),$$

whence $\psi$ is a map of simplicial sets. We have thus constructed an isomorphism $\phi: X \to NC$ of simplicial sets, as desired.
3.9. A characterization of maps between nerves. Maps between nerves are the same as functors between categories.

3.10. Proposition. The nerve functor \( N : \text{Cat} \to s\text{Set} \) is fully faithful. That is, every simplicial set map \( g : N(C) \to N(D) \) between nerves is of the form \( g = N(f) \) for a unique functor \( f : C \to D \).

Proof sketch. We need to show that \( \text{Hom}_{\text{Cat}}(C,D) \to \text{Hom}_{s\text{Set}}(N(C),N(D)) \) is a bijection for all categories \( C \) and \( D \). Injectivity is clear, as a functor \( f \) is determined by its action on objects and morphisms, which is exactly the effect of \( N(f) \) on 0- and 1-simplices of the nerves.

For surjectivity, observe that for any map \( g : N(C) \to N(D) \) of simplicial sets, we can define a candidate functor \( f : C \to D \), defined on objects and morphisms by the action of \( g \) on 0-simplices and 1-simplices. That \( F \) has the correct action on identity maps follows from the fact that \( g \) commutes with the simplicial operator \( \langle 0 \rangle : [1] \to [0] \). That \( f \) preserves composition uses (3.4) and the fact that \( g \) commutes with the simplicial operator \( \langle 01 \rangle : [2] \to [1] \).

Note that given \( g : N(C) \to N(D) \) and \( f : C \to D \) as constructed above, the maps \( g, N(f) : N(C) \to N(D) \) coincide on 0-simplices and 1-simplices by construction. It follows that \( g = N(f) \) by (3.11) below. Thus, we have shown that \( N : \text{Hom}_{\text{Cat}}(C,D) \to \text{Hom}_{s\text{Set}}(N(C),N(D)) \) is surjective as desired. \( \square \)

3.11. Exercise. Show that if \( C \) is a category and \( X \) is any simplicial set (not necessarily a nerve), then two maps \( g,g' : X \to NC \) are equal if and only if \( g_0 = g'_0 \) and \( g_1 = g'_1 \), i.e., \( g \) and \( g' \) are equal if and only if they coincide on 0-simplices and 1-simplices. (Hint: use (3.4).)

4. Spines

In this section we will restate our characterization of simplicial sets which are isomorphic to nerves, in terms of a certain “extension” condition. To state this condition we need the notion of a “spine” of a standard \( n \)-simplex.

4.1. The spine of an \( n \)-simplex. The spine of the \( n \)-simplex \( \Delta^n \) is the simplicial set \( I^n \) defined by

\[
(I^n)_k = \{ (a_0 \cdots a_k) \in (\Delta^n)_k \mid a_k \leq a_0 + 1 \}.
\]

That is, a \( k \)-simplex in \( I^n \) is a simplicial operator \( a : [k] \to [n] \) whose image is of the form either \( \{j\} \) or \( \{j,j+1\} \). The action of simplicial operators on simplices in \( I^n \) is induced by their action on \( \Delta^n \). (To see that this action is well defined, observe that for \( a : [k] \to [n] \) in \( (I^n)_k \) and \( f : [p] \to [k] \), the image of the simplicial operator \( a f \) is contained in the image of \( a \).)

There is an evident injective map \( I^n \to \Delta^n \) of simplicial sets. (In fact, \( I^n \) is another example of a subcomplex of \( \Delta^n \), see below §4.9.) Here is a picture of \( I^3 \) in \( \Delta^3 \):

\[
\begin{array}{c}
\langle 0 \rangle \\
\langle 2 \rangle
\end{array} \xrightarrow{(1)} \begin{array}{c}
\langle 1 \rangle \\
\langle 3 \rangle
\end{array} \xrightarrow{(1)} \langle 3 \rangle
\]

is the spine inside

\[
\begin{array}{c}
\langle 0 \rangle \\
\langle 2 \rangle
\end{array} \xrightarrow{(1)} \begin{array}{c}
\langle 1 \rangle \\
\langle 3 \rangle
\end{array} \xrightarrow{(1)} \langle 3 \rangle
\]

Note that \( I^0 = \Delta^0 \) and \( I^1 = \Delta^1 \).

The key property of the spine is the following.

4.2. Proposition. Given a simplicial set \( X \), for every \( n \geq 0 \) there is a bijection

\[
\text{Hom}(I^n,X) \xrightarrow{\sim} \{ (a_1,\ldots,a_n) \in (X_1)^{\times n} \mid a_i(1) = a_{i+1}(0) \},
\]

defined by sending \( f : I^n \to X \) to \((f(\langle 01 \rangle)),f(\langle 12 \rangle),\ldots,f(\langle n-1,n \rangle)) \). (In the case \( n = 0 \), the target of the bijection is taken to be the set \( X_0 \) of vertices of \( X \), and the bijection in this case sends \( f \mapsto f(0) \).)

We will give the proof at the end of this section, after we describe \( I^n \) as a colimit of a diagram of standard simplices; specifically, as a collection of 1-simplices “glued” together at their ends.
4.3. **Nerves are characterized by unique spine extensions.** We can now state our new characterization of nerves: they are simplicial sets such that every map \( I^n \to X \) from a spine extends uniquely along \( I^n \subseteq \Delta^n \) to a map from the standard \( n \)-simplex. That is, nerves are precisely the simplicial sets with “unique spine extensions”.

4.4. **Proposition.** A simplicial set \( X \) is isomorphic to the nerve of some category if and only if the restriction map \( \text{Hom}(\Delta^n, X) \to \text{Hom}(I^n, X) \) along \( I^n \subseteq \Delta^n \) is a bijection for all \( n \geq 2 \).

**Proof.** Immediate from (4.2) and (3.8).

4.5. **Colimits of sets and simplicial sets.** Given any functor \( F: C \to \text{Set} \) from a small category to sets, there is a “simple formula” for its colimit. First consider the coproduct (i.e., disjoint union) \( \coprod_{c \in \text{ob} C} F(c) \) of the values of the functor; I’ll write \((c, x)\) for a typical element of this coproduct, with \( c \in \text{ob} C \) and \( x \in F(c) \). Consider the relation \( \sim \) on this defined by \((c, x) \sim (c', x')\) if \( \exists \alpha: c \to c' \) in \( C \) such that \( F(\alpha)(x) = x' \).

Define \( X := (\coprod_{c \in \text{ob} C} F(c))/\sim \), the set obtained as the quotient by the equivalence relation “\( \sim \)” which is generated by the relation “\( \sim \)”. For each object \( c \) of \( C \) we have a function \( i_c: F(c) \to X \) defined by \( i_c(x) := [(c, x)] \), sending \( x \) to the equivalence class of \((c, x)\). Then the data \((X, \{i_c\})\) is a colimit of the functor \( F \): i.e., for any set \( S \) and collection of functions \( f_c: F(c) \to S \) such that \( f_c' \circ F(\alpha) = f_c \) for all \( \alpha: c \to c' \) there exists a unique function \( f: X \to S \) such that \( f \circ i_c = f_c \).

4.6. **Example.** Verify that \((X, \{i_c\})\) is in fact a colimit of \( F \).

We write \( \text{colim}_C F \) for a chosen colimit of \( F \).

Note: an easy way to satisfy the hypothesis of (4.8) is to show that \( A \) is closed under finite intersection.
4.9. **Subcomplexes.** Given a simplicial set $X$, a **subcomplex** is just a subfunctor of $X$; i.e., a collection of subsets $A_n \subseteq X_n$ which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion $A \to X$ is a morphism of simplicial sets. We typically write $A \subseteq X$ when $A$ is a subcomplex of $X$.

4.10. **Example.** Examples we have already seen include the spines $I^n \subseteq \Delta^n$ and the $\Delta^S \subseteq \Delta^n$ associated to subsets $S \subseteq \llbracket n \rrbracket$.

4.11. **Exercise.** For any map $f: X \to Y$ of simplicial sets, the image $f(X) \subseteq Y$ of $f$ is a subcomplex of $Y$.

For every set $S$ of simplices in a simplicial set, there is a smallest subcomplex which contains the set, namely the intersection of all subcomplexes containing $S$.

4.12. **Example.** For a vertex $x \in X_0$, we write $\{x\} \subseteq X$ for the smallest subcomplex which contains $x$. This subcomplex has exactly one $n$-simplex for each $n \geq 0$, namely $x(0 \cdots 0)$, and thus is isomorphic to $\Delta^0$.

More generally, for a collection of vertices $a, b, c, \ldots \in X_0$, we write $\{a, b, c,\ldots\} \subseteq X$ for the smallest subcomplex which contains $a, b, c,\ldots$. This subcomplex is a discrete simplicial set. This choice of notation is supported by our informal identification of discrete sets with sets (2.5).

The result (4.8) carries over to simplicial sets, where the role of subsets is replaced by subcomplexes.

4.13. **Proposition.** Let $A$ be a collection of subcomplexes of a simplicial set $X$, which is a partially ordered set under “$\subseteq$” and thus can be regarded as a category. Suppose $A$ has the following property: for all $n \geq 0$, all $x \in X_n$, and all $K, L \in A$ such that $x \in K_n \cap L_n$, there exists $M \in A$ such that $x \in M_n$ and $M \subseteq K \cap L$. Then the tautological map

$$\text{colim}_{K \in A} K \to \bigcup_{K \in A} K$$

is a bijection.

**Proof.** Because simplicial sets are actually functors $\Delta^{op} \to \text{Set}$, colimits in simplicial sets are “computed degreewise”. That is, if $F: C \to \text{sSet}$ is a functor with colimit $Y = \text{colim}_{c \in C} F(c) \in \text{sSet}$, then for each $n \geq 0$ there is a canonical bijection

$$Y_n \approx \text{colim}_{c \in C} F(c)_n.$$

The proposition follows using this observation and (4.8). □

4.14. **Remark (Pushouts of subcomplexes).** A special case of (4.13) applied to simplicial sets which we will use constantly is the following. If $K$ and $L$ are subcomplexes of a simplicial set $X$, then so are both $K \cap L$ and $K \cup L$, and furthermore the evident commutative square

$$\begin{array}{ccc}
K \cap L & \rightarrow & L \\
\downarrow & & \downarrow \\
K & \rightarrow & K \cup L
\end{array}$$

is a pushout square in simplicial sets. (Proof: $A = \{K, L, K \cap L\}$.)

4.15. **Subcomplexes of $\Delta^n$.** For each $S \subseteq \llbracket n \rrbracket$ we have a subcomplex $\Delta^S \subseteq \Delta^n$. The following says that every subcomplex of $\Delta^n$ is a union of $\Delta^S$s.

4.16. **Lemma.** Let $K \subseteq \Delta^n$ be a subcomplex. If $(f: [m] \to \llbracket n \rrbracket) \in K_m$ with $f([m]) = S$, then $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$.

This the proof uses the following elementary fact.
4.17. Lemma. Any order preserving surjection \( f : S \to T \) between finite totally ordered sets admits an order preserving section, i.e., \( s : T \to S \) such that \( fs = \text{id}_T \).

Proof. Let \( s(t) = \min \{ s \in S \mid f(s) = t \} \).

Proof of (4.16). Given an order preserving map \( g : [k] \to S \) a dotted arrow \( s \) exists making

\[
\begin{array}{ccc}
[k] & \xrightarrow{s} & [m] \\
\downarrow & & \downarrow
g & \xrightarrow{\sigma} & \xrightarrow{\bar{g}} S \\
[n] & \xrightarrow{f_{\text{sur}}} & \xrightarrow{f_{\text{inj}}} \Delta^n
\end{array}
\]

commute by (4.17). Therefore any simplex \( \bar{g} \in (\Delta^n)_K \) is a simplex \( \bar{g} = fs \in K_k \), and it is clear that \( f \in (\Delta^n)_m \). □

4.18. Remark. Thus, a subcomplex \( K \subseteq \Delta^n \) determines and is determined by a collection \( \mathcal{K} \) of subsets of \([n]\) with the property that \( T \subseteq S \) and \( S \in \mathcal{K} \) implies \( T \in \mathcal{K} \): namely,

\[
\mathcal{K} = \{ S \subseteq [n] \mid \Delta^S \subseteq K \} \quad \text{and} \quad K = \bigcup_{S \in \mathcal{K}} \Delta^S.
\]

In other words, a subcomplex of \( \Delta^n \) is the “same thing” as an abstract simplicial complex whose vertex set is a subset of \([n]\).

We can sharpen the above: every subcomplex of \( \Delta^n \) is a colimit of subcomplexes \( \Delta^S \).

4.19. Proposition. Let \( K \subseteq \Delta^n \) be a subcomplex. Let \( \mathcal{A} \) be the poset of all non-empty subsets \( S \subseteq [n] \) such that the inclusion map \( f : S \to [n] \) represents a (\(|S| - 1\))-simplex in \( K \). Then the tautological map

\[
\text{colim}_{S \in \mathcal{A}} \Delta^S \to K
\]

is an isomorphism.

Proof. We must show that for each \( m \geq 0 \), the map \(\text{colim}_{S \in \mathcal{A}} (\Delta^S)_m \to K_m \) is a bijection. Each \((\Delta^S)_m = \{ f : [m] \to [n] \mid f([m]) \subseteq S \}\) is a distinct subset of \( K_m \subseteq (\Delta^n)_m \); i.e., \( S \neq S' \) implies \((\Delta^S)_m \neq (\Delta^{S'})_m \). In view of (4.13), it suffices to show that for each \( f \in K_m \) there is a minimal \( S \) in \( \mathcal{A} \) such that \( f \in (\Delta^S)_m \). This is immediate from (4.16), which says that \( f \in (\Delta^S)_m \) and \( \Delta^S \subseteq K \) where \( S = f([m]) \), and it is obvious that this \( S \) is minimal with this property. □

4.20. Proof of (4.2). Now we can prove our claim about maps out of a spine, using an explicit description of a spine as a colimit.

Proof of (4.2). Let \( \mathcal{A} \) be the poset of all non-empty \( S \subseteq [n] \) which are simplices of \( I^n \); i.e., subsets of \([n]\) of the form \( \{j\} \) or \( \{j, j + 1\} \). Explicitly the poset \( \mathcal{A} \) has the form

\[
\{0\} \to \{0, 1\} \leftarrow \{1\} \to \{1, 2\} \leftarrow \{2\} \to \cdots \leftarrow \{n - 1\} \to \{n - 1, n\} \leftarrow \{n\}.
\]

By (4.19), \(\text{colim}_{S \in \mathcal{A}} \Delta^S \to I^n \) is an isomorphism. Thus \(\text{Hom}(I^n, X) \approx \text{Hom}(\text{colim}_{S \in \mathcal{A}} \Delta^S, X) \approx \lim_{S \in \mathcal{A}} \text{Hom}(\Delta^S, X)\), and an elementary argument gives the result. □

5. Horns and inner horns

We now are going to give another (less obvious!) characterization of nerves, in terms of “extending inner horns”, rather than “extending spines”. It will be this characterization that we “weaken” to obtain the definition of a quasicategory.
5.1. **Definition of horns.** We define a collection of subobjects of the standard simplices, called “horns”. For each $n \geq 1$, these are subsimplicial sets $\Lambda^n_j \subset \Delta^n$ for each $0 \leq j \leq n$. The horn $\Lambda^n_j$ is the subcomplex of $\Delta^n$ defined by

$$(\Lambda^n_j)_k = \{ f: [k] \to [n] \mid ([n] \setminus \{j\}) \not\subseteq f([k]) \}.$$ 

Using the fact (4.19) that subcomplexes of $\Delta^n$ are always unions of $\Delta^S$s, we see that $\Lambda^n_j$ is the union of “faces” $\Delta^{[n]\setminus i}$ of $\Delta^n$ other than the $j$th face:

$$\Lambda^n_j = \bigcup_{i \neq j} \Delta^{[n]\setminus i} \subset \Delta^n.$$ 

When $0 < j < n$ we say that $\Lambda^n_j \subset \Delta^n$ is an **inner horn**. We also say it is a **left horn** if $j < n$ and a **right horn** if $0 < j$.

5.2. **Example.** The horns inside $\Delta^1$ are just the vertices viewed as subobjects: $\Lambda^1_0 = \Delta^{0} = \{0\} \subset \Delta^1$ and $\Lambda^1_1 = \Delta^{1} = \{1\} \subset \Delta^1$. Neither is an inner horn, the first is a left horn, and the second is a right horn.

5.3. **Example.** These are the three horns inside the 2-simplex.

5.4. **Exercise.** Visualize the four horns inside the 3-simplex. The simplicial set $\Lambda^3_j$ actually kind of looks like a horn: you blow into the vertex $\langle j \rangle$, and sound comes out of the opposite missing face $\Delta^{[3]\setminus j}$.

5.5. **Exercise.** Show that $\Lambda^n_j$ is the largest subobject of $\Delta^n$ which does not contain the simplex $\langle 0, \cdots, \hat{j}, \cdots, n \rangle \in (\Delta^n)_{n-1}$, the “face opposite the vertex $j$”.

We note that inner horns always contain spines: $I^n \subseteq \Lambda^n_j$ if $0 < j < n$. This is also true for non-inner horns if $n \geq 3$, but not for non-inner horns with $n = 1$ or $n = 2$.

5.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit “unique inner horn extensions”; this is different than, but analogous to, the characterization in terms of unique spine extensions (4.4).

5.7. **Proposition.** A simplicial set $X$ is isomorphic to the nerve of a category, if and only if $\text{Hom}(\Delta^n, X) \to \text{Hom}(\Lambda^n_j, X)$ is a bijection for all $n \geq 2$, $0 < j < n$.

The proof will take up the rest of the section.

5.8. **Nerves have unique inner horn extensions.** First we show that nerves have unique inner horn extensions.

5.9. **Proposition.** If $C$ is a category, then for every inner horn $\Lambda^n_j \subset \Delta^n$ the evident restriction map

$$\text{Hom}(\Delta^n, NC) \to \text{Hom}(\Lambda^n_j, NC)$$

is a bijection.
Proof. Since inner horns contain spines, we can consider restriction along \( I^n \subseteq \Lambda^n_j \subseteq \Delta^n \). The composite

\[
\text{Hom}(\Delta^n, NC) \rightarrow \text{Hom}(\Lambda^n_j, NC) \xrightarrow{r} \text{Hom}(I^n, NC)
\]

of restriction maps is a bijection (4.4), so \( r \) is a surjection. Thus, it suffices to show that \( r \) is injective. This is immediate when \( n = 2 \), since \( \Lambda^2_1 = I^2 \), so we can assume \( n \geq 3 \).

We will show that for any inner horn \( \Lambda^n_j \) with \( n \geq 3 \) there exists a finite chain

\[
I^n = F_0 \subset F_1 \subset \cdots \subset F_d = \Lambda^n_j
\]

of subcomplexes, together with a list of subsets \( S_1, \ldots, S_d \subseteq [n] \), such that (i) \( F_i = F_{i-1} \cup \Delta^{S_i} \) and (ii) \( \Delta^{S_i} \subseteq F_{i-1} \cap \Delta^{S_{i-1}} \); here \( \Delta^{S_i} \) denotes the spine of \( \Delta^{S_i} \). Given this, we see by (4.14) that \( F_i \) is isomorphic to a pushout:

\[
F_i \approx \text{colim}(F_{i-1} \leftarrow F_{i-1} \cap \Delta^{S_i} \rightarrow \Delta^{S_i}).
\]

We then obtain a commutative diagram of sets

\[
\begin{array}{ccc}
\text{Hom}(F_i, NC) & \xrightarrow{b} & \text{Hom}(F_{i-1}, NC) \\
\downarrow & & \downarrow \\
\text{Hom}(\Delta^{S_i}, NC) & \xrightarrow{a} & \text{Hom}(F_{i-1} \cap \Delta^{S_i}, NC) \xrightarrow{r} \text{Hom}(I^{S_i}, NC)
\end{array}
\]

where all maps are induced by restriction, in which the square is a pullback (because \( F_i \) is a pushout), and such that the horizontal composition on the bottom is a bijection. It immediately follows that \( a \), and hence \( b \), are injective. We can thus conclude that \( \text{Hom}(\Lambda^n_j, NC) \rightarrow \text{Hom}(I^n, NC) \) is injective as desired, since it is a composite of injective functions such as \( b \).

Now we prove the claim about the filtration of \( \Lambda^n_j \) by suitable subcomplexes \( F_i \).

When \( n = 3 \), we can “attach” simplices in order explicitly:

\[
\Lambda^3_1 = ((I^3 \cup \Delta^{(0,1,2)}) \cup \Delta^{(1,2,3)}) \cup \Delta^{(0,1,3)}, \quad \Lambda^3_2 = ((I^3 \cup \Delta^{(0,1,2)}) \cup \Delta^{(1,2,3)}) \cup \Delta^{(0,2,3)}.
\]

Note that, for instance, in building \( \Lambda^3_1 \), we must add \( \Delta^{(0,1,3)} \) after adding \( \Delta^{(1,2,3)} \), so that the spine \( I^{(0,1,3)} \) of \( \Delta^{(0,1,3)} \) is already present.

When \( n \geq 4 \), we have that \( (\Lambda^n_j)_1 = (\Delta^n)_1 \) and \( (\Lambda^n_j)_2 = (\Delta^n)_2 \). The procedure to “build” \( \Lambda^n_j \) from \( I^n \) by adding subsimplices is: (1) first attach 2-simplices one at a time, in an allowable order; then (2) attach all needed higher dimensional subsimplices. In step (2) the order doesn’t matter since all 1-simplices (and hence all spines) are present in what has already been built. We leave the details of step (1) to the reader. \( \square \)

5.10. **Nerves are characterized by unique inner horn extension.** Let \( X \) be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each \( \text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda^n_j, X) \) is a bijection for all \( 0 < j < n \) with \( n \geq 2 \).
Considering the unique extensions along $\Lambda_2^3 \subset \Delta^2$, we see that this defines a “composition law” on the set $X_1$. That is, given $f, g \in X_1$ such that $f_1 = g_0$ in $X_0$, there is a unique map $u$

$$u: \Lambda_2^3 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \to X, \quad (01) \mapsto f \in X_1, \quad (12) \mapsto g \in X_1.$$ 

Let $\tilde{u}: \Delta^2 \to X$ be the unique extension of $u$ along $\Lambda_2^3 \subset \Delta^2$, and define

$$g \circ f := \tilde{u}_{02}.$$ 

The 2-simplex $\tilde{u}$ is uniquely characterized by: $\tilde{u}_{01} = f$, $\tilde{u}_{12} = g$, $\tilde{u}_{02} = g \circ f$.

This composition law is automatically unital. Given $x \in X_0$, write $1_x := x(00) \in X_1$, so that $(1_x)_0 = x = (1_x)_1$. Then applying the composition law gives $1_x \circ f = f$ and $g \circ 1_x = g$. (Proof: consider the 2-simplices $f(01), g(001) \in X_2$, and use the fact that their representing maps $\Delta^2 \to X$ are the unique extensions of their restrictions to $\Lambda_2^3 \subset \Delta^2$.)

Now consider $\Lambda_3^3 \subset \Delta^3$. Recall (4.19) that $\Lambda_3^3$ is a union (and colimit) of $\Delta^S \subset \Delta^3$ such that $S \not\subset \{0, 2, 3\}$. A map $\Lambda_3^3 \to X$ can be pictured as

so that the planar 2-cells in the picture correspond to non-degenerate 2-simplices in $\Delta^3$, which are contained in $\Lambda_3^3$, while the edges are labelled according to their images in $X$, using the composition law defined above. Let $v: \Delta^3 \to X$ be any extension of the above picture along $\Lambda_3^3 \subset \Delta^3$, and consider the restriction $w := v(023): \Delta^2 \to X$ to the face $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$. Then $w_{01} = g \circ f$, $w_{12} = h$, and $w_{02} = (h \circ g) \circ f$, and thus the existence of $w$ demonstrates that

$$h(g \circ f) = (h \circ g)f.$$ 

In other words, the existence of extensions along $\Lambda_3^3 \subset \Delta^3$ implies that the composition law we defined above is associative. (We could carry out this argument using $\Lambda_2^3 \subset \Delta^3$ instead.)

Thus, given an $X$ with unique inner horn extensions, we can construct a category $C$, so that objects of $C$ are elements of $X_0$, morphisms of $C$ are elements of $X_1$, and composition is given as above.

Next we construct a map $X \to NC$ of simplicial sets. There are obvious maps $\alpha_n: X_n \to (NC)_n$, corresponding to restriction along spines $I^n \subset \Delta^n$; i.e., $\alpha(x) = (x_{01}, \ldots, x_{n-1,n})$. These maps are compatible with simplicial operators, so that they define a map $\alpha: X \to NC$ of simplicial sets. 

**Proof:** For any $n$-simplex $x \in X_n$, all of its edges are determined by edges on its spine via the composition law: $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \cdots \circ x_{i,i+1}$, for all $0 \leq i \leq j \leq n$. Thus for $f: [m] \to [n]$ we have $\alpha(f) = ((f(01), \ldots, (f(n-1,n))) = (x_{f_0 f_1}, \ldots, x_{f_{n-1} f_n}) = (x_{01}, \ldots, x_{n-1,n})f_0 \cdots f_n = (\alpha x)f$.

Now we can prove that nerves are characterized by unique extension along inner horns.

**Proof of** (5.7). We have already shown (5.9) that nerves have unique extensions for inner horns. Consider a simplicial set $X$ which has unique inner horn extension. By the discussion above, we obtain a category $C$ and a map $\alpha: X \to NC$ of simplicial sets, which is clearly a bijection in degrees $\leq 2$. We will show $\alpha_n: X_n \to (NC)_n$ is bijective by induction on $n$.

---

4Recall that $f_1 = f(1)$ and $g_0 = g(0)$, regarded as maps $\Delta^0 \to X$ and thus as elements of $X_0$, using the notation discussed in §2.3.
Fix \( n \geq 3 \), and consider the commutative square
\[
\begin{array}{ccc}
\operatorname{Hom}(\Delta^n, X) & \xrightarrow{\sim} & \operatorname{Hom}(\Lambda^n_1, X) \\
\alpha_{\Delta^n} \downarrow & & \downarrow \alpha_{\Lambda^n_1} \\
\operatorname{Hom}(\Delta^n, NC) & \xrightarrow{\sim} & \operatorname{Hom}(\Lambda^n_1, NC)
\end{array}
\]
The horizontal maps are induced by restriction, and are bijections (top by hypothesis, bottom by (5.9)). Because \( \Lambda^n_1 \) is a colimit of standard simplices of dimension \(< n \) (4.19), the map \( \alpha_{\Lambda^n_1} \) is a bijection by the induction hypothesis. Therefore so is \( \alpha_{\Delta^n} \). \( \square \)

6. Quasicategories

We can now define the notion of a quasicategory, by removing the uniqueness part of the inner horn extension criterion for nerves.

6.1. Identifying categories with their nerves. From this point on, I will (at least informally) often not distinguish a category \( C \) from its nerve. In particular, I may assert something like “let \( C \) be a simplicial set which is a category”, which should be read as “\( C \) is a simplicial set which is isomorphic to the nerve of some category”. This should not lead to much confusion, due to the fact that the nerve functor is a fully faithful embedding of \( \text{Cat} \) into \( s\text{Set} \) (3.10).

6.2. Definition of quasicategory. A quasicategory is a simplicial set \( C \) such that for every map \( f: \Lambda^n_j \to C \) from an inner horn, there exists an extension of it to \( g: \Delta^n \to C \). That is, \( C \) is a quasicategory if the function \( \operatorname{Hom}(\Delta^n, C) \to \operatorname{Hom}(\Lambda^n_j, C) \) induced by restriction along \( \Lambda^n_j \subset \Delta^n \) is surjective for all \( 0 < j < n \), \( n \geq 2 \), so there always exists a dotted arrow in any commutative diagram of the form

Any category (more precisely, the nerve of any category) is a quasicategory. In fact, by what we have shown (5.7) a category is precisely a quasicategory for which there exist unique extensions of inner horns.

Let \( C \) be a quasicategory. We refer to elements of \( C_0 \) as the objects of \( C \), and elements of \( C_1 \) as the morphisms of \( C \). Every morphism \( f \in C_1 \) has a source and target, namely its vertices \( f_0, f_1 \in C_0 \). We write \( f: f_0 \to f_1 \), just as we would for morphisms in a category. Likewise, for every object \( x \in C_0 \), there is a distinguished morphism \( 1_x: x \to x \), called the identity morphism, defined by \( 1_x = x_{00} \). When \( C \) is (the nerve of) a category, all the above notions coincide with the usual ones. (Note, however, that we cannot generally define composition of morphisms in a quasicategory in the same way we do for a category.)

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

6.3. Products of quasicategories. Simplicial sets are functors, so the product of simplicial sets \( X \) and \( Y \) is just the product of the functors. Thus, \((X \times Y)_n = X_n \times Y_n\).

6.4. Proposition. The product of two quasicategories (as simplicial sets) is a quasicategory.

Proof. Exercise, using the bijective correspondence between the sets of (i) maps \( K \to X \times Y \) and (ii) pairs of maps \( (K \to X, K \to Y) \). \( \square \)

6.5. Exercise. If \( C \) and \( D \) are categories, then \( N(C \times D) \approx NC \times ND \). Thus, the notion of product of quasicategories generalizes that of categories.
6.6. **Coproduts of quasicategories.** The coproduct of simplicial sets \( X \) and \( Y \) is just the coproduct of functors, whence \( (X \sqcup Y)_n = X_n \sqcup Y_n \), i.e., the set \( n \)-simplices of the coproduct is the disjoint union of the sets of \( n \)-simplices of \( X \) and \( Y \). More generally, \( \left( \coprod_s X_s \right)_n = \coprod_s (X_s)_n \) for an indexed collection \{\( X_s \)\} of simplicial sets.

6.7. **Proposition.** The coproduct of any indexed collection of quasicategories is a quasicategory.

To prove this, we introduce the set of **connected components** of a simplicial set. Given a simplicial set \( X \), define an equivalence relation \( \approx \) on the set \( \coprod_{n \geq 0} X_n \) of all simplices, generated by the relation

\[
a \approx af \quad \text{for all } n \geq 0, \ a \in X_n, \ f : [m] \to [n].
\]

Thus two simplices are related when you can get from one to another by a sequence of simplicial operators. An equivalence class for \( \approx \) is called a **connected component** of \( X \), and we write \( \pi_0 X \) for the set of connected components. This construction defines a functor \( \pi_0 : sSet \to Set \).

6.8. **Exercise** (Connected components are path components). Show that there is a canonical bijection

\[
(X_0 / \approx_1) \sim X,
\]

where the left-hand side denotes the set of equivalence classes in the vertex set \( X_0 \) with respect to the equivalence relation \( \approx_1 \) which is generated by the relation \( \sim_1 \) on \( X_0 \), defined by

\[
a \sim_1 b \quad \text{iff there exists } e \in X_1 \text{ such that } a = e_0, b = e_1.
\]

6.9. **Exercise.** Show that there is a bijection \( \text{colim}_{\Delta^{op}} X \sim \pi_0 X \), between the set of connected components of \( X \) and the colimit of the functor \( X : \Delta^{op} \to Set \).

6.10. **Exercise** (Connected components respect colimits). Show that if \( X \) is the colimit of a functor \( F : D \to sSet \) of some small category \( D \), then \( \pi_0 X \approx \text{colim}_D \pi_0 F \). In particular, \( \pi_0 \left( \coprod_s X_s \right) \approx \coprod_s \pi_0 (X_s) \) for any collection \{\( X_s \)\} of simplicial sets.

We say that a simplicial set \( X \) is **connected** if \( \pi_0 X \) is a singleton.

6.11. **Exercise.** Show that every standard simplex \( \Delta^n \) is connected, and that every horn \( \Lambda^p_j \) is connected.

6.12. **Exercise** (Every simplicial set is a coproduct of its connected components). Let \( X \) be a simplicial set. Given \( a \in \pi_0 X \), let \( C_a \) denote its equivalence class (regarded as a subset of the set \( \coprod_{n \geq 0} X_n \) of simplices).

(1) Show that \( C_a \) is closed under the action of simplicial operators, and thus describes a subcomplex of \( X \).

(2) Show that the evident map

\[
\prod_{a \in \pi_0 X} C_a \to X
\]

is an isomorphism of simplicial sets.

**Proof of (6.7).** If \( X = \coprod_s X_s \) is a coproduct of simplicial sets, then any connected component of \( X \) must be contained in exactly one of the \( X_s \) summands, by (6.10). The proof is now straightforward, using (6.12) and the fact that horns and standard simplices are connected (6.11).

6.13. **Exercise** (Important). Show that the evident map \( \pi_0 (X \times Y) \to \pi_0 X \times \pi_0 Y \) is a bijection.
6.14. **Full subquasicategories.** Given a category $C$ and a set of objects $S \subseteq \text{ob} C$, the **full subcategory spanned by** $S$ is the subcategory $C' \subseteq C$ with $\text{ob} C' = S$ and with $\text{mor} C' = \{ f \in \text{mor} C \mid \text{source}(f), \text{target}(f) \in S \}$.

This has a straightforward generalization to quasicategories. Given a simplicial set $C$ and a set $S \subseteq C_0$ of vertices, let

$$C'_n = \{ a \in C_n \mid a_j \in S \text{ for all } j = 0, \ldots, n \},$$

the set of $n$-simplices in $C$ all of whose vertices are in $S$.

6.15. **Exercise.** Show that $C'$ is a subcomplex of $C$, and that if $C$ is a quasicategory then so is $C'$.

When $C$ is a quasicategory, the subcomplex $C'$ is called the **full subcategory spanned by** $S$.

(Note: it would be more logical to say “subquasicategory”, but it is a mouthful. When constructing new terminology like this we often leave out the “quasi” if it won’t cause confusion.)

6.16. **Opposite of a quasicategory.** Given a category $C$, the **opposite category** $C^{\text{op}}$ has $\text{ob} C^{\text{op}} = \text{ob} C$, and $\text{Hom}_{C^{\text{op}}}(x, y) = \text{Hom}_C(y, x)$, and the sense of composition is reversed: $g \circ_{C^{\text{op}}} f = f \circ_C g$.

This concept also admits a generalization to quasicategories, which we define using a non-trivial involution $\text{op}: \Delta \rightarrow \Delta$ of the category $\Delta$. This is the functor which on objects sends $[n] \mapsto [n]$, and on morphisms sends $(f_0, \ldots, f_n): [n] \rightarrow [m]$ to $(m - f_n, \ldots, m - f_0)$.

6.17. **Remark.** You can visualize this involution as the functor which “reverses the ordering” of the totally-ordered sets $[n]$. Note that the totally ordered set “$[n]$ with the order of its elements reversed” isn’t actually an object of $\Delta$, but rather is uniquely isomorphic to $[n]$, via the function $x \mapsto n - x$.

The **opposite** of a simplicial set $X: \Delta^{\text{op}} \rightarrow \text{Set}$ is the composite functor $X^{\text{op}} := X \circ \text{op}$. We have that $(\Delta^n)^{\text{op}} = \Delta^n$, while $(\Lambda^j_n)^{\text{op}} = \Lambda^{n-j}_n$, so that the opposite of an inner horn is another inner horn. As a consequence, the opposite of a quasicategory is a quasicategory. It is straightforward to verify that $(\text{NC})^{\text{op}} = N(C^{\text{op}})$, so the notion of opposite quasicategory generalizes the notion of opposite category. The functor $\text{op}: \Delta \rightarrow \Delta$ satisfies $\text{op} \circ \text{op} = \text{id}_\Delta$, so $(X^{\text{op}})^{\text{op}} = X$.

7. **Functors and natural transformations**

7.1. **Functors.** A **functor** between quasicategories is merely a map $f: C \rightarrow D$ between the simplicial sets.

We write $\text{QCat}$ for the category of quasicategories and functors between them.\(^5\) Clearly $\text{QCat} \subset \text{sSet}$ is a full subcategory. Because the nerve functor is a full embedding of $\text{Cat}$ into $\text{QCat}$, any functor between ordinary categories is also a functor between quasicategories.

7.2. **Exercise** (Mapping property of a full subcategory). Let $C$ be a quasicategory, and $C' \subseteq C$ the full subcategory spanned by some subset $S \subseteq C_0$. Show that a functor $f: D \rightarrow C$ factors through a functor $f': D \rightarrow C'$ if and only if $f(D_0) \subseteq S$.

7.3. **Natural transformations.** Given functors $F, G: C \rightarrow D$ between categories, a **natural transformation** $\phi: F \Rightarrow G$ is a choice, for each object $c$ of $C$, of a map $\phi(c): F(c) \rightarrow G(c)$ in $D$, such that for every morphism $\alpha: c \rightarrow c'$ in $C$ the square

$$\begin{array}{ccc}
F(c) & \xrightarrow{\phi(c)} & G(c) \\
\downarrow{f(\alpha)} & & \downarrow{g(\alpha)} \\
F(c') & \xrightarrow{\phi(c')}& G(c')
\end{array}$$

commutes in $D$.

\(^5\)Lurie [Lur09] denotes this category by $\text{Cat}_\Delta$. 


There is a standard convenient reformulation of this: a natural transformation \( \phi : F \Rightarrow G \) is the same thing as a functor

\[ H : C \times [1] \to D, \]

so that \( H[C \times \{0\}] = F, \ H[C \times \{1\}] = G, \) and \( H[\{c\}] \times [1] = \alpha(c) \) for each \( c \in \text{ob} \, C. \) (Here we make implicit use of the evident isomorphisms \( C \times \{0\} \approx C \approx C \times \{1\}. \))

This reformulation admits a straightforward generalization to quasicategories. A natural transformation \( f_0 \Rightarrow f_1 \) of functors \( f_0, f_1 : C \to D \) between quasicategories is defined to be a map

\[ f : C \times N[1] = C \times \Delta^1 \to D \]

of simplicial sets such that \( f[C \times \{i\}] = f_i \) for \( i = 0, 1. \) For ordinary categories this coincides with the classical notion.

8. Examples of Quasicategories

There are many ways to produce quasicategories, as we will see. Unfortunately, “hands-on” constructions of quasicategories are relatively rare. Here I give a few reasonably explicit examples to play with.

8.1. Large vs. small. I have been implicitly assuming that certain categories are small; i.e., they have sets of objects and morphisms. For instance, for the nerve of a category \( C \) to be a simplicial set, we need \( C_0 = \text{ob} \, C \) to be a set.

However, in practice many categories of interest are only locally small; i.e., the collection of objects is not a set but is a “proper class”, although for any pair of objects \( \text{Hom}_C(X,Y) \) is a set. For instance, the category Set of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These are called large categories.

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) “universes”. A universe \( U \) is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe \( U \), there is a larger universe \( U' \) such that \( U \in U' \). Thus, if by “set” we mean “\( U \)-set”, then the category Set is a “\( U \)-category”. This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable strongly inaccessible cardinals.

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the simplices form a set), while the nerve of a large category is a large simplicial set.

I’m not going to be pedantic about this. I’ll usually assume categories like Set, Cat, sSet, etc., are categories whose objects are “small” sets/categories/simplicial sets/whatever, i.e., are built from sets in a fixed universe \( U \) of “small sets”. However, I sometimes need to consider examples of sets/categories/simplicial sets/whatever which are not small. I leave it to the reader to determine when this is the case.

In practice, a main point of concern involves constructions limits and colimits. Many typical examples of categories \( C = \text{Set}, \text{Cat}, \text{sSet}, \) etc., in which objects are built out of small sets are small complete and small cocomplete: any functor \( F : D \to C \) from a small category \( D \) has a limit and a colimit in \( C. \) This is not true if \( D \) is not assumed to be small. In this case care about the small/large distinction is necessary.
8.2. The Morita quasicategory. This is an example of a quasicategory in which objects are associative rings, morphisms between two rings are bimodules for the pair of rings, and 2-simplices are given by certain isomorphisms of bimodules.

Define a simplicial set \( C \), so that \( C_n \) is a set whose elements are data \( x := (A_i, M_{ij}, f_{ijk}) \), where

- for each \( i \in [n] \), \( A_i \) is an associative ring,
- for each \( i < j \) in \([n] \), \( M_{ij} \) is an \((A_i, A_j)\)-bimodule,
- for each \( i < j < k \) in \([n] \), \( f_{ijk} : M_{ij} \otimes A_j M_{jk} \rightarrow M_{ik} \) is an isomorphism of \((A_i, A_k)\)-bimodules, such that
- for each \( i < j < k < \ell \), the diagram

\[
\begin{array}{ccc}
M_{ij} \otimes M_{jk} \otimes M_{k\ell} & \xrightarrow{id \otimes f_{jk\ell}} & M_{ij} \otimes M_{j\ell} \\
\downarrow f_{ij\ell} & & \downarrow f_{ij\ell} \\
M_{ik} \otimes M_{k\ell} & \xrightarrow{f_{ik\ell}} & M_{i\ell}
\end{array}
\]

(8.3)

commutes.

Here is a picture of the data of an \( n \)-simplex for \( n \in \{0, 1, 2, 3\} \):

\[
\begin{array}{ccc}
A_0 & \xrightarrow{M_{01}} & A_1 \\
\downarrow M_{01} & & \downarrow M_{12} \\
A_1 & \xrightarrow{f_{012}} & A_2 \\
\downarrow f_{012} & & \downarrow f_{123} \\
A_2 & \xrightarrow{M_{023}} & A_3
\end{array}
\]

For an simplicial operator \( \delta : [m] \rightarrow [n] \), we define \( x\delta := (A_{\delta(i)}, M_{\delta(i)\delta(j)}, f_{\delta(i)\delta(j)\delta(k)}) \). When \( \delta \) is injective this stands as it is, but if \( \delta \) is not injective, we must set \( M_{ij} := A_{\delta(i)} \) when \( \delta(i) = \delta(j) \), and set \( f_{ijk} \) to the canonical isomorphism \( A_{\delta(i)} \otimes A_{\delta(j)} M_{\delta(j)\delta(k)} \rightarrow M_{\delta(i)\delta(k)} \) if \( \delta(i) = \delta(j) \) or \( M_{\delta(i)\delta(j)} \otimes A_{\delta(j)} M_{\delta(k)} \rightarrow M_{\delta(i)\delta(k)} \) if \( \delta(j) = \delta(k) \).

I claim that \( C \) is a quasicategory. Fillers for \( \Lambda^n_1 \subset \Delta^2 \) always exist: a map \( \Lambda^n_1 \rightarrow C \) is a choice of \((A_0, M_{01}, A_1, M_{12}, A_2)\), and an extension to \( \Delta^2 \) can be given by setting \( M_{02} \) to be the tensor product, and \( f_{012} \) the identity map. Note that there can be more than one choice: even keeping \( M_{02} \) the same, there is a choice of isomorphism \( f_{012} \).

Fillers for \( \Lambda^n_2 \subset \Delta^3 \) and \( \Lambda^n_3 \subset \Delta^3 \) always exist, and are unique: finding a filler amounts to choosing isomorphisms \( f_{023} = f_{ik\ell} \) (for \( \Lambda^n_1 \)) or \( f_{013} = f_{ij\ell} \) (for \( \Lambda^n_2 \)) making (8.3) commute, and such choices are unique. Similarly, all fillers in higher dimensions \( \Lambda^n_j \subset \Delta^n \) with \( n \geq 4 \) exist and are unique.

8.4. Quasicategory of categories. Define a simplicial set \( C \) so that \( C_n \) is a set whose elements are data \( x := (C_i, F_{ij}, \phi_{ijk}) \) where
- for each \( i \in [n] \), \( C_i \) is a (small) category,
- for each \( i < j \) in \([n] \), \( F_{ij} : C_i \rightarrow C_j \) is a functor,
- for each \( i < j < k \) in \([n] \), \( \phi_{ijk} : F_{j} F_{ij} \Rightarrow F_{ik} \) is a natural isomorphism of functors \( C_i \rightarrow C_k \), such that
- for each \( i < j < k < \ell \), the diagram

\[
\begin{array}{ccc}
F_{ik} F_{ij} & \xrightarrow{\phi_{ijk} \id_{F_{ij}}} & F_{ij} F_{ijk} \\
\id_{F_{ik}} \phi_{ijk} & & \phi_{ijk} \\
F_{ik} F_{ik} & \xrightarrow{\phi_{ik\ell}} & F_{ik} F_{ik}
\end{array}
\]

is commutative.
commutes.
The action of simplicial operators is defined exactly as in the previous example, as is the proof that $C$ is a quasicategory.

8.5. **Nerve of a crossed module.** A **crossed module** is data $(G, H, \phi, \rho)$, consisting of groups $G$ and $H$, and homomorphisms $\phi: H \to G$ and $\rho: G \to \text{Aut} H$, such that

$$
\phi(\rho(g)(h)) = g\phi(h)g^{-1}, \quad \rho(\phi(h))(h') = hh'h^{-1},
$$

for all $g \in G$, $h, h' \in H$.

(For instance: $G = H = \text{the cyclic group of order 4, with } \phi(x) = x^2$ and $\rho$ the non-trivial action.) From this we can construct a quasicategory (in fact, a "quasigroupoid") much as in the last example: an $n$-simplex is data $(g_{ij}, h_{ijk})$ with $g_{ij} \in G$, $h_{ijk} \in H$, satisfying identities

$$
g_{ij}g_{jk} = \phi(h_{ijk})g_{ik}, \quad h_{ijk}h_{ik\ell} = \rho(g_{ij})(h_{\ell jk})h_{ijk}.
$$

8.6. **Spans.** (See [Bar14, §§2–3], where this is called the effective Burnside ∞-category.) For each object $[n]$ of $\Delta$, define $[n]^\text{tw}$ to be the category with

- **objects** pairs $(i, j)$ with $0 \leq i \leq j \leq n$, and
- a unique **morphism** $(i, j) \to (i', j')$ whenever $i' \leq i \leq j'$.

The construction $[n] \mapsto [n]^\text{tw}$ defines a functor $\Delta \to \text{Cat}$. (The category $[n]^\text{tw}$ is called the twisted arrow category of $[n]$; in fact you can define a twisted arrow category $C^\text{tw}$ for any category $C$.)

Let $C$ be a category which has pullbacks; for an explicit example, think of the category of finite sets. Let $\mathcal{R}(C)$ be the simplicial set defined so that

$$
\mathcal{R}(C)_n := \{\text{functors } ([n]^\text{tw})^\text{op} \to C\}.
$$

Elements of $\mathcal{R}(C)_0$ are just objects of $C$. Elements of $\mathcal{R}(C)_1$, $\mathcal{R}(C)_2$, $\mathcal{R}(C)_3$ are respectively diagrams in $C$ of shape

$$
\begin{array}{cccc}
X_{00} & X_{01} & X_{02} & X_{03} \\
X_{11} & X_{12} & X_{13} & X_{13} \\
X_{22} & X_{23} & X_{23} & X_{33}
\end{array}
$$

Let $\mathcal{A}(C)_n \subseteq \mathcal{R}(C)_n$ denote the subset whose $n$-simplices are functors $X: ([n]^\text{tw})^\text{op} \to C$ such that for every $i' \leq i \leq j \leq j'$ the square

$$
\begin{array}{ccc}
X_{i'j'} & \to & X_{ij'} \\
\downarrow & & \downarrow \\
X_{i'j} & \to & X_{ij}
\end{array}
$$

is a pullback in $C$. Then $\mathcal{A}(C)$ is a subcomplex, and in fact is a quasicategory. This is another example in which extensions along inner horns $\Lambda^n_j \subseteq \Delta^n$ exist for $n \geq 2$, and are unique for $n \geq 3$.

8.7. **Singular complex of a space.** The **topological $n$-simplex** is

$$
\Delta^n_{\text{top}} := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, \ x_i \geq 0 \right\},
$$

the convex hull of the standard basis vectors. We get a functor $\Delta_{\text{top}}: \Delta \to \text{Top}$ by $\Delta_{\text{top}}([n]) = \Delta^n_{\text{top}}$.

For a topological space $T$, we define $\text{Sing} T$ to be the simplicial set $[n] \mapsto \text{Hom}_{\text{Top}}(\Delta^n_{\text{top}}, T)$.

Define **topological horns**

$$
(\Lambda^n_j)_{\text{top}} := \{ x \in \Delta^n_{\text{top}} \mid \exists i \in [n] \setminus \{j\} \text{ such that } x_i = 0 \} \subset \Delta^n_{\text{top}},
$$

and observe that continuous maps $(\Lambda^n_j)_{\text{top}} \to T$ correspond in a natural way with maps $\Lambda^n_j \to \text{Sing} T$.

*(Exercise. This is a consequence of the fact that $\Lambda^n_j$ is a colimit of all the $\Delta^S \subseteq \Delta^n_j$, and that $(\Lambda^n_j)_{\text{top}}$...*
is similarly a colimit of all the $\Delta^n_\text{top} \subseteq (\Lambda^n_j)_\text{top}$.) There exists a continuous retraction $\Delta^n_\text{top} \to (\Lambda^n_j)_\text{top}$ (Exercise: describe such a retraction), and thus we see that

$$\text{Hom}(\Delta^n, \text{Sing}T) \to \text{Hom}(\Lambda^n_j, \text{Sing}T)$$

is surjective for every horn (not just inner ones).

8.8. **Remark** (Kan complexes). A simplicial set $X$ which has extensions for all horns is called a **Kan complex**. Thus, Sing$T$ is a Kan complex, and so in particular is a quasicategory (and as we will see below, a “quasigroupoid” (10.11)).

8.9. **Eilenberg-MacLane object**. Fix an abelian group $A$ and an integer $d \geq 0$. We define a simplicial set $K = K(A, d)$, so that $K_n$ is a set whose elements are data $a = (a_{i_0,...,i_d})$ consisting of

- for each $0 \leq i_0 \leq \cdots \leq i_d \leq n$, an element $a_{i_0,...,i_d} \in A$, such that
- $a_{i_0,...,i_d} = 0$ if $i_{u-1} = i_u$ for any $u$, and
- for each $0 \leq j_0 \leq \cdots \leq j_{d+1} \leq n$ we have $\sum u(-1)^u a_{j_0,...,\hat{i}_u,...,j_{d+1}} = 0$.

(Here “$j_0 \cdots \hat{j}_u \cdots j_{d+1}$” is shorthand for the subsequence $j_0, j_1, \ldots, j_{u-1}, j_{u+1}, \ldots, j_d, j_{d+1}$ with $j_u$ omitted.

For a map $\delta: [m] \to [n]$ we define

$$(a\delta)_{i_0,...,i_d} = a_{\delta(i_0)\cdots\delta(i_d)}.$$ 

The object $K(A, d)$ is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid). When $d = 0$, this is just a discrete simplicial set, equal to $A$ in each dimension.

8.10. **Exercise**. Show that $K(A, 1)$ is isomorphic to the nerve of a category, namely the nerve of the group $A$ regarded as a category with one object.

8.11. **Exercise**. Show that $K(A, d)$ is a Kan complex, i.e., that $\text{Hom}(\Delta^n, K(A, d)) \to \text{Hom}(\Lambda^n_j, K(A, d))$ is surjective for all horns $\Lambda^n_j \subseteq \Delta^n$. In fact, this map is bijective unless $n = d$. (Hint: there are four distinct cases to check, namely $n < d$, $n = d$, $n = d + 1$, and $n > d + 1$.)

8.12. **Exercise**. Given a simplicial set $X$, a **normalized $d$-cocycle** with values in an abelian group $A$ is a function $f: X_d \to A$ such that

\begin{enumerate}
  \item $f(x_{i_0,...,i_d,...,d-1}) = 0$ for all $x \in X_{d-1}$ and $0 \leq i > d - 1$, and
  \item $\sum (-1)^i f(x_{i_0,...,\hat{i}_u,...,i_d,...,d+1}) = 0$ for all $x \in X_{d+1}$ and $0 \leq i < d + 1$.
\end{enumerate}

Show that the set $Z^n_{\text{norm}}(X; A)$ of normalized $d$-cocycles on $X$ is in bijective correspondence with $\text{Hom}_{\text{Set}}(X, K(A, d))$. (Hint: an element $a \in K_n$ is uniquely determined by the collection of elements $a\delta \in K_d = A$, as $\delta$ ranges over injective maps $[d] \to [n]$.)

8.13. **Remark**. Eilenberg-MacLane objects are an example of a **simplicial abelian group** the map $+: K \times K \to K$ defined in each dimension by $(a + b)_{i_0,...,i_d} = a_{i_0,...,i_d} + b_{i_0,...,i_d}$ is a map of simplicial sets which satisfies the axioms of an abelian group, reflecting the fact that $Z^n_{\text{norm}}(X; A)$ is an abelian group.

9. **Homotopy category of a quasicategory**

Our next goal is to define the notion of an **isomorphism** in a quasicategory. This notion behaves much like that of **homotopy equivalence** in topology. We will define isomorphism by means of the **homotopy category** of a quasicategory. If we think of a quasicategory as “an ordinary category with higher structure”, then its homotopy category is the ordinary category obtained by “flattening out the higher structure”.
9.1. The fundamental category of a simplicial set. The homotopy category of a quasicategory is itself a special case of the notion of the fundamental category of a simplicial set, which we turn to first.

A fundamental category for a simplicial set $X$ consists of (i) a category $hX$, and (ii) a map $\alpha: X \to N(hX)$ of simplicial sets, such that for every category $C$, the map

$$\alpha^*: \text{Hom}(N(hX), NC) \to \text{Hom}(X, NC)$$

induced by restriction along $\alpha$ is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

9.2. Proposition. Every simplicial set has a fundamental category.

Proof sketch. Given $X$, we construct $hX$ by generators and relations. First, consider the free category $F$, whose objects are the set $X_0$, and whose morphisms are finite “composable” sequences $[a_n, \ldots, a_1]$ of edges of $X_1$. Thus, morphisms in $F$ are “words”, whose “letters” are 1-simplices $a_i$ with $(a_{i+1})_0 = (a_i)_1$, and composition is concatenation of words; the element $[a_n, \ldots, a_1]$ is then a morphism $(a_1)_0 \to (a_n)_1$. (Note: we also suppose that there is an empty sequence $[]_x$ in $F$ for each vertex $x \in X_0$; these correspond to identity maps in $F$.)

Then $hX$ is defined to be the largest quotient category of $F$ subject to the following relations on morphisms:

- $[a] \sim []_x$ for each $x \in X_0$ where $a = x_{00} \in X_1$, and
- $[g, f] \sim [h]$ whenever there exists $a \in X_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$.

The map $\alpha: X \to N(hX)$ sends $x \in X_0$ to the equivalence class of $[x_{n-1,n}, \ldots, x_{0,1}]$. Given this, verifying the desired universal property of $\alpha$ is formal.

(We will give another construction of the fundamental category in (13.18).) \qed

9.3. Exercise. Complete the proof of (9.2) by showing that $\alpha^*: \text{Hom}(N(hX), NC) \to \text{Hom}(X, NC)$ is a bijection for any category $C$.

As a consequence: the fundamental category construction describes a functor $h: s\text{Set} \to \text{Cat}$, which is left adjoint to the nerve functor $N: \text{Cat} \to s\text{Set}$.

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of explicitly, because it is difficult to give an explicit description of a “quotient category” induced by a relation on its morphisms. We will not be making much use of it. When $C$ is a quasicategory, there is a more concrete construction of $hC$, which in this context is called the homotopy category of $C$. Warning: Sometimes people will not distinguish “fundamental category” from “homotopy category” as I have here, and just call either the homotopy category.

9.4. The homotopy relation on morphisms. Fix a quasicategory $C$. For $x, y \in C_0$, let $\text{hom}_C(x, y) := \{f \in C_1 \mid f_0 = x, f_1 = y\}$ denote the set of “morphisms” in $C$ from $x$ to $y$. We write $1_x$ for the degenerate element $x_{00} \in \text{hom}_C(x, x)$.

Define relations $\sim_\ell$, $\sim_r$ on $\text{hom}_C(x, y)$ (called left homotopy and right homotopy) by

- $f \sim_\ell g$ iff there exists $a \in C_2$ with $a_{01} = 1_x$, $a_{02} = f$, $a_{12} = g$,
- $f \sim_r g$ iff there exists $b \in C_2$ with $b_{12} = 1_y$, $b_{01} = f$, $b_{02} = g$.

Pictorially:

$$f \sim_\ell g: \quad \begin{array}{c} x \\ \downarrow \ell \\ a \\ \downarrow \ell \\ y \end{array} \quad \quad \quad \quad \quad f \sim_r g: \quad \begin{array}{c} x \\ \downarrow r \\ b \\ \downarrow r \\ y \end{array}$$

Note that $f \sim_\ell g$ in $\text{hom}_C(x, y)$ coincides with $g \sim_r f$ in $\text{hom}_{C^{op}}(y, x)$.

9.5. Remark. If $C$ is an ordinary category, then the left homotopy and right homotopy relations reduce to the equality relation on morphisms $x \to y$. 
9.6. **Proposition.** The relations $\sim_{\ell}$ and $\sim_r$ are identical, and are an equivalence relation on $\text{hom}_C(x, y)$.

**Proof.** Given $f, g, h : x \to y$ in a quasicategory $C$, we will prove

(1) $f \sim_{\ell} f$,
(2) $f \sim_{\ell} g$ and $g \sim_{\ell} h$ imply $f \sim_{\ell} h$,
(3) $f \sim_{\ell} g$ implies $f \sim_r g$,
(4) $f \sim_r g$ implies $g \sim_{\ell} f$.

These show that $\sim_{\ell}$ is an equivalence relation, and also that $\sim_r$ and $\sim_{\ell}$ coincide. The idea is to use the inner-horn extension condition for $C$ to produce the appropriate relations.

(1) $f \sim_{\ell} f$ is exhibited by $f_{001} \in C_2$.

(2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of $\Delta^3$ (respectively $\Lambda^3_1, \Lambda^3_1$, and $\Lambda^3_2$) to $C$ constructed from the given data. The restriction of any extension to $\Delta^3$ along the remaining face gives the conclusion.

9.7. **Composition of homotopy classes of morphisms.** We now define $f \approx g$ to mean $f \sim_{\ell} g$ (equivalently $f \sim_r g$). We speak of **homotopy classes** $[f]$ of morphisms $f \in \text{hom}_C(x, y)$, meaning equivalence classes under $\approx$. Next we observe that we can compose homotopy classes.

Given $f \in \text{hom}_C(x, y)$, $g \in \text{hom}_C(y, z)$, $h \in \text{hom}_C(x, z)$, we say that $h$ is a **composite** of $(g, f)$ if there exists a 2-simplex $a \in C_2$ with $a(01) = f$, $a(12) = g$, $a(02) = h$; thus composition is a three-fold relation on $\text{hom}(x, y) \times \text{hom}(y, z) \times \text{hom}(x, z)$. The composite relation is compatible with the homotopy relation, as shown by the following.

9.8. **Lemma.** If $f \approx f'$, $g \approx g'$, $h$ a composite of $(g, f)$, and $h'$ a composite of $(g', f')$, then $h \approx h'$.

**Proof.** Since $\approx$ is an equivalence relation, it suffices prove the special cases (a) $f = f'$, and (b) $g = g'$. We prove case (b).

Let $a \in C_2$ exhibit $f \sim_{\ell} f'$, and let $b, b' \in C_2$ exhibit $h$ as a composite of $(g, f)$ and $h'$ as a composite of $(g', f')$ respectively. The inner horn $\Lambda^3_2 \to C$ defined by

extends to $u : \Delta^3 \to C$, and $u|\Delta^{(0,1,3)}$ exhibits $h \sim_{\ell} h'$.
Thus, composites of \((g, f)\) live in a unique homotopy class of morphisms in \(C\), which only depends on the homotopy classes of \(g\) and \(f\). I will write \([g] \circ [f]\) for this class.

I’ll leave the following as exercises; the proofs are much like what we have already seen.

**9.9. Lemma.** Given \(f : x \to y\), we have \([f] \circ [1_x] = [f] = [1_y] \circ [f]\).

**9.10. Lemma.** If \([g] \circ [f] = [u], [h] \circ [g] = [v]\), then \([h] \circ [u] = [v] \circ [f]\).

**9.11. The homotopy category of a quasicategory.** For any quasicategory, we define its homotopy category \(hC\), so that \(\text{ob}(hC) := C_0\), while \(\text{hom}_hC(x, y) := \text{hom}_C(x, y)/ \approx\), with composition defined by \([g] \circ [f]\). The above lemmas (9.9) and (9.10) exactly imply that \(hC\) is a category.

We define a map \(\pi : C \to N(hC)\) of simplicial sets as follows. On 0 simplices, \(\pi\) is the identity map \(C_0 = N(hC)_0 = \text{ob} hC\). On 1-simplices, the map is defined by the tautological quotient maps \(\text{hom}_C(x, y) \to \text{hom}_C(x, y)/ \approx\) sending \(f \mapsto [f]\). The map \(\pi\) sends an \(n\)-simplex \(a \in C_n\) to the unique \(n\)-simplex \(\pi(a) \in N(hC)_n\) such that \(\pi(a)_{i-1, i} = \pi(a_{i-1, i})\). These functions are seen to be compatible with simplicial operators using the following exercise.

**9.12. Exercise.** Let \(C\) be a quasicategory and \(a \in C_0\) a simplex, and define \(f_i := a_{i-1, i} \in C_1\) for \(i = 1, \ldots, n\) and \(g := a_{0, n} \in C_1\). Show that \([f_i] \circ \cdots \circ [f_1] = [g]\) in the homotopy category \(hC\).

Note that if \(C\) is an ordinary category, then \(f \approx g\) if and only if \(f = g\). Thus, \(\pi : C \to N(hC)\) is an isomorphism of simplicial sets if and only if \(C\) is isomorphic to the nerve of a category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation “\(hC\)”.

**9.13. Proposition.** Let \(C\) be a quasicategory and \(D\) a small category, and let \(\phi : C \to N(D)\) be a map of simplicial sets. Then there exists a unique map \(\psi : N(hC) \to N(D)\) such that \(\psi \pi = \phi\).

**Proof.** We first show existence, by constructing a suitable map \(\psi\), which being a map between nerves can be described as a functor \(hC \to D\). On objects, let \(\psi\) send \(x \in \text{ob}(hC) = C_0\) to \(\phi(x) \in \text{ob}(D) = (ND)_0\). On morphisms, let \(\psi\) send \([f] \in \text{hom}_hC(x, y)\) to \(\phi(f) \in \text{hom}_D(\phi(x), \phi(y)) \subseteq (ND)_1\). Observe that the function on morphisms is well-defined since if \(f \sim f'\), exhibited by some \(a \in C_2\), then \(\phi(a) \in (ND)_2\) exhibits the identity \(\phi(f) = \phi(f')\phi(1_x) = \phi(f')\) in \(D\). It is straightforward to show that \(\psi\) so defined is actually a functor, and that \(\psi \pi = \phi\) as maps \(C \to N(D)\).

The functor \(\psi\) defined above is the unique solution: the value of \(\psi\) on objects and morphisms is uniquely determined, and \(\pi : C_k \to (hC)_k\) is bijective for \(k = 0\) and surjective for \(k = 1\).

In particular, the homotopy category construction gives a pair of adjoint functors

\[ h : \text{QCat} \rightleftarrows \text{Cat} : N. \]

**9.14. Exercise.** Understand the homotopy categories of the various examples described in §8.

**9.15. Exercise (Easy but important).** Show that for quasicategories \(C\) and \(D\) there is an isomorphism \(hC \times hD \approx h(C \times D)\).

**9.16. A criterion for composition.** We have observed that for morphisms \(f : x \to y\) and \(g : y \to z\) in a quasicategory that we can define a composite “\(g \circ f\)” using extension along \(\Delta_2^\top \subseteq \Delta^2\), and that though such compositions are not unique, they are unique up to homotopy, so we get a well-defined homotopy class \([g] \circ [f]\). The following proposition says that every element in this homotopy class is obtained from this construction.

**9.17. Proposition.** If \(f : x \to y, g : y \to z,\) and \(h : x \to z\) are morphisms in a quasicategory \(C\), then \([h] = [g] \circ [f]\) if and only if there exists \(u : \Delta^2 \to C\) such that

\[ u|_{\Delta^0} = f, \quad u|_{\Delta^1} = g, \quad u|_{\Delta^2} = h. \]

Thus, every morphism in the homotopy class of \(h\) can be interpreted as a composite of \(g\) with \(f\).
Proof. Clearly if \( u \) exists then \([h] = [g] \circ [f]\). Conversely, suppose given \( f, g, h \) with \( h \in [g] \circ [f] \), and choose \( a: \Delta^2 \to C \) with \( a_{01} = f \) and \( a_{12} = g \), whence \([g] \circ [f] = [h']\) for \( h' = a_{02} \). Since \( h \in [h'] \) there is a \( b \in C_2 \) witnessing the relation \( h' \sim_r h \), and using this we can construct a map \( \Lambda_3^{2} \to C \) according to the diagram

\[
\begin{array}{c}
1 \\
\downarrow g_{011} \\
2 \\
\downarrow g \\
3 \\
\downarrow h \\
0 \\
\end{array}
\]

Extend to a map \( v: \Delta^3 \to C \); then \( u = v|\Delta^{0,1,3} \) exhibits \( h \) as a composite of \((g,f)\) as desired. \( \square \)

10. ISOMORPHISMS IN A QUASICATEGORY

Let \( C \) be a quasicategory. We say that an edge \( f \in C_1 \) is an isomorphism\(^6\) if its image in the homotopy category \( hC \) is an isomorphism in the usual sense of category theory.

Explicitly, \( f: x \to y \) is an isomorphism if and only if there exists an edge \( g: y \to x \) such that \([g] \circ [f] = [1_x]\) and \([f] \circ [g] = [1_y]\), where equality is in the homotopy category \( hC \).

10.1. Example. Consider \( f \in C_1 \). If we can produce \( g \in C_1 \) and \( a, b \in C_2 \) such that

\[
\begin{align*}
a_{01} &= f = b_{12}, & a_{12} &= g = b_{01}, & a_{02} &= x_{00}, & b_{02} &= y_{00}; \\
g &\xrightarrow{a} x & f &\xrightarrow{1_x} x & g &\xrightarrow{f} y \\
&\downarrow b & &\downarrow 1_x & &\downarrow a &\downarrow g \\
y &\xrightarrow{g} & x &\xrightarrow{1_y} & y &\xrightarrow{g} & y
\end{align*}
\]

then \([g] \circ [f] = [1_x]\) and \([f] \circ [g] = [1_y]\), so \( f \) isomorphism. The converse also holds: if \( f \) is an isomorphism, then there exist \( g \in C_1 \) and \( a, b \in C_2 \) as above, which can be proved using (9.17).

10.2. Example (Identity maps are isomorphisms). For every \( x \in C_0 \) the identity map \( 1_x: x \to x \) is an isomorphism: for instance, use \( a = b = x_{000} \) in the above diagram.

10.3. Preinverses and postinverses. Let \( C \) be a quasicategory. Given \( f: x \to y \in C_1 \), a postinverse\(^7\) of \( f \) is a \( g: y \to x \in C_1 \) such that \([g] \circ [f] = [1_x]\), and a preinverse\(^8\) of \( f \) is an \( e: y \to x \in C_1 \) such that \([f] \circ [e] = [1_y]\). An inverse is an \( f' \in C_1 \) which is both a postinverse and a preinverse. The following is trivial, but very handy.

10.4. Proposition. In a quasicategory \( C \) consider \( f \in C_1 \). The following are equivalent.

\begin{itemize}
  \item \( f \) is an isomorphism.
  \item \( f \) admits an inverse \( f' \).
  \item \( f \) admits a postinverse \( g \) and a preinverse \( e \).
  \item \( f \) admits a postinverse \( g \) and \( g \) admits a postinverse \( h \).
  \item \( f \) admits a preinverse \( e \) and \( e \) admits a preinverse \( d \).
\end{itemize}

If these equivalent conditions hold, then \( f \approx d \approx h \) and \( f' \approx e \approx g \), and all of them are isomorphisms.

\(^6\)Lurie [Lur09, §1.2.4] uses the term “equivalence” for this. I prefer to go with “isomorphism” here, because it is in fact a generalization of the classical notion of isomorphism, and because so many other things also get to be called some kind of equivalence. Other authors also use “isomorphism” in this context.

\(^7\)or left inverse, or retraction,

\(^8\)or right inverse, or section,
Proof. All of these are equivalent to the corresponding statements about morphisms in the homotopy category \( hC \), where they are seen to be equivalent by elementary arguments. \( \square \)

Note that inverses to a morphism in a quasicategory are generally not unique, though necessarily they are unique up to homotopy.

10.5. **Quasigroupoids.** A quasigroupoid is a quasicategory \( C \) such that \( hC \) is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism.

10.6. **Exercise.** If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

10.7. **The core of a quasicategory.** For an ordinary category \( A \), the core (or maximal subgroupoid) of \( A \) is the subcategory \( A^{\text{core}} \subseteq A \) consisting of all the objects, and all the isomorphisms between the objects.

For a quasicategory \( C \), we define the core \( C^{\text{core}} \subseteq C \) to be the subsimplicial set consisting of simplices all of whose edges are all isomorphisms. That is, \( C^{\text{core}} \) is defined so that the diagram

\[
\begin{array}{ccc}
C^{\text{core}} & \longrightarrow & C \\
\downarrow & & \downarrow \pi \\
(hC)^{\text{core}} & \longrightarrow & hC
\end{array}
\]

is a pullback of simplicial sets. Observe that \( N(A^{\text{core}}) = (NA)^{\text{core}} \) for a category \( A \).

10.8. **Proposition.** For a quasicategory \( C \), its core \( C^{\text{core}} \) is a quasigroupoid, and every subcomplex of \( C \) which is a quasigroupoid is contained in \( C^{\text{core}} \).

Proof. First, note that \( C^{\text{core}} \) is a subcomplex by construction: if \( a \in C_n \) is such that all edges are isomorphisms, then the same is true for \( af \in C_m \) for any \( f : [m] \to [n] \), since \( (af)_{i,j} = a_{f(i),f(j)} \) for any \( 0 \leq i \leq j \leq m \).

Next, we show that \( C^{\text{core}} \) is a quasicategory. In fact, we show that given \( f : \Delta^n \to C \) such that \( f(\Lambda^j_n) \subseteq C^{\text{core}} \) for some \( 0 < j < n \), then \( f(\Delta^n) \subseteq C^{\text{core}} \), so that inner-horn-filling for \( C \) implies inner-horn-filling for \( C^{\text{core}} \). For \( n = 2 \) this is the fact that composites of isomorphisms are isomorphisms (true in any quasicategory because it is true in its homotopy category), while for \( n \geq 3 \) it is just the fact that an inner horn of \( \Delta^n \) contains all the edges of \( \Delta^n \).

Thus, \( C^{\text{core}} \) is a quasicategory, and is easily seen to be a quasigroupoid, since an inverse of an isomorphism in \( C \) is also an isomorphism.

The final statement is clear: if \( G \subseteq C \) is a subcomplex which is a quasigroupoid, then every edge in \( G \) has in inverse in \( G \), and hence an inverse in \( C \). \( \square \)

10.9. **Kan complexes.** Recall that a Kan complex (8.8) is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is, \( K \) is a Kan complex iff

\[
\text{Hom}(\Delta^n, K) \to \text{Hom}(\Lambda^n_j, K)
\]

is surjective for all \( 0 \leq j \leq n, n \geq 1 \).

10.10. **Exercise.** Show that every simplicial set \( X \) has extensions for 1-dimensional horns; i.e., every \( \Lambda^1_j \to X \) extends over \( \Lambda^1_j \subseteq \Delta^1 \), where \( j \in \{0,1\} \). Thus, \( X \) is a Kan complex if and only if it has extensions just for the horns inside the simplices of dimension \( \geq 2 \).

10.11. **Proposition.** Every Kan complex is a quasigroupoid.

\footnote{Lurie uses the notation \( C^\circ \) for what we are calling \( C^{\text{core}} \).}
Proof. It is immediate that a Kan complex $K$ is a quasicategory. To show $K$ is a quasigroupoid, note that the extension condition for $\Lambda^2_0 \subset \Delta^2$ implies that every morphism in $hK$ admits a postinverse. Explicitly, if $f: x \to y$ is an edge in $K$, let $u: \Lambda^2_0 \to K$ with $u_{01} = f$ and $u_{02} = f_{00} = 1_x$, so there is an extension $v: \Delta^2 \to K$ and $g := v_{12}$ satisfies $gf \approx 1_x$. Use (10.6).

This proposition has a converse.

A. Deferred Proposition. Quasigroupoids are precisely the Kan complexes.

This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will give the proof in (29.2).

Recall (§8.7) that we observed that the singular complex $\text{Sing} T$ of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of $\text{Sing} T$ as the fundamental quasigroupoid of the space $T$.

10.12. Exercise (for topologists). Show that if $T$ is a space, then $h \text{Sing} T$, the homotopy category of the singular complex of $T$, is precisely the usual fundamental groupoid of $T$.

10.13. Quasigroupoids, components, and isomorphism classes. We say that two objects in a quasicategory are isomorphic if there exists an isomorphism between one. This is clearly an equivalence relation on $C_0$, and thus we speak of isomorphism classes of objects.

Recall (6.8) that the set of connected components of a simplicial set is given by

$$\pi_0 X \approx \left( \prod_{n \geq 0} X_n \right) / \sim \approx (X_0 / \sim_1),$$

the equivalence classes of simplices of $X$ under the equivalence relation generated by “related by a simplicial operator”, or equivalently the equivalence classes of vertices of $X$ under the equivalence relation generated by “connected by an edge”. Note that if $T$ is a topological space, then elements of $\pi_0 \text{Sing} T$ correspond exactly to path components of $T$.

For quasigroupoids, $\pi_0$ recovers the set of isomorphism classes of objects.

10.14. Proposition. If $C$ is a quasicategory, then

$$\pi_0(C^{\text{core}}) \approx \text{isomorphism classes of objects of } C.$$

Proof. Straightforward: edges in $C^{\text{core}}$ are precisely the isomorphisms in $C$. \hfill \Box

10.15. Exercise. Show that for a quasicategory $C$, $\pi_0(C^{\text{core}}) \approx \pi_0(h(C^{\text{core}})) \approx \pi_0((hC)^{\text{core}})$.

11. Function complexes and the functor quasicategory

Given ordinary categories $C$ and $D$, the functor category $\text{Fun}(C, D)$ has

- as objects, the functors $C \to D$, and
- as morphisms $f \to f'$, natural transformations of functors.

Furthermore, for any category $A$ there is a bijective correspondence between sets of functors

$$\{ A \times C \to D \} \longleftrightarrow \{ A \to \text{Fun}(C, D) \}.$$

Explicitly, a functor $\phi: A \to \text{Fun}(C, D)$ corresponds to $\tilde{\phi}: A \times C \to D$, given on objects by $\tilde{\phi}(a, c) = \phi(a)(c)$ for $a \in \text{ob} A$ and $c \in \text{ob} C$, and on morphisms by $\tilde{\phi}(\alpha, \gamma) = \phi(a')(\gamma) \circ \phi(a)(c) = \phi(a)(c') \circ \phi(a)(\gamma): \phi(a)(c) \to \phi(a')(c')$ for $\alpha: a \to a' \in \text{mor} A$ and $\gamma: c \to c' \in \text{mor} C$.

The generalization of the functor category to quasicategories admits a similar adjunction, and in fact can be defined for arbitrary simplicial sets.
11.1. **Function complexes.** Given simplicial sets $X$ and $Y$, we may form the **function complex** (or **mapping space**) $\operatorname{Map}(X,Y)$. This is a simplicial set with

$$\operatorname{Map}(X,Y)_n = \operatorname{Hom}(\Delta^n \times X, Y),$$

so that the action of a simplicial operator $\delta: [m] \to [n]$ on $\operatorname{Map}(X,Y)$ is induced by $\operatorname{Hom}(\delta \times \text{id}_X, Y): \operatorname{Hom}(\Delta^n \times X, Y) \to \operatorname{Hom}(\Delta^m \times X, Y)$. In particular, the set $\operatorname{Map}(X,Y)_0$ of vertices of the function complex is precisely the set of maps $X \to Y$ of simplicial sets.

11.2. **Proposition.** The function complex construction defines a functor

$$\operatorname{Map}: \mathsf{sSet}^{\text{op}} \times \mathsf{sSet} \to \mathsf{sSet}.$$

**Proof.** Left as an exercise. □

By construction, for each $n$, there is a bijective correspondence

$$\{\Delta^n \times X \to Y\} \leftrightarrow \{\Delta^n \to \operatorname{Map}(X,Y)\}.$$

In fact, we can replace $\Delta^n$ with an arbitrary simplicial set.

11.3. **Proposition.** For simplicial sets $X$, $Y$, $Z$, there is a bijection

$$\operatorname{Hom}(X \times Y, Z) \approx \operatorname{Hom}(X, \operatorname{Map}(Y,Z))$$

natural in all three variables.

**Proof.** The bijection sends $f: X \times Y \to Z$ to $\tilde{f}: X \to \operatorname{Map}(Y,Z)$ defined so that for $x \in X_n$, the simplex $\tilde{f}(x) \in \operatorname{Map}(Y,Z)_n$ is represented by the composite

$$\Delta^n \times Y \xrightarrow{x \times \text{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends $g: X \to \operatorname{Map}(Y,Z)$ to $\tilde{g}: X \times Y \to Z$, defined so that for $(x,y) \in X_n \times Y_n$, the simplex $\tilde{g}(x,y) \in Z_n$ is represented by

$$\Delta^n \xrightarrow{(\text{id},y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both $\tilde{f}$ and $\tilde{g}$ are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise, as is the proof of naturality. □

11.4. **Exercise.** Show, using the previous proposition, that there are natural isomorphisms

$$\operatorname{Map}(X \times Y, Z) \approx \operatorname{Map}(X, \operatorname{Map}(Y,Z)).$$

of simplicial sets. This implies that the function complex construction makes $\mathsf{sSet}$ into a **cartesian closed category**. (Hint: show that both objects represent isomorphic functors $\mathsf{sSet}^{\text{op}} \to \mathsf{Set}$, and apply the Yoneda lemma.)

11.5. **Remark.** The construction of the function complex is not special to simplicial sets. The construction of $\operatorname{Map}(X,Y)$ (and its properties as described above) works the same way in any category of functors $C^{\text{op}} \to \mathsf{Set}$, where $C$ is a small category (e.g., $C = \Delta$). In this general setting, the role of the standard $n$-simplices is played by the representable functors $\operatorname{Hom}_C(-, c): C^{\text{op}} \to \mathsf{Set}$. 
11.6. **Functor quasicategories.** Thus, we may expect the generalization of functor category to quasicategories to be defined by the function complex. In fact, if $C$ and $D$ are quasicategories, then the vertices of $\text{Map}(C,D)$ are precisely the functors $C \to D$, and the edges of $\text{Map}(C,D)$ are precisely the natural transformations. Furthermore, for ordinary categories, the function complex recovers the functor category.

11.7. **Exercise.** Show that for ordinary categories $C$ and $D$ that $N \text{Fun}(C,D) \approx \text{Map}(NC,ND)$. (Hint: use that $N([n]) = \Delta^n$, and the fact that the nerve preserves finite products (6.5).)

It turns out that a function complex between quasicategories is again a quasicategory. In fact, we have the following.

**B. Deferred Proposition.** Let $K$ be any simplicial set and $C$ a quasicategory. Then $\text{Map}(K,C)$ is a quasicategory.

For this reason, we will sometimes write $\text{Fun}(K,C)$ for $\text{Map}(K,C)$ when $C$ is a quasicategory.

To prove (B), we need a to take a detour to develop some technology about “weakly saturated” classes of maps and “lifting properties”. After this, we will complete the proof in §16.

**Part 2. Lifting properties**

12. **Weakly saturated classes and inner-anodyne maps**

Quasicategories are defined by an “extension property”: they are the simplicial sets $C$ such that any map $K \to C$ extends over $L$, whenever $K \subset L$ is an inner horn inclusion $\Lambda^n_j \subset \Delta^n$. The set of inner horns “generates” a larger class of maps (which will be called the class of *inner anodyne maps*), which “automatically” shares the extension property of the inner horns. This class of inner anodyne maps is called the *weak saturation* of the set of inner horns.

For instance, we will observe that the spine inclusions $I^n \subset \Delta^n$ are inner anodyne, so that quasicategories admit “spine extensions”, i.e., any $I^n \to C$ extends over $I^n \subset \Delta^n$ to a map $\Delta^n \to C$.

12.1. **Weakly saturated classes.** Consider a category (such as s$\text{Set}$) which has all small colimits. A *weakly saturated class* is a class $A$ of morphisms in the category, which

1. contains all isomorphisms,
2. is closed under cobase change,
3. is closed under composition,
4. is closed under transfinite composition,
5. is closed under coproducts, and
6. is closed under retracts.

Given a class of maps $S$, its *weak saturation* $\overline{S}$ is the smallest weakly saturated class containing $S$.

We need to explain some of the elements of this definition.

- **Closed under cobase change** is also called **closed under pushout**: it means that if $f'$ is the pushout of $f: X \to Y$ along some map $g: X \to Z$, then $f \in A$ implies $f' \in A$.
- **Closed under composition** means that if $g, f \in A$ and $gf$ is defined, then $gf \in A$.
- We say that $A$ is **closed under countable composition** if given a countable sequence of composable morphisms, i.e., maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

such that each $f_k \in A$ for all $k \in \mathbb{Z}_{>0}$, the induced map $X_0 \to \text{colim}_k X_k$ to the colimit is in $A$. 

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The notion **closed under transfinite composition** is the generalization of this, in which \( \mathbb{N} \) is replaced by an arbitrary ordinal \( \lambda \) (i.e., a well-ordered set). This means that for any ordinal \( \lambda \) and any functor \( X: \lambda \to sSet \), if for every \( i \in \lambda \) with \( i \neq 0 \) the evident map
\[
\text{colim}_{j < i} X(j) \to X(i)
\]
is in \( \mathcal{A} \), then the induced map \( X(0) \to \text{colim}_{j \in \lambda} X(j) \) is in \( \mathcal{A} \).

- **Closed under coproducts** means that if \( \{ f_i: X_i \to Y_i \} \) is a set of maps in \( \mathcal{A} \), then
\[
\coprod_i f_i: \coprod_i X_i \to \coprod_i Y_i
\]
is in \( \mathcal{A} \).

- We say that \( f \) is a **retract** of \( g \) if there exists a commutative diagram in \( C \) of the form

\[
\begin{array}{ccc}
\bullet & \xrightarrow{f} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{g} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{id} & \bullet
\end{array}
\]

This is really a special case of the notion of a retract of an object in the functor category \( \text{Fun}([1], sSet) \). We say that \( \mathcal{A} \) is **closed under retracts** if for every diagram as above, \( g \in \mathcal{A} \) implies \( f \in \mathcal{A} \).

12.2. **Remark.** This list of properties is not minimal: (3) is actually the special case of (4) when \( \lambda = [2] \), and (5) can be deduced from (2) and (4).

12.3. **Example.** Consider the category of sets. The class of all surjective maps is weakly saturated, and in fact is the weak saturation of \( \{ \{0,1\} \to \{1\} \} \). Likewise, the class of injective maps is weakly saturated, and in fact is the weak saturation of \( \{ \emptyset \to \{1\} \} \).

12.4. **Example.** The classes of monomorphisms and epimorphisms of simplicial sets are weakly saturated classes. Later we will identify the class of monomorphisms of simplicial sets as the weak saturation of the set of “cell inclusions” (15.19).

There is a dual notion of a **weakly cosaturated class**: a weakly cosaturated class is the same thing as a weakly saturated class in the opposite category, and is characterized by being closed under properties formally dual to (1)–(6).

12.5. **Classes of “anodyne” morphisms.** We use the following notation for sets of types of horns:

- \( \text{InnHorn} := \{ \Lambda^n_k \subset \Delta^n \mid 0 < k < n, n \geq 2 \} \), (inner horns),
- \( \text{LHorn} := \{ \Lambda^n_k \subset \Delta^n \mid 0 \leq k < n, n \geq 1 \} \), (left horns),
- \( \text{RHorn} := \{ \Lambda^n_k \subset \Delta^n \mid 0 < k \leq n, n \geq 1 \} \), (right horns),
- \( \text{Horn} := \{ \Lambda^n_k \subset \Delta^n \mid 0 \leq k \leq n, n \geq 1 \} \), (horns).

The weak saturation of each of these sets will play an important role in what follows. Right now, we focus on the weak saturation \( \text{InnHorn} \) of the set of inner horns, which is called the class of **inner anodyne**\(^{10} \) morphisms. Note that inner anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets themselves form a weakly saturated class.

12.6. **Proposition.** If \( C \) is a quasicategory and \( A \subseteq B \) is an inner anodyne inclusion, then any \( f: A \to C \) admits an extension to \( g: B \to C \) so that \( g|A = f \).

\(^{10}\)The “anodyne” terminology for the weak saturation of a set of horns was introduced by Gabriel and Zisman [GZ67]. “Anodyne” derives from ancient Greek, meaning “without pain”; we leave it to the reader to decide whether this choice of terminology is appropriate.
Proof. It suffices to show that the collection $\mathcal{A}$ of monomorphisms $i : A \to B$ such that every map from $A$ to a quasicategory extends along $i$ is weakly saturated. Since $\text{InnHorn} \subseteq \mathcal{A}$ it then follows that $\text{InnHorn} \subseteq \mathcal{A}$. To prove this claim is a relatively straightforward exercise, which we leave for the reader: check that the class $\mathcal{A}$ satisfies each of the conditions (1)–(6) of a weakly saturated class. It is highly recommended that you work through this argument if you haven’t seen it before. □

12.7. Exercise (Easy but important). Show that every inner anodyne map induces a bijection on vertices. (Hint: the class of maps of simplicial set which are a bijection on vertices is weakly saturated.)

12.8. Examples of inner anodyne morphisms. It is crucial to be able to prove that certain explicit maps are inner anodyne.

Let $S \subseteq [n]$. A generalized horn the subcomplex $\Lambda^n_S \subset \Delta^n$ defined by

$$\Lambda^n_S := \bigcup_{i \in S} \Delta^{[n] \setminus i},$$

i.e., the union of codimension one faces of the $n$-simplex indexed by elements of $S$. In particular, $\Lambda^{|n| \setminus \{j\}}_j$ is the usual horn $\Lambda^n_j$. I’ll generalize this notation to arbitrary totally ordered sets, so $\Lambda^n_T = \bigcup_{i \in S} \Delta^{T \setminus i}$ when $S \subseteq T$.

We call $\Lambda^n_S \subset \Delta^n$ a generalized inner horn if $S$ is not an “interval” in $[n]$, i.e., if there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$.

12.9. Example. The union $\Lambda^n_{\{0, n\}} = \Delta^{\{0,1,\ldots,n-1\}} \cup \Delta^{\{1,2,\ldots,n\}}$ of the “first and last” faces of $\Delta^n$ is an inner generalized horn when $n \geq 2$.

12.10. Lemma. Generalized inner horn inclusions $\Lambda^n_S \subset \Delta^n$ are inner anodyne.

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (58.1).

12.11. Example. Consider $\Lambda^3_{\{0,3\}}$, which can be pictured as the solid diagram in

$$\begin{array}{ccc}
0 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
1 & \rightarrow & 3
\end{array}$$

We can get from this to $\Delta^3$ in two steps:

$$\begin{array}{ccc}
\Lambda^3_{\{0,3\}} & \rightarrow & \Delta^{\{0,2,3\}} \\
\downarrow & & \downarrow \\
\Lambda^3_{\{0,3\}} & \rightarrow & \Delta^3
\end{array}$$

The square is a pushout of subcomplexes since $\Lambda^3_{\{0,3\}} \cap \Delta^{\{0,2,3\}} = \Lambda^3_{\{0,3\}}$, and the map along the top is isomorphic to $\Lambda^2_2 \subset \Delta^2$, an inner horn inclusion. This proves that $\Lambda^3_{\{0,3\}} \subset \Delta^3$ is inner anodyne.

Recall that every standard $n$-simplex contains a spine $I^n \subseteq \Delta^n$.

12.12. Lemma. The spine inclusions $I^n \subset \Delta^n$ are inner anodyne for all $n$. Thus, for a quasicategory $C$, any $I^n \to C$ extends to $\Delta^n \to C$.

This is proved in [Joy08a, Prop. 2.13]; we give the proof in the appendix (58.2).
12.13. Example. To show that \( I^3 \subset \Delta^3 \) is inner anodyne, observe that we can get from \( I^3 \) to a generalized inner horn two steps by gluing 2-simplices along inner horns inclusions:

\[
\begin{array}{cc}
\Lambda^{\{0,1,2\}} & \Delta^{\{0,1,2\}} \\
\downarrow & \\
I^3 & I^3 \cup \Delta^{\{0,1,2\}}
\end{array}
\quad
\begin{array}{cc}
\Lambda^{\{1,2,3\}} & \Delta^{\{1,2,3\}} \\
\downarrow & \\
I^3 & I^3 \cup \Delta^{\{0,1,2\}} \to \Lambda^{\{0,3\}}
\end{array}
\]

since \( I^3 \cap \Delta^{\{0,1,2\}} = \Lambda^{\{0,1,2\}} \) and \( (I^3 \cup \Delta^{\{0,1,2\}}) \cap \Delta^{\{1,2,3\}} = \Lambda^{\{1,2,3\}} \).

12.14. Exercise. Use (12.12) to show that the tautological map \( \pi: C \to N(hC) \) from a quasicategory to (the nerve of) its homotopy category is surjective in every degree.

13. Lifting Calculus and Inner Fibrations

We have defined quasicategories by an “extension property”: in general, we say that \( X \) has satisfies the extension property for \( A \to B \) if in any diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & & \downarrow{s} \\
B & \xleftarrow{s} & Y
\end{array}
\]

there exists a morphism \( s \) making the diagram commute. In this section, we discuss a “relative” version of this, called a “lifting property”.

13.1. The lifting relation. Given morphisms \( f \) and \( g \) in a category, a lifting problem for \( (f, g) \) is a pair of morphisms \((u, v)\) such that \( vf = gu \). That is, a lifting problem is any commutative square of solid arrows of the form

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & & \downarrow{s} \\
B & \xleftarrow{s} & Y
\end{array}
\]

A lift for the lifting problem is a morphism \( s \) such that \( sf = u \) and \( gs = v \), i.e., a dotted arrow making the diagram commute.

We may thus define the lifting relation on morphisms in our category: we write “\( f \Box g \)” if every lifting problem for \( (f, g) \) admits a lift\(^\text{11}\). Equivalently, \( f \Box g \) exactly if

\[
\Hom(B, X) \xrightarrow{s \mapsto (sf, gs)} \Hom(A, X) \times_{\Hom(A, Y)} \Hom(B, Y)
\]

is a surjection, where the target is the set of pairs \((u: A \to X, v: B \to Y)\) such that \( gu = vf \).

When \( f \Box g \) holds, one sometimes says \( f \) has the left lifting property relative to \( g \), or that \( g \) has the right lifting property relative to \( f \). Or we just say that \( f \) lifts against \( g \).

We extend the notation to classes of maps, so “\( A \Box B \)” means: \( a \Box b \) for all \( a \in A \) and \( b \in B \).

Given a class of morphisms \( A \), define the right complement \( A^\Box \) and left complement \( ^\Box A \) by

\[
A^\Box = \{ g \mid a \Box g \text{ for all } a \in A \}, \quad ^\Box A = \{ f \mid f \Box a \text{ for all } a \in A \}.
\]

13.2. Proposition. For any class \( B \), the left complement \( ^\Box B \) is a weakly saturated class.


\(^{11}\) Sometimes one sees the notation “\( f \perp g \)” or “\( f \pitchfork g \)” used instead. Our notation is taken from [Rie14, §11].
The above proposition (13.2) has a dual statement: any right complement $B^{\square}$ is weakly cosaturated, i.e., satisfies dual versions of the closure properties of a weakly saturated class, or equivalently, corresponds to a weakly saturated class in the opposite category.

13.4. Exercise (Easy). Prove that if $A \subseteq B$, then $A^{\square} \supseteq B^{\square}$ and $\square A \supseteq B$. Use this to show $A^{\square} = (\square(A^{\square}))^{\square}$ and $\square A = \square((\square A)^{\square})$.

13.5. Exercise (for those who know a little homological algebra). Fix an abelian category $C$ (e.g., the category of modules over some ring $R$). Let $A$ be the class of morphisms in $C$ of the form $0 \to P$ where $P$ is projective, and let $B$ be the class of epimorphisms in $C$. Show that $A \sqsupseteq B$; also, show that $B = A^{\square}$ if $C$ has enough projectives.

13.6. Exercise. In the setting of the previous exercise, identify the class $B^{\square}$.

13.7. Inner fibrations. A map $p$ of simplicial sets is an inner fibration if $\text{InnHorn} p$. The class of inner fibrations $\text{InnFib} = \text{InnHorn}^{\square}$ is thus the right complement of the set of inner horns. Note that $C$ is a quasicategory if and only if $C \to *$ is an inner fibration.

Because $\text{InnFib}$ is a right complement, it is weakly cosaturated. In particular, it is closed under composition. This implies that if $p: C \to D$ is an inner fibration and $D$ is a quasicategory, then $C$ is also a quasicategory.

13.8. Exercise. Show that if $f: C \to D$ is any functor from a quasicategory $C$ to a category $D$, then $f$ is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have unique extensions to simplices.)

13.9. Factorizations. It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation.

13.10. Proposition (“Small object argument”). Let $S$ be a set of morphisms in $s$Set. Every map $f$ between simplicial sets admits a factorization $f = pj$ with $j \in S$ and $p \in S^{\square}$.

The proof of this proposition is by means of what is known as the “small object argument”. I’ll give the proof in the next section. For now we record a consequence.

13.11. Corollary. For any set $S$ of morphisms in $s$Set, we have that $S = (S^{\square})^{\square}$.

Proof. That $S \subseteq (S^{\square})^{\square}$ is immediate from (13.2). Given $f$ such that $f \sqsupseteq S^{\square}$, use the small object argument (13.10) to choose $f = pj$ with $j \in S$ and $p \in S^{\square}$. We have a commutative diagram of solid arrows

A map $s$ exists making the diagram commute, because $f \sqsupseteq p$, so there is a lift in

The diagram exhibits $f$ as a retract of $j$, whence $f \in S$ since weak saturations are closed under retracts.
The proof of the corollary is called the “retract trick”: given \( f = pj, f \boxdot p \) implies that \( f \) is a retract of \( j \), while \( j \boxdot f \) implies that \( f \) is a retract of \( p \).

In the case we are currently interested in, we have that \( \text{InnHorn} = \mathcal{S} \text{InnFib} \) and \( \text{InnHorn} \mathcal{S} = \text{InnFib} \), and thus any map can be factored into an inner anodyne map followed by an inner fibration.

### 13.12. Weak factorization systems

A weak factorization system in a category is a pair \((\mathcal{L}, \mathcal{R})\) of classes of maps such that

- every map \( f \) admits a factorization \( f = r\ell \) with \( r \in \mathcal{R} \) and \( \ell \in \mathcal{L} \), and
- \( \mathcal{L} = \mathcal{R}^{\mathcal{S}} \) and \( \mathcal{R} = \mathcal{L}^{\mathcal{S}} \).

Thus, in any weak factorization the “left” class \( \mathcal{L} \) is weakly saturated and the “right” class \( \mathcal{R} \) is weakly cosaturated. The small object argument implies that \((\mathcal{S}, \mathcal{S}^{\mathcal{S}})\) is a weak factorization in \( \text{sSet} \) for every set of maps \( S \).

#### 13.13. Exercise

(for those who know some homological algebra). Consider the classes \( \mathcal{A} = \{0 \to \text{projective}\} \) and \( \mathcal{B} = \{\text{epimorphism}\} \) of maps in an abelian category described in (13.5). Show directly that the pair \((\mathcal{A}, \mathcal{B})\) is a weak factorization system if and only if the category has enough projectives.

#### 13.14. Uniqueness of liftings

The relation \( f \boxdot g \) says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map \( f: A \to B \), let \( f^{\mathcal{S}} := (f, f): B \amalg_A B \to B \) be the evident “fold” map. It is straightforward to show that for a map \( g: X \to Y \) we have that \( \{f, f^{\mathcal{S}}\} \boxdot g \) if and only if in every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & \nearrow s & \downarrow g \\
B & \xrightarrow{g} & Y
\end{array}
\]

there exists a unique lift \( s \).

#### 13.15. Example

Consider the category of topological spaces. Let \( \mathcal{A} \) be the class of morphisms of the form \( A \times \{0\} \to A \times [0,1] \), where \( A \) is an arbitrary space. Then \((\mathcal{A} \cup \mathcal{A}^{\mathcal{S}})^{\mathcal{S}}\) contains all covering maps (by the “Covering Homotopy Theorem”).

A weak factorization system \((\mathcal{L}, \mathcal{R})\) in which liftings of type \( \mathcal{L} \boxdot \mathcal{R} \) are always unique is called an orthogonal factorization system.

#### 13.16. Exercise

Show that in an orthogonal factorization system, the factorizations \( f = r\ell \) are unique up to unique isomorphism.

#### 13.17. Exercise

Show that \((\{\text{surjections}\}, \{\text{injections}\})\) is an orthogonal factorization system for \( \text{Set} \).

The small object argument implies that \((\mathcal{S} \cup \mathcal{S}^{\mathcal{S}}, (\mathcal{S} \cup \mathcal{S}^{\mathcal{S}})^{\mathcal{S}})\) is an orthogonal factorization system for every set \( S \) of morphisms.

#### 13.18. Example

(The fundamental category via a weak factorization system). In simplicial sets, the projection map \( C \to * \) is in the right complement to \( S := \text{InnHorn} \cup \text{InnHorn}^{\mathcal{S}} \) if and only if \( C \) is isomorphic to a nerve of a category (5.7). The small object argument using \( S \), applied to a projection \( X \to * \), thus produces a morphism \( \pi: X \to Y \) in \( \mathcal{S} \) with \( Y \) the nerve of a category.

Uniqueness of liftings in this case implies that \( \pi: X \to Y \) has precisely the universal property of the fundamental category of \( X \) defined in §9.1: given \( f: X \to C \) with \( C \) a category, a unique extension of \( f \) over \( X \to Y \) exists. Thus, the small object argument applied to \( S \) gives another construction of the fundamental category (9.1) of an arbitrary simplicial set \( S \).
13.19. Exercise. Prove that if \( f: X \to Y \) is any inner anodyne map, then the induced functor \( h(f): hX \to hY \) between fundamental categories is an isomorphism. (Hint: use the universal property of fundamental categories to construct an inverse to \( h(f) \).)

14. The small object argument

In this section we give the proof of (13.10), i.e., that given a fixed set \( S = \{ s_i: A_i \to B_i \} \) of maps of simplicial sets, we can factor any map \( f: X \to Y \) as \( f = pj \) with \( j \in \bar{S} \) and \( p \in S^{\square} \). For the reader: it may be helpful to first work through the special case where \( Y = \Delta^0 \) (the terminal object in simplicial sets).

14.1. A factorization construction. Given any map \( f: X \to Y \), we first produce a factorization 
\[
X \xrightarrow{L_f} Ef \xrightarrow{R_f} Y,
\]
using the diagram
\[
\begin{array}{ccc}
\coprod A_i & \xrightarrow{(u)} & X \\
\downarrow \downarrow & & \downarrow f \\
\coprod B_i & \xrightarrow{(v)} & Ef \\
\end{array}
\]
\[
\begin{array}{ccc}
Ef & \xrightarrow{R_f} & Y \\
\uparrow \uparrow & & \uparrow \uparrow \\
X & \xrightarrow{L_f} & Ef \\
\end{array}
\]

where the the coproducts are indexed by the set \([S, f]\), and the square is a pushout. Note that \( Lf \in \bar{S} \) by construction; however, we do not expect that \( Rf \) in \( S^{\square} \).

We can iterate the construction:

\[
\begin{array}{ccc}
X & \xrightarrow{L f} & Ef \\
& \xrightarrow{L^2 f} & E^2 f \\
& \xrightarrow{R f} & R^2 f \\
& \xrightarrow{R^3 f} & E^3 f \\
& \xrightarrow{R^\alpha f} & \cdots \\
\end{array}
\]

Here each triple \((E^\alpha f, L^\alpha f, R^\alpha f)\) is obtained by factoring the "\( R \)" map of the previous one, so that
\[
E^{\alpha + 1} f := E(R^\alpha f), \quad L^{\alpha + 1} f := L(R^\alpha f) \circ (L^\alpha f), \quad R^{\alpha + 1} f := R(R^\alpha f).
\]

Taking direct limits gives a factorization 
\[
X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y,
\]
with \( E^\omega f = \operatorname{colim}_{n \to \infty} E^n f \).

We can go even further, using the magic of transfinite induction, and define compatible factorizations \((E^\lambda f, L^\lambda f, R^\lambda f)\) for each ordinal\(^{12}\) \( \lambda \). For successor ordinals \( \alpha + 1 \) use the prescription of (14.2), while for limit ordinals \( \beta \) take a direct limit \( E^\beta f := \operatorname{colim}_{\alpha < \beta} E^\alpha f \) as in the construction of \( E^\omega f \) above.

\(^{12}\)For a treatment of ordinals, see for instance the chapter on sets in [TS14].
It is immediate that every $L^\alpha f \in \mathcal{S}$, because weak saturations are closed under transfinite composition. The maps $R^\alpha f$ are not generally contained in $S^2$, though they do satisfy a “partial lifting property”: whenever $\alpha < \beta$ there exists by construction a dashed arrow making

$$
\begin{array}{c}
A_i \\
\downarrow \quad s_i
\end{array}
\xymatrix{ & E^\alpha f \ar[r] & E^{\alpha+1} f \ar[r] & E^\beta f \\
& B_i \ar[u] \ar[ru] & \ar[ru] Y \ar[u] & \\
\downarrow v & & & \downarrow R^\beta f
\end{array}
$$

commute, for any $u'$ and $v$ making the square commute. Thus, we get a solution to a lifting problem $(u, v)$ of $s_i$ against $R^\beta f$ whenever the map $u: A_i \to E^\beta f$ on the top of a commutative square that we want a lift for can be factored through one of the maps $E^\alpha f \to E^\beta f$ with $\alpha < \beta$. This is so exactly because $E^{\alpha+1} f$ was obtained from $E^\alpha f$ by “formally adjoining” a solution to every such lifting problem.

The “small object argument” amounts to the following.

Claim. There exists an ordinal $\kappa$ such that for every domain $A_i$ of a map in $S$, every map $A_i \to E^\alpha f$ factors through some $E^\alpha f \to E^\beta f$ with $\alpha < \kappa$.

Given this, it follows from the “partial lifting property” that $S \boxtimes R^\kappa f$, and so we obtain the desired factorization: $f = (R^\kappa f) \circ (L^\kappa f)$ with $L^\kappa f \in \mathcal{S}$ and $R^\kappa f \in S^2$.

It remains to prove the claim, which we will do by choosing $\kappa$ to be a regular cardinal which is “bigger” than all the simplicial sets $A_i$.

14.3. Regular cardinals. The cardinality of a set $X$ is the smallest ordinal $\lambda$ such that there exists a bijection between $X$ and $\lambda$; we write $|X|$ for this. Ordinals which can appear this way are called cardinals. For instance, the first infinite ordinal $\omega$ is the countable cardinal.

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol $\aleph_\alpha$ denotes the $\alpha$th infinite cardinal, e.g., $\aleph_0 = \omega$.

Say that $\lambda$ is a regular cardinal\footnote{In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.} if it is an infinite cardinal, and if for every set $A$ of ordinals such that (i) $\alpha < \lambda$ for all $\alpha \in A$, and (ii) $|A| < \lambda$, then $\sup A < \lambda$. For instance, $\omega$ is a regular cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular\footnote{For instance, $\aleph_\omega = \sup \{ \aleph_k \mid k < \omega \}$ is not regular.}; however, there exist arbitrarily large regular cardinals\footnote{For instance, every successor cardinal $\aleph_{\alpha+1}$ is regular.}.

Every ordinal $\alpha$ defines a category, which is the poset of ordinals strictly less than $\alpha$. Colimits of functors $Y: \kappa \to Set$ with $\kappa$ a regular cardinal have the following property: the map

$$
\text{colim}_{\alpha < \kappa} \text{Hom}(X, Y_\alpha) \to \text{Hom}(X, \text{colim}_{\alpha < \kappa} Y_\alpha)
$$

is a bijection whenever $|X| < \kappa$. This generalizes the familiar case of $\kappa = \omega$: any map of a finite set into the colimit of a countable sequence factors through a finite stage.

14.5. Exercise. Prove that (14.4) is a bijection when $|X| < \kappa$.

14.6. Small simplicial sets. Given a regular cardinal $\kappa$, we say that a simplicial set is $\kappa$-small if it is isomorphic to the colimit of some functor $F: C \to s\text{Set}$, such that (i) $|\text{ob } C|, |\text{mor } C| < \kappa$, and (ii) each $F(c)$ is isomorphic to a standard simplex $\Delta^n$. Morally, we are saying that a simplicial set is $\kappa$-small if it can be “presented” with fewer than $\kappa$ generators and fewer than $\kappa$ relations.

Given a functor $Y: \kappa \to s\text{Set}$ and a $\kappa$-small simplicial set $X$, we have a bijection as in (14.4). (This is sometimes phrased as: $\kappa$-small simplicial sets are $\kappa$-compact.) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal $\kappa$ greater than $\sup\{|A_i|\}$.\footnote{In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.}
14.7. Example. The standard simplices $\Delta^n$, as well as any subcomplex such as the horns $\Lambda^n_k$, are $\omega$-small: this is a consequence of (4.19). Thus, when we carry out the small object argument for $S = \text{InnHorn}$, we can take $(E^\omega f, L^\omega f, R^\omega f)$ to be the desired factorization.

14.8. Functoriality. The construction $f \mapsto (X \xrightarrow{L_f} Ef \xrightarrow{R_f} Y)$ is a functor $\text{Fun}([1], s\text{Set}) \to \text{Fun}([2], s\text{Set})$, and it follows that so is $f \mapsto (X \xrightarrow{L^\alpha_f} E^\alpha f \xrightarrow{R^\alpha f} Y)$ for any $\alpha$. Because the choice of regular cardinal $\kappa$ depends only on $S$, not on the map $f$, we see that the small object argument actually produces a functorial factorization of a map into a composite of an element of $\mathcal{S}$ with an element $S^{\omega_2}$. We will have use of this later.

15. Non-degenerate simplices and the skeletal filtration

We have noted that monomorphisms of simplicial sets form a weakly saturated class. Here we identify an important set of maps called Cell, so that the weak saturation of Cell is precisely the class of monomorphisms. We do so by getting a very explicit handle on monomorphisms of simplicial sets.

15.1. Boundary of a standard simplex. For each $n \geq 0$, we define

$$\partial \Delta^n := \bigcup_{k \in [n]} \Delta^{[n] \setminus \{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the $n$-simplex. Equivalently,

$$(\partial \Delta^n)_k = \{ f : [k] \to [n] \mid f([k]) \neq [n] \}.$$

We call $\partial \Delta^n$ the boundary of $\Delta^n$. Note that $\partial \Delta^0 = \emptyset$ and $\partial \Delta^1 = \Delta^{\{0\} \amalg \{1\}}$.

15.2. Exercise. Show that $\partial \Delta^n$ is the largest subcomplex of $\Delta^n$ which does not contain the “generator” $\langle 0 \ldots n \rangle \in (\Delta^n)_n$. In other words, $\partial \Delta^n$ is the maximal proper subcomplex of $\Delta^n$.

15.3. Exercise. Show that if $C$ is a category, then the evident maps $\text{Hom}(\Delta^n, C) \to \text{Hom}(\partial \Delta^n, C)$ defined by restriction are isomorphisms when $n \geq 3$, but not necessarily when $n \leq 2$.

15.4. Trivial fibrations and monomorphisms. Let Cell be the set consisting of the inclusions $\partial \Delta^n \subset \Delta^n$ for $n \geq 0$. The resulting right complement is $\text{TrivFib} := \text{Cell}^{\omega_2}$, the class of trivial fibrations (also sometimes called acyclic fibrations).

Since the elements of Cell are monomorphisms, and the class of all monomorphisms is weakly saturated, we see that all elements of Cell are monomorphisms. We are going to prove the converse, i.e., we will show that Cell is precisely equal to the class of monomorphisms.

15.5. Degenerate and non-degenerate simplices. Recall $\Delta^{\text{surj}}, \Delta^{\text{inj}} \subset \Delta$, the subcategories of the category $\Delta$ of simplicial operators, consisting of all the objects and the surjective and injective order-preserving maps respectively.

A simplex $a \in X_n$ is said to be degenerate if there exists a non-identity operator $\sigma \in \Delta^{\text{surj}}$ and simplex $b$ in $X$ such that $a = b \sigma$. Equivalently, $a$ is degenerate if there exists a non-injective $f \in \Delta$ and $b$ in $X$ such that $a = bf$; this is because any $f \in \Delta$ factors as $f = f^{\text{inj}}f^{\text{surj}}$, a surjection followed by an injection, and if $f$ is non-injective then $f^{\text{surj}}$ is necessarily non-identity.

Likewise, a simplex $a \in X_n$ is non-degenerate if whenever $a = b \sigma$ for some $\sigma \in \Delta^{\text{surj}}$ and simplex $b$ in $X$, we must have that $\sigma$ is an identity map. Equivalently, $a \in X_n$ is non-degenerate if $a = bf$ for some $f \in \Delta$ and $b$ in $X$ implies that $f \in \Delta^{\text{inj}}$.

We write $X_n^{\text{deg}}, X_n^{\text{nd}} \subset X_n$ for the complementary subsets of degenerate and non-degenerate $n$-simplices in a simplicial set. Note that if $f : A \to X$ is a map of simplicial sets, then $f(A_n^{\text{deg}}) \subseteq X_n^{\text{deg}}$, while $f^{-1}(X_n^{\text{nd}}) \subseteq A_n^{\text{nd}}$. (Neither $X_n^{\text{deg}}$ nor $X_n^{\text{nd}}$ assemble to give a subcomplex of $X$.)
15.6. Exercise (easy). For any simplicial set \( X \), we have \( X^{\text{nd}}_0 = X_0 \), while \( X^{\text{nd}}_1 \subseteq X_1 \) is the complement of the image of \((01)^* : X_0 \to X_1\).

15.7. Exercise. Describe the degenerate and non-degenerate simplices of the standard \( n \)-simplices \( \Delta^n \).

15.8. Exercise. Show that if \( C \) is an ordinary category, then a simplex \( a \in N(C)_k \) of the nerve is non-degenerate if and only if it is represented by a composable sequence of non-identity maps \( c_0 \to \cdots \to c_k \) in the category \( C \).

15.9. Proposition. If \( X \) is a simplicial set and \( A \subseteq X \) is a subcomplex, then \( A^{\text{nd}}_n = X^{\text{nd}}_n \cap A_n \).

Proof. The inclusion \( X^{\text{nd}}_n \cap A_n \subseteq A^{\text{nd}}_n \) is obvious. Conversely, let \( a \in A^{\text{nd}}_n \), and suppose there exists \( b \in X_k \) and \( \sigma \colon [m] \to [k] \in \Delta^{\text{surj}} \) such that \( b\sigma = a \). Any surjection in \( \Delta \) has a section \((4.17)\), so there exists \( \delta \colon [k] \to [m] \) such that \( \sigma\delta = 1_{[k]} \). Thus \( b = b\sigma\delta = a\delta \in A_k \), so \( b \) is a simplex of \( A \). Therefore \( a \in A^{\text{nd}}_n \) implies that \( \sigma \) is an identity map. Thus, we have proved that if \( a \) is nondegenerate in \( A \), it must also be nondegenerate in \( X \). \( \square \)

15.10. Simplicial sets are canonically free with respect to surjective operators. The key observation is that degenerate simplices in a simplicial set are precisely determined by knowledge of the non-degenerate simplices.

15.11. Proposition (Eilenberg-Zilber lemma). Let \( a \) be a simplex in \( X \). Then there exists a unique pair \((b, \sigma)\) consisting of a non-degenerate simplex \( b \) and a map \( \sigma \) in \( \Delta^{\text{surj}} \) such that \( a = b\sigma \).

Proof. \([GZ67, \S I.3]\). Given \( \sigma \colon [n] \to [m] \), let \( \Gamma(\sigma) = \{ \delta \colon [m] \to [n] \mid \sigma\delta = \text{id}_{[m]} \} \) denote the set of sections of \( \sigma \). The sets \( \Gamma(\sigma) \) is non-empty when \( \sigma \in \Delta^{\text{surj}} (4.17) \). We note the following elementary observation, whose proof is left for the reader:

If \( \sigma, \sigma' \in \Delta^{\text{surj}} \) are such that \( \Gamma(\sigma) = \Gamma(\sigma') \), then \( \sigma = \sigma' \).

Let \( a \in X_n \) be such that \( a = b_i\sigma_i \) for \( b_i \in X^{\text{nd}}_{m_i}, \sigma_i \in \Delta^{\text{surj}}([n], [m_i]) \), for \( i = 1, 2 \). We want to show that \( m_1 = m_2, b_1 = b_2, \) and \( \sigma_1 = \sigma_2 \).

Pick any \( \delta_1 \in \Gamma(\sigma_1) \) and \( \delta_2 \in \Gamma(\sigma_2) \). Then we have

\[
\begin{align*}
\delta_1 &= b_1\sigma_1\delta_1 = a\delta_1 = b_2\sigma_2\delta_1; \\
\delta_2 &= b_2\sigma_2\delta_2 = a\delta_2 = b_1\sigma_1\delta_2,
\end{align*}
\]

so \( b_1 \) and \( b_2 \) are related by the simplicial operators \( \sigma_2\delta_1 \) and \( \sigma_1\delta_2 \). Since \( b_1 \) and \( b_2 \) are both non-degenerate, \( \sigma_2\delta_1 \colon [m_1] \to [m_2] \) and \( \sigma_1\delta_2 \colon [m_2] \to [m_1] \) must be injective. This implies \( m_1 = m_2 \), and since the order-preserving injective map \([m] \to [m]\) is the identity map, we must have \( \sigma_2\delta_1 = \text{id} = \sigma_1\delta_2 \), from which it follows that \( b_1 = b_2 \). This also shows that \( \delta_1 \in \Gamma(\sigma_2) \) and \( \delta_2 \in \Gamma(\sigma_1) \). Since \( \delta_1 \) and \( \delta_2 \) were arbitrarily chosen sections, we have shown \( \Gamma(\sigma_1) = \Gamma(\sigma_2) \), and therefore \( \sigma_1 = \sigma_2 \). \( \square \)

Here is an illustration of what can happen. Suppose \( a \in X_2 \) is a degeneracy of two simplices \( x \) and \( y \).

- If \( x, y \in X_0 \), and \( a = x_{000} = y_{000} \), then \( x = a_0 = y \).
- If \( x \in X_0, y \in X_1 \), and \( a = x_{000} = y_{001} \), then \( y = a_{02} = x_{00} \), whence \( y \) is degenerate.
- If \( x, y \in X_1 \) and \( a = x_{001} = y_{001} \), then \( x = a_{01} = y \).
- If \( x, y \in X_1 \) and \( a = x_{001} = y_{011} \), then \( x = a_{12} = y_{11} \) and \( y = a_{01} = x_{00} \), whence \( x \) and \( y \) are degenerate.

15.12. Corollary. For any simplicial set \( X \), the evident maps

\[
\prod_{k \geq 0} X^{\text{nd}}_k \times \text{Hom}_{\Delta^{\text{surj}}}([n], [k]) \to X_n
\]

defined by \((k, x, \sigma) \mapsto kx\sigma\) are bijections. Furthermore, these bijections are natural with respect to surjective simplicial operators \([n] \to [n']\).
Another way to say this: the restricted functor $X |(\Delta^{\text{surj}})_{\text{op}}: (\Delta^{\text{surj}})_{\text{op}} \to \text{Set}$ is canonically isomorphic to a coproduct of representable functors $\text{Hom}_{\Delta^{\text{surj}}}(\_, [k])$ indexed by the nondegenerate simplices of $X$. Or more simply: simplicial sets are canonically free with respect to surjective simplicial operators.

15.13. Example. Let $X = \Delta^k/\partial \Delta^k := \text{colim}(\Delta^k \leftarrow \partial \Delta^k \to *)$. The simplicial set $X$ has exactly two non-degenerate simplices: $* \in X_0$, and $\tau \in X_k$ (the image of the generator $\iota \in (\Delta^k)_k$). The set $X_n$ has exactly $1 + \binom{n}{k}$ elements: $1 = |\Delta^{\text{surj}}([n], [0])|$ simplex associated to $*$, and $\binom{n}{k} = |\Delta^{\text{surj}}([n], [k])|$ simplices associated to $\tau$.

15.14. Remark. A simplicial set can be recovered up to isomorphism if you only know (i) its sets of non-degenerate simplices, and (ii) the faces of the non-degenerate simplices. The proposition we proved above tells how to reconstruct the degenerate simplices; simplicial operators on degenerate simplices are computed using the fact that any simplicial operator factors into a surjection followed by an injection.

Warning. The faces of a non-degenerate simplex can be degenerate; this happens for instance in (15.13). If it happens that all faces of non-degenerate simplices in $X$ are also non-degenerate, then we get a functor $X^{\text{nd}}: (\Delta^{\text{inj}})_{\text{op}} \to \text{Set}$, and the full simplicial set $X$ can be recovered from $X^{\text{nd}}$. For instance, this is so for the standard simplicies $\Delta^n$, as well as any subcomplexes of such. Functors $(\Delta^{\text{inj}})_{\text{op}} \to \text{Set}$ are the combinatorial data behind the notion of a $\Delta$-complex, as seen in Hatcher’s textbook on algebraic topology [Hat02, Ch. 2.1].

The following exercises give a different point of view of this principle.

15.15. Exercise. Fix an object $[n]$ in $\Delta$, and consider the category $\Delta^{\text{surj}}_{[n]/}$, which has

- objects the surjective morphisms $\sigma: [n] \to [k]$ in $\Delta$, and
- morphisms commutative triangles in $\Delta$ of the form

$$
\begin{array}{ccc}
[n] & \xrightarrow{\sigma} & [k] \\
\downarrow \sigma & & \downarrow \tau \\
[k'] & \xrightarrow{\sigma'} & [k']
\end{array}
$$

Show that the category $\Delta^{\text{surj}}_{[n]/}$ is isomorphic to the poset $\mathcal{P}(n)$ of subsets of the set $n = \{1, \ldots, n\}$. In particular, $\Delta^{\text{surj}}_{[n]/}$ is a lattice (i.e., has finite products and coproducts, called meets and joins in this context).

15.16. Exercise. Let $X$ be a simplicial set. Given $n \geq 0$ and $\sigma: [n] \to [k]$ in $\Delta^{\text{surj}}$, let $X^n_\sigma := \sigma^*(X_k)$, the image of the operator $\sigma^*$ in $X_n$. Show that $X^{\sigma \vee \sigma'}_n = X^n_\sigma \cap X^n_{\sigma'}$, where $\sigma \vee \sigma'$ is join in the lattice $\Delta^{\text{surj}}_{[n]/}$. Conclude that for each $x \in X_n$ there exists a maximal $\sigma$ such that $x \in X^n_\sigma$.

15.17. Skeleta. Given a simplicial set $X$, the $k$-skeleton $\text{Sk}_k X \subseteq X$ is the subcomplex with $n$-simplices

$$(\text{Sk}_k X)_n = \bigcup_{0 \leq j \leq k} \{ yf \mid y \in X_j, f: [n] \to [j] \in \Delta \}.$$ 

It is immediate that this defines a subcomplex of $X$, which is in fact the smallest subcomplex containing all simplices of dimensions $\leq k$.

In view of (15.11) and (15.12), we see that

$$(\text{Sk}_k X)_n \approx \coprod_{0 \leq j \leq k} X^{\text{nd}}_j \times \text{Hom}_{\Delta^{\text{surj}}}(\_, [j]).$$

Note that $X = \text{colim}_{k \to \infty} \text{Sk}_k X$. The complement of $\text{Sk}_{k-1} X$ in $\text{Sk}_k X$ consists precisely of the nondegenerate $k$-simplices of $X$ (in simplicial degree $k$) together with their degeneracies (in simplicial degrees $> k$).
15.18. **Proposition.** The evident square
\[
\prod_{a \in X} \partial \Delta^k \longrightarrow \text{Sk}_{k-1} X \\
\downarrow \\
\prod_{a \in X} \Delta^k \longrightarrow \text{Sk}_k X
\]
is a pushout of simplicial sets. More generally, for any subcomplex \(A \subseteq X\), the evident square
\[
\prod_{a \in X \setminus A} \partial \Delta^k \longrightarrow A \cup \text{Sk}_{k-1} X \\
\downarrow \\
\prod_{a \in X \setminus A} \Delta^k \longrightarrow A \cup \text{Sk}_k X
\]
is a pushout.

**Proof.** In each of the above squares, the complements of the vertical inclusions coincide precisely. □

15.19. **Corollary.** \(\text{Cell}\) is precisely the class of monomorphisms.

**Proof.** We know all elements of \(\text{Cell}\) are monomorphisms. Any monomorphism is isomorphic to an inclusion \(A \subseteq X\) of a subcomplex, so we only need show that such inclusions are contained in \(\text{Cell}\). Since \(X \approx \colim_k A \cup \text{Sk}_k X\), (15.18) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of Cell. □

15.20. **Geometric realization.** Recall the singular complex functor \(\text{Sing} : \text{Top} \to \text{sSet}\) (8.7). This functor has a left adjoint \(|-| : \text{sSet} \to \text{Top}\), called **geometric realization**, constructed explicitly by

\[
|X| := \text{Cok} \left[ \prod_{f : [m] \to [n]} X_n \times \Delta^n_{\text{top}} \Rightarrow \prod_{[p]} X_p \times \Delta^p_{\text{top}} \right];
\]

that is, take a collection of topological simplices indexed by elements of \(X\), and make identifications according to the simplicial operators in \(X\). (Here the symbol “Cok” represents taking a “coequalizer”, i.e., the colimit of a diagram of shape \(\bullet \rightrightarrows \bullet\).)

15.22. **Exercise.** Describe the two unlabelled maps in (15.21). Then show that \(|-|\) is in fact left adjoint to \(\text{Sing}\).

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that \(|\Delta^n| \approx \Delta^n_{\text{top}}\), and that \(|\partial \Delta^n| \approx \partial \Delta^n_{\text{top}}\). Applying this to the skeletal filtration, we discover that there are pushouts
\[
\prod_{a \in X} \partial \Delta^k_{\text{top}} \longrightarrow |\text{Sk}_{k-1} X| \\
\downarrow \\
\prod_{a \in X} \Delta^k_{\text{top}} \longrightarrow |\text{Sk}_k X|
\]
of spaces, and that \(|X| = \bigcup |\text{Sk}_k X|\) with the direct limit topology. Thus, \(|X|\) is presented to us as a \(\text{CW-complex}\), whose cells are in an evident bijective correspondence with the set of non-degenerate simplices of \(X\).
16. Pushout-product and pullback-power

We are going to prove several “enriched” versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we’ll be able to prove that function complexes of quasicategories are themselves quasicategories.

16.1. Definition of pushout-product and pullback-power. Given maps \( f: A \to B, g: K \to L \) and \( h: X \to Y \) of simplicial sets, we define new maps called the **pushout-product**

\[
f \Box g: (A \times L) \amalg_{A \times K} (B \times K) \xrightarrow{(f \times L, B \times g)} B \times L
\]

and **pullback-power**

\[
h \Box g: \text{Map}(L, X) \xrightarrow{(\text{Map}(g, X), \text{Map}(L, h))} \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y).
\]

16.2. Remark. Typically we form \( f \Box g \) when \( f \) and \( g \) are monomorphisms, in which case \( f \Box g \) is also a monomorphism. In this case, the simplices \((b, \ell) \in B_n \times L_n \) which are not in the image of \( f \Box g \) are exactly those such that \( b \in B_n \setminus A_n \) and \( \ell \in L_n \setminus K_n \).

16.3. Remark (Important!). On 0-simplices, the pullback-power \( h \Box g \) is just the “usual” map \( \text{Hom}(L, X) \to \text{Hom}(K, X) \times_{\text{Hom}(K, Y)} \text{Hom}(L, Y) \) sending \( s \mapsto (sg, hs) \). Thus, \( h \Box g \) is surjective on 0-simplicies if and only if \( g \sqsubset h \).

We think of the pullback-power as encoding an “enriched” version of the lifting problem for \((g, h)\). Thus, the target of \( h \Box g \) is an object which “parameterizes families” of commutative squares involving \( g \) and \( h \). Similarly, the source of \( h \Box g \) “parameterizes families” of commutative squares together with lifts.

The product/mapping object adjunction gives rise to the following relationship between lifting problems.

16.4. Proposition. We have that \((f \Box g) \sqsubset h\) if and only if \( f \sqsubset (h \Box g) \).

Proof. Compare the two lifting problems using the product/map adjunction.

\[
\begin{array}{ccc}
(A \times L) \amalg_{A \times K} (B \times K) & \xrightarrow{(u, v)} & X \\
(f \Box g) \downarrow & & \downarrow h \\
B \times L & \xrightarrow{w} & Y
\end{array}
\quad \iff \quad
\begin{array}{ccc}
A & \xrightarrow{\tilde{u}} & \text{Map}(L, X) \\
\downarrow f & & \downarrow h \Box g \\
B & \xrightarrow{(\tilde{v}, \tilde{w})} & \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)
\end{array}
\]

The data of \((u, v, w)\) giving a commutative square as on the left corresponds bijectively to data \((\tilde{u}, \tilde{v}, \tilde{w})\) giving a commutative square as on the right. Similarly, lifts \( s \) correspond bijectively to lifts \( \tilde{s} \).

It is important to note the special cases where one or more of \( A = \emptyset, K = \emptyset, \) or \( Y = \ast \) hold. For instance, if \( K = \emptyset \) and \( Y = \ast \), the proposition implies

\[
(A \times L) \xrightarrow{f \times L} B \times L \sqsubset (X \to \ast) \iff (A \xrightarrow{f} B) \sqsubset (\text{Map}(L, X) \to \ast).
\]

---

16. This is sometimes called the **box-product**. Some also call it the **Leibniz-product**, as its form is that of the Leibniz rule for boundary of a product space: \( \partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y) \) (which is itself reminiscent of the original Leibniz rule \( D(fg) = (Df)g + f(Dg) \) of calculus).

17. Sometimes called the **box-power** or **pullback-hom**. A common alternate notation is \( g \sqcap h \). This may also be called the **Leibniz-hom**, though I don’t know what rule of calculus it is related to.

18. This notation for pullback-powers is kinda awkward, and I’d like to change it. However, a new notation ought to admit compatible variants to describe the “pullback-slice” and “alternate pullback-slice” constructions which appear later on. I don’t see a good way to do this.
This is the kind of case we are interested in for proving that $\text{Map}(K, C)$ is a quasicategory whenever $C$ is. The more general statement of the proposition is a kind of “relative” version of the thing we want; it is especially handy for carrying out inductive arguments.

16.5. Exercise (if you like monoidal categories). Let $C := \text{Fun}([1], s\text{Set})$, the “arrow category” of simplicial sets. Show that $\square: C \times C \to C$ defines a symmetric monoidal structure on $C$, with unit object $(\varnothing \subset \Delta^0)$. Furthermore, show that this is a closed monoidal structure, with $-\square g$ left adjoint to $(-)^{C} : C \to C$.

16.6. Inner anodyne maps and pushout-products. The key fact we want to prove is the following.

16.7. Proposition. We have that $\text{InnHorn} \square \text{Cell} \subseteq \text{InnHorn}$.

Proof. This will follow from (16.8) and (16.9) below. □

To set up the proof we need the following.

16.8. Proposition. For any sets of maps $S$ and $T$, we have $S \square T \subseteq S \square T$.

Proof. Let $F := (S \square T)^\square$. By the small object argument (13.11), $S \square T = F^\square$. First we show that $S \square T \subseteq S \square T$. Consider

$$A := \{ a | (a \square T) \square F \} \approx \{ a | a \square (F^\square T) \}$$

by the previous proposition. Thus $A$ is a left complement, and so is weakly saturated. Since $S \subseteq A$ then $S \subseteq A$, i.e., $S \square T \subseteq F^\square T = S \square T$. Now consider

$$B := \{ b | (S \square b) \square F \} \approx \{ b | b \square (F^\square S) \},$$

which is likewise weakly saturated. We have just shown that $T \subseteq B$, whence $T \subseteq B$, i.e., $S \square T \subseteq S \square T$. □

16.9. Lemma. We have $\text{InnHorn} \square \text{Cell} \subseteq \text{InnHorn}$.

Proof. This is a calculation, given in [Joy08a, App. H], and presented in the appendix (59.3). □

Let’s carry out a proof of (16.9) explicitly in one case, by showing that $(\Lambda^2_1 \subset \Delta^1) \square (\partial \Delta^1 \subset \Delta^1)$ is inner anodyne. This map is the inclusion

$$(\Lambda^2_1 \times \Delta^1) \cup_{\Lambda^2_1 \times \partial \Delta^1} (\Delta^2 \times \partial \Delta^1) \subset \Delta^2 \times \Delta^1,$$

whose target is a “prism”, and whose source is a “trough”. To show this is in $\text{InnHorn}$, we’ll give an explicit procedure for constructing the prism from the trough by successively attaching simplices along inner horns.

Note that $\Delta^2 \times \Delta^1 = N([2] \times [1])$, so we are working inside the nerve of a poset, whose elements (objects) are “$i j$” with $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. Here is a picture of the trough, showing all the non-degenerate simplices.

```
01  11  21
\  /  \
00 10 20
```

The complement of this in the prism consists of three non-degenerate 3-simplices, five non-degenerate 2-simplices (two of which form the “lid” of the trough, while the other three are in the interior of the prism), and one non-degenerate 1-simplex (separating the two 2-simplices which form the lid).

The following chart lists all non-degenerate simplices in the complement of the trough, along with their codimension one faces (in order). The “√” marks simplices which are contained in the trough.
17. Function complexes of quasicategories are quasicategories

17.1. Enriched lifting properties. We record the immediate consequences of $\text{InnHorn} \Box \text{Cell} \subseteq \text{InnHorn}$ (16.7).

17.2. Proposition.

1. If $i: A \to B$ is inner anodyne and $j: K \to L$ a monomorphism, then
   \[
   i \Box j: (A \times L) \cup_{A \times K} (B \times K) \to B \times L
   \]
   is inner anodyne.

2. If $j: K \to L$ is a monomorphism and $p: X \to Y$ is an inner fibration, then
   \[
   p^{\Box j}: \text{Map}(L, X) \to \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)
   \]
   is an inner fibration.

3. If $i: A \to B$ is inner anodyne and $p: X \to Y$ is an inner fibration, then
   \[
   p^{\Box i}: \text{Map}(B, X) \to \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)
   \]
   is a trivial fibration.

These can be summarized as

\[
\text{InnHorn} \Box \text{Cell} \subseteq \text{InnHorn}, \quad \text{InnFib} \Box \text{Cell} \subseteq \text{InnFib}, \quad \text{InnFib} \Box \text{InnHorn} \subseteq \text{TrivFib}.
\]

Statement (1) is just restating (16.7). Statements (2) and (3) can be thought of as “enriched” lifting properties. In particular, \( \text{InnFib} \Box \text{InnHorn} \subseteq \text{TrivFib} \) contains $\text{InnHorn} \Box \text{InnFib}$ as a special case, since trivial fibrations are necessarily surjective on vertices.

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert “$\text{InnHorn} \Box \text{Cell} \subseteq \text{InnHorn}$” without other explanation; alternately, to indicate an application of statements (2) and (3), I will call it “enriched lifting”. The following gives the most general statement, of which (16.7) amounts to the special case of $S = U = \text{InnHorn}$ and $T = \text{Cell}$.

17.3. Proposition. Let $S$, $T$, and $U$ be sets of morphisms in $sSet$. Write $\overline{S}$, $\overline{T}$, and $\overline{U}$ for the weak saturations of these sets, and let $S\text{Fib} := S^{\Box}$, $T\text{Fib} := T^{\Box}$, and $U\text{Fib} := U^{\Box}$ denote the respective right complements. If $S \Box T \subseteq U$, then

\[
\overline{S} \Box \overline{T} \subseteq \overline{U}, \quad U\text{Fib} \Box \overline{T} \subseteq S\text{Fib}, \quad U\text{Fib} \Box \overline{S} \subseteq T\text{Fib}.
\]

Proof. Exercise. \(\square\)

There are many useful special cases of (17.2), obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If $i: A \to B$ is inner anodyne, so is $i \times \text{id}_{L}: A \times L \to B \times L$.
- If $p: X \to Y$ is an inner fibration, then so is $\text{Map}(L, p): \text{Map}(L, X) \to \text{Map}(L, Y)$.
- If $j: K \to L$ is a monomorphism and $C$ a quasicategory, then $\text{Map}(j, C): \text{Map}(L, C) \to \text{Map}(K, C)$ is an inner fibration.
- If \( i: A \to B \) is inner anodyne and \( C \) a quasicategory, then \( \text{Map}(i, C): \text{Map}(B, C) \to \text{Map}(A, C) \) is a trivial fibration.
- If \( C \) is a quasicategory, so is \( \text{Map}(L, C) \). Thus we have proved (B).

Let’s spell out the proof of (B) in a little more detail. Because \( \text{InnHorn} \sqcap \text{Cell} \subseteq \text{InnHorn} \), we have (16.7) that
\[
(\Lambda^n_j \subset \Delta^n) \sqcap (\emptyset \subseteq K) = (\Lambda^n_j \times K \to \Delta^n \times K)
\]
is inner anodyne for any \( K \) and \( 0 < j < n \). Thus, for any diagram
\[
\begin{array}{ccc}
\Lambda^n_j \times K & \longrightarrow & C \\
\downarrow & & \downarrow \\
\Delta^n \times K & & \\
\end{array}
\]
with \( C \) a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend \( \Lambda^n_j \to \text{Map}(K, C) \) along \( \Lambda^n_j \subset \Delta^n \). That is, we have proved that \( \text{Map}(K, C) \) is a quasicategory.

17.4. Remark. Most weakly saturated classes \( \overline{S} \) that we will explicitly discuss in these notes will have the property that \( S \sqcap \text{Cell} \subseteq \overline{S} \), and thus analogues of the above remarks will hold for such classes.

17.5. Exercise (Important). Show that \( \text{Cell} \sqcap \text{Cell} \subseteq \overline{\text{Cell}} \). (Hint: (15.19).) State the analogue of (17.2) associated to this inclusion.

17.6. Composition functors. We can use the above theory to construct “composition functors”. If \( C \) is an ordinary category, the operation of composing a sequence of \( n \) maps can be upgraded to a functor:
\[
\text{Fun}([1], C) \times_C \cdots \times_C \text{Fun}([1], C) \to \text{Fun}([1], C)
\]
which on \textit{objects} describes composition of a sequence of maps. The source of this functor is the evident inverse limit in \text{Cat}; it can be identified (using simplicial set language) with \( \text{Fun}(I^n, C) \).

We can generalize such composition functors to quasicategories. We use the following observation: any trivial fibration admits a section, since \( (\emptyset \to Y) \sqcap (p: X \to Y) \) if \( p \) is a trivial fibration (17.2).

Let \( C \) be a quasicategory. Then map \( r: \text{Fun}(\Delta^n, C) \to \text{Fun}(I^n, C) \) induced by restriction along \( I^n \subset \Delta^n \) is a trivial fibration by (17.2), since the spine inclusion is inner anodyne (12.12). Therefore \( r \) admits a section \( s \), so we get a diagram
\[
\frac{\text{Fun}(I^n, C)}{r'} \leftarrow \text{Fun}(\Delta^n, C) \xrightarrow{r} \text{Fun}(\Delta^{(0,n)}, C)
\]
where \( r' \) is restriction along \( \Delta^{(0,n)} \subset \Delta^n \). The composite \( r's \) can be thought of as a kind of “\( n \)-fold composition” functor. It is not unique, since \( s \) isn’t, but we’ll see (??) that this is ok: all functors constructed this way are “naturally isomorphic” to each other.

17.7. A useful variant. The proof of (16.7) actually proves something a little stronger.

17.8. Proposition ([Joy08a, §2.3.1], [Lur09, §2.3.2]). We have that \( (\Lambda^2_1 \subset \Delta^2) \sqcap \text{Cell} = \text{InnHorn} \).

Proof. We give a proof in the appendix (59.3).

A consequence of this is another characterization of quasicategories.

17.9. Corollary. A simplicial set \( C \) is a quasicategory if and only if \( \text{Map}(\Delta^2, C) \to \text{Map}(\Lambda^2_1, C) \) is a trivial fibration.

Proof. First notice that \( (\partial \Delta^k \subset \Delta^k) \sqcap f \) for all \( k \geq 0 \) iff \( (\partial \Delta^k \subset \Delta^k) \sqcap (\Lambda^2_1 \subset \Delta^2) \sqcap (C \to *) \) for all \( k \geq 0 \), since \( f = (C \to *) \sqcap (\Lambda^2_1 \subset \Delta^2) \). Therefore \( f \in \text{TrivFib} = \text{InnHorn} \sqcap \) if and only if \( C \to * \) \( \in (\text{Cell} \sqcap (\Lambda^2_1 \subset \Delta^2)) \sqcap \). The conclusion immediately follows using (17.8).
### 18. Natural isomorphisms

#### 18.1. Natural isomorphisms of functors

Let $C$ and $D$ be quasicategories. Recall that a natural transformation between functors $f_0, f_1 : C \to D$ is defined to be a morphism $\alpha : f_0 \to f_1$ in the functor quasicategory $\text{Fun}(C, D)$, or equivalently a map $\bar{\alpha} : C \times \Delta^1 \to D$ such that $\bar{\alpha}|C \times \Delta^1(i) = f_i$, $i = 0, 1$.

Say that $\alpha : f_0 \to f_1$ is a natural isomorphism if $\alpha$ is an isomorphism in the quasicategory of functors $\text{Fun}(C, D)$. Thus, $\alpha$ is a natural isomorphism iff there exists a natural transformation $\beta : f_1 \to f_0$ such that $\beta \alpha \sim 1_{f_0}$ and $\alpha \beta \sim 1_{f_1}$, where “$\sim$” is homotopy between morphisms in the quasicategory $\text{Fun}(C, D)$.

This notion of natural isomorphism corresponds with the usual one for ordinary categories, since in that case homotopy of morphisms is the same as equality.

Observe that “there exists a natural isomorphism $f_0 \to f_1$” is an equivalence relation on the set of all functors $C \to D$, as this relation precisely coincides with “there exists an isomorphism $f_0 \to f_1$” in the category $\text{h Fun}(C, D)$.

Furthermore, the “naturally isomorphic” relation is compatible with composition: if $f, f'$ are naturally isomorphic and $g, g'$ are naturally isomorphic, then so are $gf$ and $g'f'$. You can read this off from the fact that the operation of composition of functors extends to a functor $\text{Fun}(D, E) \times \text{Fun}(C, D) \to \text{Fun}(C, E)$ between quasicategories, and so induces a functor

$$h \text{Fun}(D, E) \times h \text{Fun}(C, D) \approx h(\text{Fun}(D, E) \times \text{Fun}(C, D)) \to h \text{Fun}(C, E).$$

(This uses (9.15) to identify the homotopy category of the product with the product of homotopy categories.)

#### 18.2. Objectwise criterion for natural isomorphisms

Recall that if $A$ and $B$ are ordinary categories, a natural transformation $\alpha : f_0 \to f_1$ between functors $f_0, f_1 : A \to B$ is a natural isomorphism iff and only if $\alpha$ is “an isomorphism objectwise”; i.e., if for each object $a$ of $A$ the evident map $\alpha(a) : f_0(a) \to f_1(a)$ is an isomorphism in $B$. That natural isomorphisms are “objectwise isomorphisms” is immediate. The opposite implication follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its “values on objects”. Thus, given $\alpha : f_0 \to f_1$ such that each $\alpha(a) : f_0(a) \to f_1(a)$ is an isomorphism, we may explicitly construct an inverse transformation $\beta : f_1 \to f_0$ by setting $\beta(a) := (\alpha(a))^{-1} : f_1(a) \to f_0(a)$. Note that this $\beta$ is in fact the unique inverse to $\alpha$ (since inverses to morphisms are unique when they exist).

One of these directions is straightforward for quasicategories.

#### 18.3. Proposition

Let $C$ and $D$ be quasicategories. If $\alpha : C \times \Delta^1 \to D$ is a natural isomorphisms between functors $f_0, f_1 : C \to D$, then for each object $c$ of $C$ the induced map $\alpha(c) : f_0(c) \to f_1(c)$ is an isomorphism in $D$.

**Proof.** The restriction map $\text{Fun}(C, D) \to \text{Fun}(\{c\}, D) = D$ is a functor between quasicategories, so it takes isomorphisms to isomorphisms. It sends $\alpha$ to $\alpha(c)$. \(\square\)

The converse to this proposition is also true.

#### C. Deferred Proposition

A natural transformation $\alpha : C \times \Delta^1 \to D$ of functors between quasicategories is a natural isomorphism if and only if each of the maps $\alpha(c)$ are isomorphisms in $D$.

Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the $\alpha(c)$’s to produce an inverse to $\alpha$, which though it exists is not at all unique. We will prove this converse later (29.9).

#### 18.4. Remark

An immediate consequence of (C) is that if $D$ is a quasigroupoid, then so is $\text{Fun}(C, D)$. 

18.5. Remark. The objectwise criterion (C) can be reformulated in terms of homotopy categories. The homotopy category construction takes quasicategories to categories, and takes functors to functors. Furthermore, given a natural transformation \( \alpha : f_0 \to f_1 \) of functors \( f_0, f_1 : C \to D \) between quasicategories (i.e., a functor \( \alpha : C \times \Delta^1 \to D \) such that \( \alpha|C \times \{j\} = f_j \)), we obtain an induced transformation \( h\alpha : hf_0 \to hf_1 \) of functors \( hf_0, hf_1 : hC \to hD \) between their homotopy categories (so that the value of \( h\alpha \) at an object \( c \in \text{ob} hC = C_0 \) is the homotopy class of the edge \( \alpha(\{c\} \times \Delta^1) \subseteq D \)). Then (C) asserts that \( \alpha \) is a natural isomorphism of functors between quasicategories if and only if \( h\alpha \) is a natural isomorphism of functors between ordinary categories.

19. Categorical equivalence

We are now in position to define the correct generalization of the notion of “equivalence” of categories. This will be called categorical equivalence of quasicategories, and will be a direct generalization of the classical notion.

Given this, we use it to define a notion of categorical equivalence which applies to arbitrary maps of simplicial sets. Finally, we will show that the two definitions agree for maps between quasicategories.

19.1. Categorical equivalences between quasicategories. A categorical inverse to a functor \( f : C \to D \) between quasicategories is a functor \( g : D \to C \) such that \( gf \) is naturally isomorphic to \( 1_C \) and \( fg \) is naturally isomorphic to \( 1_D \). We provisionally say that a functor \( f \) between quasicategories is a categorical equivalence if it admits a categorical inverse.

19.2. Remark. Categorical equivalence between quasicategories is a kind of “homotopy equivalence”, where homotopies are natural isomorphisms between functors.

If \( C \) and \( D \) are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, and that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

If quasicategories are equivalent, then their homotopy categories are equivalent.

19.3. Proposition. If \( f : C \to D \) is a categorical equivalence between quasicategories, then \( h(f) : hC \to hD \) is an equivalence of categories.

Proof. Immediate, given that natural transformations \( f \Rightarrow g : C \to D \) induce natural transformations \( h(f) \Rightarrow h(g) : hC \to hD \).

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to \( \Delta^0 \), but whose homotopy categories are: e.g., \( \text{Sing} T \) for any non-contractible simply connected space \( T \), or \( K(A,d) \) for any non-trivial abelian group \( A \) and \( d \geq 2 \).

19.4. General categorical equivalence. We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map \( f : X \to Y \) between arbitrary simplicial sets is a categorical equivalence if for every quasicategory \( C \), the induced functor \( \text{Fun}(f, C) : \text{Fun}(Y, C) \to \text{Fun}(X, C) \) of quasicategories admits a categorical inverse.

We claim that on maps between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

19.5. Lemma. For a map \( f : C \to D \) between quasicategories, the two notions of categorical equivalence described above coincide. That is, the following are equivalent:

(1) \( f \) admits a categorical inverse.

(2) For every quasicategory \( E \), the functor \( \text{Fun}(f, E) : \text{Fun}(D, E) \to \text{Fun}(C, E) \) admits a categorical inverse.
To prove this, we will need the following observation. The construction \( X \mapsto \text{Map}(X, E) \) is a functor \( s\text{Set}^{op} \to s\text{Set} \), and so in particular induces a natural map
\[
\gamma_0: \text{Hom}(X, Y) \to \text{Hom}(\text{Map}(Y, E), \text{Map}(X, E))
\]
of sets. The observation we need is that this construction admits an “enrichment”, to a map
\[
\gamma: \text{Map}(X, Y) \to \text{Map}(\text{Map}(Y, E), \text{Map}(X, E)),
\]
which coincides with \( \gamma_0 \) on vertices. The map \( \gamma \) is defined to be adjoint to the “composition” map \( \text{Map}(X, Y) \times \text{Map}(Y, E) \to \text{Map}(X, E) \). (Exercise: Describe explicitly what \( \gamma \) does to \( n \)-simplices.) We say that the functor \( \text{Map}(-, E) \) is an enriched functor, as it gives not merely a map between hom-sets (i.e., acts on 0-simplices in function complexes), but in fact gives a map between function complexes.

**Proof.** (1) \( \implies \) (2). When \( C, D, \) and \( E \) are quasicategories so are the function complexes between them. In this case, the above map \( \gamma \) takes functors \( C \to D \) to functors \( \text{Map}(D, E) \to \text{Map}(C, E) \), natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Together with the fact that \( \text{Map}(-, E) \) is itself a functor \( s\text{Set}^{op} \to s\text{Set} \), it follows that a categorical inverse \( g: D \to C \) to \( f: C \to D \) gives rise to a categorical inverse \( \text{Map}(g, E) \) to the induced functor \( \text{Map}(f, E): \text{Map}(D, E) \to \text{Map}(C, E) \).

(2) \( \implies \) (1). Conversely, suppose \( f: C \to D \) is a categorical equivalence in the general sense, so that \( f^* = \text{Map}(f, E) \) admits a categorical inverse for every quasicategory \( E \), which implies that each functor
\[
h(f^*): h \text{Fun}(D, E) \to h \text{Fun}(C, E)
\]
is an equivalence of ordinary categories (19.3). In particular, it follows that \( f^* \) induces a bijection of sets
\[
f^*: \pi_0(\text{Fun}(D, E)^{\text{core}}) \sim \pi_0(\text{Fun}(C, E)^{\text{core}});
\]
recall that \( \pi_0(\text{Fun}(D, E)^{\text{core}}) \approx \pi_0((h \text{Fun}(D, E))^\text{core}) \) is precisely the set of natural isomorphism classes of functors \( D \to E \).

Taking \( E = C \), this implies that there must exist \( g \in \text{Fun}(D, C)_0 \) such that there exists a natural isomorphism \( gf \to \text{id}_C \) in \( \text{Fun}(C, C)_1 \). Taking \( E = D \), we note that since
\[
f^*(\text{id}_D) = \text{id}_D f = f \text{id}_C \approx fgf = f^*(fg),
\]
we must have that \( \text{id}_D \approx fg \), i.e., there exists a natural isomorphism \( \text{id}_D \to fg \) in \( \text{Fun}(D, D)_1 \). Thus, we have shown that \( g \) is a categorical inverse of \( f \), as desired. \( \square \)

19.6. **Remark.** The definition of categorical equivalence we are using here is very different to the definition adopted by Lurie [Lur09, §2.2.5]. It is also slightly different from the notion of “weak categorical equivalence” used by Joyal [Joy08a, 1.20]. As we will show soon (22.11), Joyal’s weak categorical equivalence is equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] show’s that Lurie’s and Joyal’s definitions are equivalent, and so they are both equivalent to the one we have used.

19.7. **Exercise.** Let \( f: C \to D \) be a functor between quasicategories. Show that \( f \) is a categorical equivalence if and only if for all simplicial sets \( X \), the induced functor \( f_*: \text{Map}(X, C) \to \text{Map}(X, D) \) is a categorical equivalence.

20. **Trivial fibrations and inner anodyne maps**

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.
20.1. **Trivial fibrations to the terminal object.** Recall that a trivial fibration \( p: X \to Y \) of simplicial sets is a map such that \((\partial \Delta^k \subset \Delta^k) \ni p \) for all \( k \geq 0 \). That is, \( \text{TrivFib} = \text{Cell}^2 \), so \( p \) is a trivial fibration if and only if \( \text{Cell} \ni p \).

20.2. **Exercise.** Consider an indexed collection of trivial fibrations \( p_i: X_i \to Y_i \). Show that \( p := \coprod p_i: \coprod X_i \to \coprod Y_i \) is a trivial fibration. (Hint: similar to proof of (6.7).)

20.3. **Proposition.** Let \( X \) be a simplicial set and \( p: X \to \ast \) be a trivial fibration whose target is the terminal simplicial set. Then \( X \) is a Kan complex (and thus a quasigroupoid) and \( p \) is a categorical equivalence.

**Proof.** The hypotheses together with \( \Delta \ni \text{Cell} \subseteq \text{Cell} \) (17.5) and enriched lifting (17.3) imply that Map\( (B, X) \to \text{Map}(A, X) \) is a trivial fibration for all inclusions \( A \subseteq B \), and thus in particular Hom\( (B, X) \to \text{Hom}(A, X) \) is surjective. It follows that \( X \) is a Kan complex, by taking \( A \subseteq B \) to be a horn inclusion.

To show that \( p \) is a categorical equivalence, first note that \( X \) is non-empty, since Hom\( (\Delta^0, X) \to \text{Hom}(\emptyset, X) = \ast \) is surjective. Choose any \( s \in \text{Hom}(\Delta^0, X) \). Clearly \( ps = \text{id}_{\Delta^0} \). We will show that \( sp: X \to X \) is naturally isomorphic to \( \text{id}_X \). Consider the commutative diagram

\[
\begin{array}{ccc}
X \coprod X & \xrightarrow{(\text{id}_X, sp)} & X \\
\downarrow & & \downarrow p \\
X \times \Delta^1 & \xrightarrow{h} & * \\
\end{array}
\]

Since \( p \) is a trivial fibration, a lift \( h \) exists, which exhibits a natural transformation \( \text{id}_X \to sp \); note that \( h \) represents a morphism in \( \text{Fun}(X, X) \). To show that \( h \) represents an isomorphism, it’s enough to know that \( \text{Fun}(X, X) \) is actually a quasigroupoid. Observe that (since \( p \) is a trivial fibration and \( \text{Cell} \ni \text{Cell} \subseteq \text{Cell} \)), the map \( \text{Fun}(X, p): \text{Fun}(X, X) \to \text{Fun}(X, \ast) = \ast \) is also a trivial fibration, whence \( \text{Fun}(X, X) \) is a Kan complex by the argument given above.

We will prove a partial converse to this later (33.9): if \( C \) is a quasicategory which is categorically equivalent to \( \ast \), then \( C \to \ast \) is a trivial fibration.

20.4. **Preisomorphisms.** We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories.

Let \( X \) be a simplicial set. Say that an edge \( a \in X_1 \) is a **preisomorphism** if it projects to an isomorphism under \( \alpha: X \to hX \), the tautological map to the (nerve of the) fundamental category (9.1). If \( X \) is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category.

20.5. **Proposition.** An edge \( a \in X_1 \) is a preisomorphism if and only if for every map \( g: X \to C \) to a quasicategory \( C \), the image \( g(a) \) is an isomorphism in \( C \).

**Proof.** Isomorphisms in \( C \) are exactly the edges which are sent to isomorphisms under \( \gamma: C \to hC \). Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that \( hX \) is itself a category and hence a quasicategory.

As a consequence, any map \( X \to Y \) of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map from a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.
20.6. Example. Consider the subcomplex $\Lambda^3_{[0,3]} = \Delta^{0,1,2} \cup \Delta^{1,2,3}$ of $\Delta^3$. Define $X$ to be the pushout of the diagram

$$
\Lambda^3_{[0,3]} \xleftarrow{j} \Delta^{0,1} \cup \Delta^{0,2} \cup \Delta^{1,3} \cup \Delta^{2,3} \xrightarrow{p} \Delta^{y<x},
$$

where $j$ is the evident inclusion, and $p$ is the unique map to a 1-simplex given on vertices by $0, 2 \mapsto y, 1, 3 \mapsto x$. The resulting complex $X$ looks like

$$
\begin{array}{ccc}
& y \downarrow^{y_{00}} & \\
& \downarrow{b} & \downarrow{g} & \\
x & \downarrow{a} & \downarrow{g} & x_{00} \\
& \uparrow{g} & \uparrow{g} & \\
& b & x & \\
\end{array}
$$

with six non-degenerate simplices $x, y, f, g, a, b$. The simplicial set $X$ is not a quasicategory. However, any map $\phi: X \to C$ to a quasicategory sends $f$ and $g$ to morphisms $\phi(f)$ and $\phi(g)$ of $C$ which are inverse to each other. Therefore these (and thus all) edges of $X$ are preisomorphisms.

Say that vertices in a simplicial set $X$ are preisomorphic if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map $g: X \to C$ to a quasicategory takes preisomorphic vertices of $X$ to isomorphic objects of $C$.

We can apply this to function complexes. If two maps $f_0, f_1: X \to Y$ are preisomorphic (viewed as vertices in $\text{Map}(X,Y)$), then for any quasicategory $C$, the induced functors $\text{Map}(f_0, C), \text{Map}(f_1, C): \text{Map}(Y, C) \to \text{Map}(X, C)$ are naturally isomorphic. To see this, consider

$$
\Delta^1 \xrightarrow{a} \text{Map}(X, Y) \xrightarrow{b} \text{Map}(\text{Map}(Y, C), \text{Map}(X, C))
$$

where $b$ is adjoint to the composition map $\text{Map}(Y, C) \times \text{Map}(X, Y) \to \text{Map}(X, C)$. If $a$ represents a preisomorphism $f_0 \to f_1$ in $\text{Map}(X, Y)$, then $ba$ represents an isomorphism $\text{Map}(f_0, C) \to \text{Map}(f_1, C)$, since the target of $b$ is a quasicategory. As a consequence we get the following.

20.7. Lemma. If $p: X \to Y$ and $q: Y \to X$ are maps of simplicial sets such that $qp$ is preisomorphic to $\text{id}_X$ in $\text{Map}(X, X)$ and $pq$ is preisomorphic to $\text{id}_Y$ in $\text{Map}(Y, Y)$, then $p$ and $q$ are categorical equivalences.

20.8. Trivial fibrations are always categorical equivalences.

20.9. Proposition. Every trivial fibration between simplicial sets is a categorical equivalence.

Here is some notation. Given maps $f: A \to Y$ and $g: B \to Y$, we write $\text{Map}_Y(f, g)$ or $\text{Map}_Y(A, B)$ for the simplicial set defined by the pullback square

$$
\begin{array}{ccc}
\text{Map}_Y(A, B) & \xrightarrow{f} & \text{Map}(A, B) \\
\downarrow{g_*=\text{Map}(A, g)} & & \downarrow{g_*=\text{Map}(A, g)} \\
\{f\} & \xrightarrow{g_*=\text{Map}(A, g)} & \text{Map}(A, Y) \\
\end{array}
$$

Note that vertices of $\text{Map}_Y(A, B)$ correspond exactly to sections of $g$ over $f$ (i.e., to $s: A \to B$ such that $gs = f$). You can think of $\text{Map}_Y(A, B)$ as a simplicial set which “parameterizes” sections of $g$ over $f$. I’ll call this the relative function complex over $Y$.

**Proof.** Fix a trivial fibration $p: X \to S$. We regard both $X$ and $S$ as objects over $S$, via $p$ and $\text{id}_S$, and consider various relative function complexes over $S$.

Note that since $p$ is a trivial fibration, so are $\text{Map}(X, p)$ and $\text{Map}(S, p)$ since $\text{Cell} \sqsubset \text{Cell} \subseteq \text{Cell}$, and therefore both

$$
\text{Map}_S(S, X) \to \text{Map}_S(S, S) = * \quad \text{and} \quad \text{Map}_S(X, X) \to \text{Map}_S(X, S) = *
$$
are trivial fibrations (since TrivFib is weakly cosaturated and so closed under base change). It follows
from (20.3) that both $\text{Map}/S(S, X)$ and $\text{Map}/S(S, Y)$ are quasigroupoids which are non-empty, and
with the property that any pair of objects in each are isomorphic. These are contained in the
simplicial sets $\text{Map}(S, X)$ and $\text{Map}(X, X)$ respectively, which however need not be quasicategories.
Pick any $s \in \text{Map}/S(S, X) \subseteq \text{Map}(S, X)$, whence $ps = \text{id}_S$, and pick any isomorphism $a: \text{id}_X \to sp$ in $\text{Map}/S(X, X)$, which is hence a preisomorphism in $\text{Map}(X, X)$.
Thus, we have exhibited maps $p$ and $s$ whose composites are preisomorphic to identity functors,
and therefore they are categorical equivalences by (20.7).  □

20.10. Remark. Suppose that $p: C \to D$ is a trivial fibration between quasicategories. As we have
noted, the relative function complex $\text{Map}/D(C, D)$ “parameterizes sections of $p$”. According to the
proof of (20.9), this is a quasigroupoid equivalent to the terminal quasicategory. In particular, not
only is $p$ a categorical equivalence, but also
- $p$ admits a section, which is a categorical inverse to $p$, and
- any two sections of $p$ are naturally isomorphic.
We will often make use of this observation.

20.11. Inner anodyne maps are always categorical equivalences.

20.12. Proposition. Every inner anodyne map between simplicial sets is a categorical equivalence.

Proof. Let $j: X \to Y$ be a map in $\text{InnHorn}$, and let $C$ be any quasicategory. The induced map
$\text{Map}(j, C): \text{Map}(Y, C) \to \text{Map}(X, C)$ is a trivial fibration since $\text{InnHorn} \subset \text{Cell}$ (17.2),
and therefore a categorical equivalence. □

20.13. Every simplicial set is categorically equivalent to a quasicategory.


(1) There exists a quasicategory $C$ and an inner anodyne map $f: X \to C$, which is therefore a
categorical equivalence.

(2) For any two $f_1: X \to C_i$ as in (1), there exists a categorical equivalence $g: C_1 \to C_2$ such
that $gf_1 = f_2$.

Here is some more notation. Given maps $f: X \to A$ and $g: X \to B$, we write $\text{Map}_X(f, g)$ or
$\text{Map}_X(A, B)$ for the simplicial set defined by the pullback square

\[
\begin{array}{ccc}
\text{Map}_X(A, B) & \longrightarrow & \text{Map}(A, B) \\
\downarrow & & \downarrow f^* = \text{Map}(f, B) \\
\{g\} & \longrightarrow & \text{Map}(X, B)
\end{array}
\]

This is the relative function complex under $X$.

Proof. (1) By the small object argument (13.10), we can factor $X \to \ast$ into $X \xrightarrow{j} C \xrightarrow{p} \ast$ where
$j \in \text{InnHorn}$ and $p \in \text{InnFib}$. The inner anodyne map $j$ is the desired categorical equivalence to a
quasicategory.

(2) For $i, j \in \{1, 2\}$, we have a restriction map $f_i^*: \text{Map}(C_i, C_j) \to \text{Map}(X, C_j)$, which is
necessarily a trivial fibration since $\text{Cell} \subset \text{Cell} \subseteq \text{Cell}$. Therefore the maps $\text{Map}_X(C_i, C_j) \to \ast$ are
all trivial fibrations, i.e., each $\text{Map}_X(C_i, C_j)$ is a quasigroupoid with only one isomorphism class
of objects (20.3). As in the proof of (20.9) we construct $g: C_1 \to C_2$ and $g': C_2 \to C_1$ which are
categorically inverse to each other; details are left to the reader. □
Thus, we can always “replace” a simplicial set $X$ by a categorically equivalent quasicategory $C$. Although such $C$ is not unique, it is unique up to categorical equivalence.

You can think of such a replacement $X \to C$ of $X$ as a quasicategory “freely generated” by the simplicial set $X$, an idea which is validated by the fact that $\text{Fun}(j, D) : \text{Fun}(C, D) \to \text{Map}(X, D)$ is a categorical equivalence for every quasicategory $D$.

21. Some examples of categorical equivalences

21.1. Free monoid on one generator. Let $F$ denote the free monoid on one generator $g$. This is a category with one object $x$, and morphism set $\{g^n \mid n \geq 0\}$.

Associated to the generator $g$ is a map

$$\gamma : S := \Delta^1/\partial \Delta^1 \to N(F)$$

sending the image of the generator $i \in (\Delta^1)_1$ in $S$ to $g$. (We use “$L/K$” as a shorthand for “$L \amalg K$” whenever $K \subseteq L$. The object $S$ is called the “simplicial circle”, which has exactly two nondegenerate simplicies, one in dimension 0 and one in dimension 1.)

It is not hard to see that $F$ is “freely generated” as a category by $S$, in the sense that $hS = F$ (the fundamental category of $S$ is $F$). It turns out that $N(F)$ is actually freely generated as a quasicategory by $S$.

21.2. Proposition. The map $\gamma : S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.

Proof. This is an explicit calculation. Let $a_k \in N(F)_k$ denote the $k$-simplex corresponding to the length $k$ chain of maps $(g, g, \ldots, g)$ in $F$ where $g$ is the generator, and let $Y_k \subseteq N(F)$ denote the subcomplex which is the image of $a_k : \Delta^k \to N(F)$. We have that $\bigcup Y_k = N(F)$, since a general $d$-simplex in $N(F)$ corresponds to a chain of maps $(g^{m_1}, \ldots, g^{m_d})$, which is evidently a face of $a_m$ if $m \geq m_1 + \cdots + m_d$.

Clearly, $Y_1 \approx S$, while $Y_2 \approx Y_1 \cup_{\Lambda^2_1} \Delta^2$. In general, you can easily verify the following.

- A simplex $x$ of $\Delta^k$ is such that $a_k(x)$ is in the subcomplex $Y_{k-1}$ of $Y_k$ if and only if $x$ is in the subcomplex $\Lambda^k_{\{0, k\}} = \Delta^{\{0, \ldots, k-1\}} \cup \Delta^{\{1, \ldots, k\}}$.
- Every simplex $y$ of $Y_k$ not in $Y_{k-1}$ is the image under $a_k$ of a unique simplex in $\Delta^k$.

In other words, the square

$$\begin{array}{ccc}
\Lambda^k_{\{0, k\}} & \longrightarrow & Y_{k-1} \\
\downarrow & & \downarrow \\
\Delta^k & \longrightarrow & Y_k \\
\downarrow_{a_k} & & \downarrow \\
& & a_k
\end{array}$$

is a pullback, and $a_k$ induces in each degree $n$ a bijection $(\Delta^k)_n \cong (\Lambda^k_{\{0, k\}})_n \cong (Y_k)_n \cong (Y_{k-1})_n$. It follows (21.3) that the square is a pushout.

The inclusion $\Lambda^k_{\{0, k\}} \subseteq \Delta^k$ is a generalized inner horn, and we have noted this is inner anodyne when $k \geq 2$ (12.10). It follows that each $Y_{k-1} \to Y_k$ is inner anodyne, whence $S \to N(F)$ is inner anodyne. \qed

In the proof, we used the following fact, which is worth recording.

21.3. Lemma. If

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow_i & & \downarrow_j \\
Y' & \longrightarrow & Y \\
\downarrow_f & & \downarrow_i \\
& & f
\end{array}$$
is a pullback of simplicial sets such that (i) $j$ is a monomorphism, and (ii) $f$ induces in each degree $n$ a bijection $Y'_{n} \to i(X'_{n}) \cong Y_{n} \setminus i(X_{n})$, then the square is a pushout square.

**Proof.** Verify the analogous statement for a pullback square of sets. 

21.4. **Free categories.** We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let $S$ be a 1-dimensional simplicial set, i.e., one such that $S = Sk_{1} S$. These are effectively the same thing as directed graphs (allowed to have multiple parallel edges and loops): $S_{0}$ corresponds to the set of vertices of the directed graph, and $S^{1}_{\text{nd}}$ corresponds to the set of edges of the directed graph.

Let $F := hS$. We call $F$ the free category on the 1-dimensional simplicial set $S$. In this case, the morphisms of the fundamental category are precisely the words in the edges $S^{1}_{\text{nd}}$ of the directed graph (including empty words for each vertex, corresponding to identity maps).

21.5. **Proposition.** The evident map $\gamma : S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.

**Proof.** This is virtually the same as the proof of (21.2). In this case, $Y_{k} \subseteq N(F)$ is the subcomplex generated by all $a : \Delta^{k} \to N(F)$ such that each spine-edge $a_{i-1,i}$ is in $S^{1}_{\text{nd}}$, and $Y_{k}$ is obtained by attaching a generalized horn to $Y_{k-1}$ for each such $a$. 

As a consequence, it is “easy” to construct functors $F \to C$ from a free category to a quasicategory: start with a map $S \to C$, which amounts to specifying vertices and edges in $C$ corresponding to elements $S_{0}$ and $S^{1}_{\text{nd}}$, and extend over $S \subseteq F$. The evident restriction map $\text{Fun}(F,C) \to \text{Map}(S,C)$ is a categorical equivalence, and in fact a trivial fibration.

21.6. **Exercise.** Describe the ordinary category $A := hA_{3}^{3}$ “freely generated” by $A_{0}^{3}$. Show that the tautological map $A_{3}^{3} \to N(A)$ is inner anodyne.

21.7. **Free commutative monoids.** Let $F$ be the free monoid on one generator again, with generator corresponding to simplicial circle $S = \Delta^{1}/\partial\Delta^{1} \subset F$. Recall that $F^{\times n}$ is the free commutative monoid on $n$ generators. Recall that the nerve functor preserves products, so $N(F^{\times n}) \approx N(F)^{\times n}$. We obtain a map

$$\delta = \gamma^{\times n} : S^{\times n} \to N(F^{\times n})$$

from the “simplicial $n$-torus”.

21.8. **Proposition.** The map $\delta : S^{\times n} \to N(F^{\times n})$ is a categorical equivalence, and in fact is inner anodyne.

This is an easy consequence of the following.

21.9. **Proposition.** For any simplicial set $K$, if $j : A \to B$ is inner anodyne, so is $j \times K : A \times K \to B \times K$.

**Proof.** The map $j \boxtimes (\varnothing \subseteq K)$ is inner anodyne, since InnHorn $\boxtimes \text{Cell} \subseteq \text{InnHorn}$ by (16.7). 

21.10. **Exercise.** Let $S \vee S \subseteq S^{\times 2}$ be the subcomplex obtained as the evident “one-point union” of the two “coordinate circles”. Suppose given a map $\phi : S \vee S \to C$ to a quasicategory $C$, corresponding to a choice of object $x \in C_{0}$ together with two morphisms $f, g : x \to x$ in $C_{1}$. Show that there exists an extension of $\phi$ along $S \vee S \subset N(F^{\times 2})$ if and only if $fg \approx gf$ in $hC$.

21.11. **Remark.** The analogue of the above exercise for $n = 3$ isn’t true. That is, given $S \vee S \vee S \to C$ corresponding to three morphisms $f, g, h : x \to x$ in $C$ such that $fg \approx gh$, $gh \approx hg$, and $hf \approx fh$ in $C$, there need not exist an extension to $N(F^{\times 3})$. (For instance, take $C = \text{Sing}T$, where $T \subseteq (S^{1})^{3}$ is the subspace of the 3-torus consisting of tuples $(x_{1}, x_{2}, x_{3})$ such that at least one $x_{i}$ is the basepoint of $S^{1}$.)
21.12. **Finite groups are not finite.** If \( A \) is any ordinary category, then \( \text{Sk}_2 N(A) \) "freely generates \( N(A) \) as a category", in the sense that \( h(\text{Sk}_2 N(A)) \approx A \), or equivalently that \( \text{Map}(N(A), N(B)) \to \text{Map}(\text{Sk}_2 N(A), N(B)) \) is an isomorphism for any category \( B \). However, it is often the case that no finite dimensional subcomplex "freely generates \( N(A) \) as a quasicategory". In fact, this is the case for every non-trivial finite group.

21.13. **Example.** Let \( G \) be a finite group, and let \( C = N(G) \). The geometric realization \( BG := |N(G)| \) is the classifying space of \( G \). I want to show that if \( G \) is not the trivial group, then \( N\!\!G \) is not categorically equivalent to any finite dimensional simplicial set \( K \) (i.e., one with no non-degenerate simplices above a certain dimension). We need to use some homotopy theory, along with a fact to be proved later\(^{10}\): if \( f: X \to Y \) is any categorical equivalence of simplicial sets, then the induced map \( |f|: |X| \to |Y| \) on geometric realizations is a homotopy equivalence of spaces.

First consider \( G = \mathbb{Z}/n \) with \( n > 1 \). A standard calculation in topology says that \( H^{2k}(|N(G)|, \mathbb{Z}) \approx \mathbb{Z}/n \neq 0 \) for all \( k > 0 \). This implies that \( |N(G)| \) cannot be homotopy equivalent to any finite dimensional complex.

Now consider a general non-trivial finite group \( G \), and choose a non-trivial cyclic subgroup \( H \leq G \). We know the fundamental group: \( \pi_1 |K| \approx \pi_1 |N(G)| = G \). Covering space theory tells us we can construct a covering map \( p: E \to |N(G)| \) so that \( \pi_1 E \to \pi_1 |N(G)| \) is the inclusion \( H \to G \). In fact, \( E \) is homotopy equivalent to the classifying space \( BH \) (because \( \pi_k E \approx 0 \) for \( k \geq 2 \)). But if \( |N(G)| \) is finite dimensional, then so is \( E \), but this would then contracting the observation that \( H^*(BH, \mathbb{Z}) \approx H^*(E, \mathbb{Z}) \approx 0 \) for all \( * \).

Thus, non-trivial finite groups are never "freely generated as a quasicategory" by finite dimensional complexes.

21.14. **Remark.** Let \( T \) be a finite CW-complex, and \( G \) a finite group. A theorem of Haynes Miller (the "Sullivan conjecture") implies that every functor \( N(G) \to \text{Sing} T \) is naturally isomorphic to a constant functor (i.e., one which factors through \( \Delta^0 \)). Thus, the singular complex of a finite CW-complex cannot "contain" any non-trivial finite group, even if its fundamental group contains a non-trivial finite subgroup.

22. **The homotopy category of QuasiCategories**

22.1. **The homotopy category of \( \text{QCat} \).** The homotopy category \( h\text{QCat} \) of quasicategories is defined as follows. The objects of \( h\text{QCat} \) are the quasicategories. Morphisms \( C \to D \) in \( h\text{QCat} \) are natural isomorphism classes of functors. That is,

\[
\text{Hom}_{h\text{QCat}}(C, D) := \text{isomorphism classes of objects in } h\text{Fun}(C, D) = \pi_0(\text{Fun}(C, D)^{\text{core}}).
\]

That this defines a category results from the fact that composition of functors passes to a functor \( h\text{Fun}(D, E) \times h\text{Fun}(C, D) \to h\text{Fun}(C, E) \), and thus is compatible with natural isomorphism.

It comes with an obvious functor \( \text{QCat} \to h\text{QCat} \). Note that a map \( f: C \to D \) of quasicategories is a categorical equivalence if and only if its image in \( h\text{QCat} \) is an isomorphism.

22.2. **Remark.** We can similarly define a category \( h\text{Cat} \), whose objects are categories and whose morphisms are isomorphism classes of functors. The nerve functor evidently induces a full embedding \( h\text{Cat} \to h\text{QCat} \).

22.3. **Warning.** Although we use the word "homotopy", the definition of \( h\text{QCat} \) given above is not an example of the notion of the homotopy category of a quasicategory defined in §9: \( \text{QCat} \) is a (large) ordinary category, so is isomorphic to its own homotopy category in that sense. Here we are using the equivalence relation on morphisms(\(=\)functors) defined by natural isomorphism.

\(^{10}\) I don’t know if this will actually get proved later. It is proved in [GJ09]
For future reference, we note that $hQCat$ has finite products, which just amount to the usual products of simplicial sets.

22.4. **Proposition.** The terminal simplicial set $\Delta^0$ is a terminal object in $hQCat$. If $C_1, C_2$ are quasicategories, then the projection maps exhibit $C_1 \times C_2$ as a product in $hQCat$.

**Proof.** This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each (6.13), and the fact that $\text{Map}(X, C_1 \times C_2) \xrightarrow{\sim} \text{Map}(X, C_1) \times \text{Map}(X, C_2)$. \[\square\]

22.5. **The 2-out-of-3 property.** A class of morphisms $W$ in a category is said to satisfy the **2-out-of-3 property** if (i) $W$ contains all identity maps, and (ii) whenever two maps in a triple of the form $(f, g, g \circ f)$ are in $W$, so is the third.

22.6. **Example.** In any category, the class of isomorphisms satisfies 2-out-of-3, as does the class of identity maps.

22.7. **Example.** In $\text{Cat}$, the category of small categories, the class of equivalences satisfies 2-out-of-3.

22.8. **Exercise.** Given a functor $f : C \to D$ between categories, let $W$ be the class of maps in $C$ that $f$ takes to isomorphisms in $D$. Show that $W$ satisfies 2-out-of-3.

22.9. **Proposition.** The class $\text{CatEq}$ of categorical equivalences in $sSet$ satisfies 2-out-of-3.

**Proof.** It is immediate that the identity map of a simplicial set is a categorical equivalence. Next, consider the special case of $h = gf$, where $f, g, h$ are functors between quasicategories. In this case, we use the following fact:

- A map $f : C \to D$ between quasicategories is a categorical equivalence if and only if its image $[f]$ in $hQCat$ is an isomorphism.

The proof of this fact amounts to showing that $g$ is a categorical inverse of $f$ exactly if $[g]$ is an inverse of $[f]$ in $hQCat$, which amounts to translating the definition of categorical inverse. Given this, 2-out-of-3 follows using the exercise (22.8) applied to the functor $h : QCat \to \text{Cat}$.

For maps $h = gf$ between arbitrary simplicial sets, we reduce to the above case by considering $\text{Fun}(h, B) = \text{Fun}(f, B) \text{Fun}(g, B)$ where $B$ is an arbitrary quasicategory. \[\square\]

22.10. **Weak categorical equivalence.** Joyal [Joy08a, 1.20] uses a variant of the notion of categorical equivalence, which turns out to be equivalent to what we are using. A map $f : X \to Y$ of simplicial sets is a **weak categorical equivalence** if for every quasicategory $C$, the induced map $h \text{Fun}(Y, C) \to h \text{Fun}(X, C)$ is an equivalence of ordinary categories. Note that, like categorical equivalences, the class of weak categorical equivalences also satisfies 2-out-of-3.

22.11. **Proposition.** A map is a categorical equivalence if and only if it is a weak categorical equivalence.

**Proof.** $(\Rightarrow)$ Straightforward. $(\Leftarrow)$ In the case that $f$ is a weak categorical equivalence between quasicategories, this is exactly what the second half of the proof of (19.5) really shows. For a general map $f$, use factorization to construct a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{v} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\]

\[20\text{This is not to be confused with “weak equivalence”, which we will talk about later (32.1).}\]
so that \( u \) and \( v \) are inner anodyne (and so categorical equivalences), and \( X' \) and \( Y' \) are quasicategories. Applying \( h \Fun(-, C) \) to the square with \( C \) a quasicategory, we see that the vertical maps become equivalences of categories, so if \( f \) is weak categorical equivalence so is \( f' \), which is then a categorical equivalence by what we have already proved, whence \( f \) is a categorical equivalence by 2-out-of-3. □

22.12. **The homotopy 2-category of QCat.** A 2-category \( E \) is a category which is itself “enriched” over \( \text{Cat} \). That is,

- for each pair of objects \( x, y \in \text{ob} E \), there is a category \( \text{Hom}_E(x, y) \), so that
- the objects of \( \text{Hom}_E(x, y) \) are precisely the set \( \text{Hom}_E(x, y) \) of morphisms of \( E \), and
- there are “composition functors” \( \text{Hom}_E(y, z) \times \text{Hom}_E(x, y) \to \text{Hom}_E(x, z) \) for all \( x, y, z \in \text{ob} E \) which on objects is just ordinary composition of morphisms in \( E \), which
  - is unital and associative in the evident sense.

One refers to the objects of \( \text{Hom}_E(x, y) \) as **1-morphisms** \( f : x \to y \) of \( E \), and the morphisms of \( \text{Hom}_E(x, y) \) as **2-morphisms** \( \alpha : f \Rightarrow g \) of \( E \). The **underlying category** of \( E \) consists of the objects and 1-morphisms only.

The standard example of a 2-category is \( \text{Cat} \), the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category \( \text{QCat} \) of quasicategories to a homotopy 2-category \( h_2 \text{QCat} \), so that \( \text{Hom}_{h_2 \text{QCat}}(C, D) := h \Fun(C, D) \).

That is,

- **objects** of \( h_2 \text{QCat} \) are quasicategories,
- **1-morphisms** of \( h_2 \text{QCat} \) are functors between quasicategories,
- **2-morphisms** of \( h_2 \text{QCat} \) are isomorphism classes of natural transformations of functors.

Note that \( \text{QCat} \) sits inside \( h_2 \text{QCat} \) as its underlying category; thus, \( h_2 \text{QCat} \) contains all the information of \( \text{QCat} \). On the other hand \( h \text{QCat} \) is obtained from \( h_2 \text{QCat} \) by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms.

**Part 3. Joins, slices, and Joyal’s extension and lifting theorems**

23. **Joins**

23.1. **Join of categories.** If \( A \) and \( B \) are ordinary categories, we can define a category \( A \star B \) called the join. This has

\[
\text{ob}(A \star B) = \text{ob} A \amalg \text{ob} B, \quad \text{mor}(A \star B) = \text{mor} A \amalg (\text{ob} A \times \text{ob} B) \amalg \text{mor} B,
\]

so that we put in a unique map from each object of \( A \) to each object of \( B \). Explicitly,

\[
\text{Hom}_{A \star B}(x, y) := \begin{cases} 
\text{Hom}_A(x, y) & \text{if } x, y \in \text{ob} A, \\
\text{Hom}_B(x, y) & \text{if } x, y \in \text{ob} B, \\
\{\ast\} & \text{if } x \in \text{ob} A, y \in \text{ob} B, \\
\emptyset & \text{if } x \in \text{ob} B, y \in \text{ob} A,
\end{cases}
\]

with composition defined so that the evident inclusions \( A \to A \star B \leftarrow B \) are functors. (Check that this really defines a category.)

23.2. **Example.** We have that \( [p] \star [q] \approx [p + 1 + q] \).

23.3. **Exercise (Functors to a join of categories).** Show that functors \( f : C \to A \star B \) are in bijective correspondence with triples of functors \( (\pi : C \to [1], f_{\{0\}} : C^{(0)} \to A, f_{\{1\}} : C^{(1)} \to B) \), where
23.4. Exercise (Functors from a join of categories). Show that functors \( f: A \star B \to C \) are in bijective correspondence with triples \( (f_A: A \to C, f_B: B \to C, \gamma: f_A \circ \pi_A \Rightarrow f_B \circ \pi_B) \), where \( f_A \) and \( f_B \) are functors, and \( \gamma \) is a natural transformation of functors \( A \times B \to C \), where \( \pi_A: A \times B \to A \) and \( \pi_B: A \times B \to B \) denote the evident projection functors.

An important special case are the left cone and right cone of a category, defined by \( A^{\sqcap} := [0] \star A \) and \( A^{\sqcup} := A \star [0] \). For instance, the right cone \( A^{\sqcup} \) is the category obtained by adjoining one additional object \( v \) to \( A \), as well as a unique map \( x \to v \) for each object \( x \) of \( A^{\sqcup} \). In this case, \( v \) becomes a terminal object for \( A^{\sqcup} \), and we can say that \( A \mapsto A^{\sqcup} \) freely adjoins a terminal object to \( A \). (Note that a terminal object of \( A \) will not be terminal in \( A^{\sqcup} \) anymore.) Likewise, \( A \mapsto A^{\sqcap} \) freely adjoins an initial object to \( A \).

Limits and colimits of functors can be characterized using cones: if \( F: A \to B \) is a functor, a colimit of \( F \) is a functor \( G: A^{\sqcap} \to B \) which is initial among functors which extend \( F \), and likewise, a limit of \( F \) is a functor \( G': A^{\sqcup} \to B \) which is terminal among functors which extend \( F \).

23.5. Remark. It is worthwhile to spell this out in detail. Given a functor \( F: A \to B \), to describe a functor \( G: A^{\sqcup} \to B \) which extends \( F \), it suffices to give

1. an object \( G(v) \) in \( B \),
2. for each object \( a \in \text{ob} A \) a morphism \( g_a = G(a \to v): F(a) = G(a) \to G(v) \) in \( B \), such that
3. for each morphism \( \alpha: a \to a' \) in \( A \) we have an equality \( g_a F(\alpha) = g_a \) of morphisms \( F(\alpha) \to G(v) \) in \( B \).

Given functors \( G, G': A^{\sqcup} \to B \), we may consider natural transformations \( \phi: G \to G' \) which extend the identity transformation of \( F \). Explicitly, such a transformation \( \phi \) is exactly determined by

1. a morphism \( \phi_v: G(v) \to G'(v) \) in \( B \) such that
2. for each object \( a \in \text{ob} A \) we have an equality \( g'_a = \phi_v g_a \) of morphisms \( F(a) \to G'(v) \) in \( B \), where \( g_a = G(a \to v) \) and \( g'_a = G'(a \to v) \).

Such a \( G \) is a colimit of \( F \) if for every \( G' \) extending \( F \) there exists a unique map \( \phi_v: G(v) \to G'(v) \) in \( B \) such that \( g'_a = \phi_v g_a \) for all \( a \in \text{ob} A \). The object \( G(v) \) is what is colloquially known as “the colimit of \( F \)”, although the full data of a colimit of \( F \) is actually the functor \( G \). We will call the functor \( G \) a colimit cone if what follows.

23.6. Ordered disjoint union. As noted above (23.2), the join operation on categories effectively descends to \( \Delta \). We will call this the ordered disjoint union. It is a functor \( \sqcup: \Delta \times \Delta \to \Delta \), defined so that \( [p] \sqcup [q] := [p + 1 + q] \), to be thought of as the disjoint union of underlying sets, ordered so that the subsets \( [p] \) and \( [q] \) retain their ordering, and elements of \( [p] \) come before elements of \( [q] \).

It is handy to extend this to the category \( \Delta_+ \), the full subcategory of ordered sets obtained by adding the empty set \([-1] := \emptyset \to \Delta \). The functor \( \sqcup \) extends in an evident way to \( \sqcup: \Delta_+ \times \Delta_+ \to \Delta_+ \). This extended functor makes \( \Delta_+ \) into a (nonsymmetric but strict) monoidal category, with unit object \([-1] \).

Note that for any map \( f: [p] \to [q_1] \sqcup [q_2] \) in \( \Delta_+ \), there is a unique decomposition \( [p] = [p_1] \sqcup [p_2] \) such that \( f = f_1 \sqcup f_2 \) for some (necessarily unique) \( f_i: [p_i] \to [q_i] \) in \( \Delta_+ \). (We need an object \([-1] \) to be able to say this, even if \( p, q_1, q_2 \geq 0 \); if \( f([p]) \subseteq [q_1] \) then \( p_2 = -1 \).)

23.7. Join of simplicial sets. Let \( X \) and \( Y \) be simplicial sets. The join of \( X \) and \( Y \) is a simplicial set \( X \star Y \) defined as follows.

The join of simplicial sets \( X \) and \( Y \) is a simplicial set \( X \star Y \) with \( n \)-simplices

\[
(X \star Y)_n := \coprod_{\left\{ n \right\} = \left\{ n_1 \cup \left\{ n_2 \right\} \right\} X_{n_1} \times Y_{n_2},
\]
where we set $X_{-1} = * = Y_{-1}$ to be a one-point set. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: for $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X \ast Y)_n$ and $f: [m] \rightarrow [n]$, we have $(x, y)f = (xf_1, yf_2) \in X_{m_1} \times Y_{m_2} \subseteq (X \ast Y)_m$, where $f = f_1 \sqcup f_2$, $f_j: [m_j] \rightarrow [n]$ is the unique decomposition of $f$ over $[n] = [n_1] \sqcup [n_2]$.

23.8. **Exercise.** Check that the above defines a simplicial set.

In particular,

$$(X \ast Y)_0 = X_0 \amalg Y_0,$$

$$(X \ast Y)_1 = X_1 \amalg X_0 \times Y_0 \amalg Y_1,$$

$$(X \ast Y)_2 = X_2 \amalg X_1 \times Y_0 \amalg X_0 \times Y_1 \amalg Y_2,$$

and so on.

The functor $\ast$ defines a monoidal structure on $sSet$, with unit object $\Delta^{-1} := \emptyset$. Note that $\ast$ is not symmetric monoidal, though it is true that $(Y \ast X)^{\text{op}} \approx X^{\text{op}} \ast Y^{\text{op}}$. We have that

$$\Delta^p \ast \Delta^q \approx \Delta^{p+1+q}.$$

An important example are the cones. The **left cone** and **right cone** of a simplicial set $X$ are

$$X^\llcorner := \Delta^0 \ast X, \quad X^\lrcorner := X \ast \Delta^0.$$

Note that outer horns are examples of cones:

$$(\partial \Delta^n)^\llcorner = \Delta^0 \ast \partial \Delta^n \approx \Lambda^n_{0+1}, \quad (\partial \Delta^n)^\lrcorner = \partial \Delta^n \ast \Delta^0 \approx \Lambda^n_{n+1}.$$ 

23.9. **Exercise.** Let $f: [m] \rightarrow [n]$ be any simplicial operator. Show that the induced map $f: \Delta^m \rightarrow \Delta^n$ is isomorphic to a join of maps $f_0 \ast f_1 \ast \cdots \ast f_n$, of the form $f_j: \Delta^m_j \rightarrow \Delta^0$, where each $m_j \geq -1$.

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets: $N(A \ast B) \approx N(A) \ast N(B)$, and thus $N(A^\llcorner) \approx (NA)^\llcorner$ and $N(A^\lrcorner) \approx (NA)^\lrcorner$. (Exercise: prove this.)

23.10. **The join of quasicategories is a quasicategory.** Here is a handy rule for constructing maps into a join. Note that every join admits a canonical map $\pi: X \ast Y \rightarrow \Delta^0 \ast \Delta^0 \approx \Delta^1$, namely the join applied to the projections $X \rightarrow \Delta^0$ and $Y \rightarrow \Delta^0$.

23.11. **Lemma** ([Joy08a, Prop. 3.5], compare (23.3)). **Maps** $f: K \rightarrow X \ast Y$ are in bijective correspondence with the set of triples

$$(\pi: K \rightarrow \Delta^1, \quad f_{\{0\}}: K^{\{0\}} \rightarrow X, \quad f_{\{1\}}: K^{\{1\}} \rightarrow Y),$$

where $K^{\{j\}} := \pi^{-1}([j]) \subseteq K$, the pullback of $\{j\} \rightarrow \Delta^1$ along $\pi$.

**Proof.** This is a straightforward exercise. In one direction, the correspondence sends $f$ to $(\pi f, f|K^{\{0\}}, f|K^{\{1\}})$, where $\pi: X \ast Y \rightarrow \Delta^0 \ast \Delta^0 = \Delta^1$.

23.12. **Proposition.** If $C$ and $D$ are quasicategories, so is $C \ast D$.

**Proof.** Use the previous lemma, together with the observation (which we leave as an exercise) that for any map $\pi: \Lambda^n_j \rightarrow \Delta^1$ from an inner horn, the preimages $\pi^{-1}([0])$ and $\pi^{-1}([1])$ are either inner horns, standard simplices, or are empty.
24. **Slices**

24.1. **Slices of categories.** Given an ordinary category $C$, and an object $x \in \text{ob} C$, we may form the **slice categories** $C_{/x}$ and $C_{x/}$, also called **undercategory** and **overcategory**, or **slice-over category** and **slice-under category**.

For instance, the slice-over category $C_{/x}$ is the category whose **objects** are maps $f : c \to x$ with target $x$, and whose **morphisms** $(f : c \to x) \to (f' : c' \to x)$ are maps $g : c \to c'$ such that $f' g = f$.

This can be reformulated in terms of joins. Let “$T$” denote the terminal category (isomorphic to $[0]$). Note that $\text{ob} C_{/x}$ corresponds to the set of functors $f : [0] \star T \to C$ such that $f|T = x$, and $\text{mor} C_{/x}$ corresponds to the set of functors $g : [1] \star T \to C$ such that $g|T = x$.

More generally, given a functor $p : A \to C$ of categories, we obtain slice categories $C_{/p}$ and $C_{{p/}}$ defined as follows. The category $C_{/p}$ has

- **objects**: functors $f : [0] \star A \to C$ such that $f|A = p$,
- **morphisms** $f \to f'$: functors $g : [1] \star A \to C$ such that $g|A = p$.

Likewise, the category $C_{{p/}}$ has

- **objects**: functors $f : A \star [0] \to C$ such that $f|A = p$,
- **morphisms** $f \to f'$: functors $g : A \star [1] \to C$ such that $g|A = p$.

24.2. **Exercise.** Describe composition of morphisms in $C_{/p}$ and $C_{{p/}}$.

24.3. **Exercise.** Show that $(C_{/p})^{\text{op}} \approx (C^{\text{op}})_{{p/}}$ (isomorphism of categories).

24.4. **Exercise.** Fix a functor $p : A \to C$, and let $B$ be a category. Construct bijections

$$\{\text{functors } f : B \to C_{/p}\} \leftrightarrow \{\text{functors } g : B \star A \to C \text{ s.t. } g|A = p\}$$

and

$$\{\text{functors } f : B \to C_{{p/}}\} \leftrightarrow \{\text{functors } g : A \star B \to C \text{ s.t. } g|A = p\}.$$ 

24.5. **Remark.** The notions of limits and colimits can be formulated very compactly in terms of the general notion of slices. Thus, given a functor $p : A \to C$, a **colimit** of $p$ is the same data as an initial object of $C_{/p}$, while a **limit** of $p$ is the same data as a terminal object of $C_{/p}$. (**Exercise:** prove this; this will be the starting case for formulating notions of limits and colimits for quasicategories. Compare (23.5).)

24.6. **Joins and colimits of simplicial sets.** The join functor $\star : s\text{Set} \times s\text{Set} \to s\text{Set}$ is in some ways analogous to the product functor $\times$, e.g., it is a monoidal functor.

The product operation $(-) \times (-)$ on simplicial sets commutes with colimits in each input, and the functors $X \times -$ and $- \times X$ admit right adjoints (in both cases, the right adjoint is $\text{Map}(X, -)$). The join functor does not commute with colimits in each variable, but **almost** does so; the only obstruction is the value on the initial object.

More precisely, the functors $X \star -$ and $- \star X : s\text{Set} \to s\text{Set}$ do not preserve the initial object, since $X \star \emptyset \approx X \approx \emptyset \star X$. However, (the identity map of) $X$ is tautologically the initial object of $s\text{Set}_{X/}$.

24.7. **Proposition.** For every simplicial set $X$, the induced functors

$$X \star -, - \star X : s\text{Set} \to s\text{Set}_{X/}$$

**preserve colimits.**

**Proof.** This is immediate from the degreewise formula for the join:

$$(X \star Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg (X_0 \times Y_{n-1}) \amalg Y_n = X_n \amalg (\text{terms which are “linear” in } Y).$$

24.8. **Exercise** (Trivial, but important). Show that the functors $X \star -$ and $- \star X : s\text{Set} \to s\text{Set}$ preserve pushouts.
24.9. Slices of simplicial sets. We have seen that the functors
\[ S \star - : sSet \to sSet_{S/} \quad \text{and} \quad - \star T : sSet \to sSet_{T/} \]
preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called slice functors, denoted
\[ (p : S \to X) \mapsto X/p : sSet_{S/} \to sSet \]
and
\[ (q : S \to X) \mapsto X/q : sSet_{S/} \to sSet. \]
I will sometimes distinguish these as slice-under and slice-over, respectively. Explicitly, there are bijections
\[
\begin{align*}
\left\{ \begin{array}{c} \downarrow S \rightarrow X \\ S \star K \downarrow \end{array} \right\} & \iff \{ K \to X/p \}, \\
\left\{ \begin{array}{c} \downarrow T \rightarrow X \\ K \star T \downarrow \end{array} \right\} & \iff \{ K \to X/q \}.
\end{align*}
\]
(24.10)
Here we write “\( S \to S \star K \)” and “\( T \to K \star T \)” for the inclusions \( S \star \emptyset \subseteq S \star K \) and \( \emptyset \star T \subseteq K \star T \), using the canonical isomorphisms \( S \star \emptyset = S \) and \( \emptyset \star T = T \).

Taking \( K = \Delta^n \) we obtain the formulas
\[
(X/p)_n = \text{Hom}_{sSet_{S/}}(S \star \Delta^n, X), \quad (X/q)_n = \text{Hom}_{sSet_{T/}}(\Delta^n \star T, X),
\]
which we regard as the definition of slices. (I.e., these formulas specify the \( n \)-simplices of the slices, and naturality in “\( \Delta^n \)” specifies the action of simplicial operators.)

24.11. Exercise. Given this explicit definition of slices in terms of their simplices and the action of simplicial operators, verify the bijective correspondences (24.10).

In particular, we note the special cases associated to \( x : \Delta^0 \to X \):
\[
(X/x)_n = \text{Hom}_{sSet_{\Delta^0/}}(\Delta^0 \star \Delta^n, X) \approx \text{Hom}_{sSet_*}(\Delta^{n+1}, (X, x)),
\]
\[
(X/x)_n = \text{Hom}_{sSet_{\Delta^0/}}(\Delta^n \star \Delta^0, X) \approx \text{Hom}_{sSet_*}(\Delta^{n+1}, (X, x)).
\]
The notation \((X, x)\) with \( x \in X_0 \) represents a pointed simplicial set, the category of which is \( sSet_* := sSet_{\Delta^0/} \). The associated adjuncts are
\[
\text{Hom}_{sSet_*}(K^\lt, (X, x)) \approx \text{Hom}_{sSet}(K, X/x), \quad \text{Hom}_{sSet_*}(K^\gt, (X, x)) \approx \text{Hom}_{sSet}(K, X/x).
\]
The slice construction for simplicial sets agrees with that for categories.

24.12. Proposition. The nerve preserves slices; i.e., if \( p : A \to C \) is a functor, then \( N(C/p) \approx (NC)_{Np/} \) and \( N(C/p) \approx (NC)_{NP} \).

Proof. Straightforward. \square

24.13. Slice as a functor. The function complex \( \text{Map}(-, -) \) is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables, though it is a little more complicated. A more precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

\[
\begin{array}{ccc}
S \xrightarrow{p} X & \xrightarrow{f} & X/p \xrightarrow{} Y/f_p \\
\downarrow j & & \downarrow & \downarrow \\
T \xrightarrow{f_{pj}} Y & & X/pj \xrightarrow{} Y/f_{pj}
\end{array}
\]

There seems to be no decent notation for the maps in the right-hand squares. The whole business of joins and slices can get pretty confusing because of this.
24.14. Remark. A very precise formulation is that each kind of slice defines a functor $\text{sSet}^{\text{tw}} \to \text{sSet}$ from the twisted arrow category of simplicial sets, whose objects are maps $p$ of simplicial sets, and whose morphisms are pairs $(j, f): p \to fpj$, where $j$ and $f$ are themselves maps of simplicial sets.

Let’s spell this out in terms of the correspondence between “maps into slices” and “maps from joins”. Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider “restriction map” $X_{p/} \to Y_{fpj/}$. The composite of a map $u: K \to X_{p/}$ with this restriction map is described in terms of the bijection of (24.10) as follows. The map $u$ corresponds to a dotted arrow in

$$
\begin{array}{c}
T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y \\
\downarrow \hspace{1cm} \downarrow \quad \downarrow \\
T \star K \xrightarrow{j \star K} S \star K
\end{array}
$$

The composite $K \xrightarrow{\tilde{u}} X_{p/} \to Y_{fpj/}$ corresponds to $f \tilde{u}(j \star K)$.

A particular special case which we will see a lot of are the “restriction” functors $X_{/p} \to X$ and $X_{p/} \to X$ induced by sequence $\emptyset \xrightarrow{j} S \xrightarrow{p} X$, using that $X_{/\emptyset} = X = X_{\emptyset/}$. For instance, $X_{/p} \to X$ sends an $n$-simplex $x \in (X_{/p})_n$ corresponding to $\tilde{x}: \Delta^n \star S \to X$ extending $p$ to the $n$-simplex of $X$ represented by the map $\tilde{x}|(\Delta^n \star \emptyset) = x$.

25. Initial and terminal objects

We now show that for a vertex $x$ in a quasicategory $C$, the slice objects $C_{/x}$ and $C_{x/}$ are also quasicategories. Using this, we can make a definition of initial and terminal object in a quasicategory.

25.1. Left and right horns, fibrations, and anodyne maps. We recall the sets left horns $L\text{Horn} := \{ \Lambda^n_k \subset \Delta^n \mid 0 \leq k < n, n \geq 1 \} = \text{InnHorn} \cup \{ \Lambda^n_0 \subset \Delta^n \mid n \geq 1 \}$

and the right horns $R\text{Horn} := \{ \Lambda^n_k \subset \Delta^n \mid 0 < k \leq n, n \geq 1 \} = \text{InnHorn} \cup \{ \Lambda^n_n \subset \Delta^n \mid n \geq 1 \}$.

The associated weak saturations $L\text{Horn}$ and $R\text{Horn}$ are the left anodyne and right anodyne maps. The associated right complements $L\text{Fib} := L\text{Horn}^\square$, $R\text{Fib} := R\text{Horn}^\square$

are the left fibrations and right fibrations. Note that $\text{InnHorn} \subseteq L\text{Horn} \cap R\text{Horn}$ and $L\text{Fib} \cup R\text{Fib} \subseteq \text{InnFib}$.

These classes correspond to each other under the opposite involution $(-)^{\text{op}}: \text{sSet} \to \text{sSet}$; i.e., $L\text{Horn}^{\text{op}} = R\text{Horn}$, $L\text{Fib}^{\text{op}} = R\text{Fib}$.

25.2. Proposition. Let $C$ be a quasicategory and $x \in C_0$. The evident maps $C_{x/} \to C$ and $C_{/x} \to C$ which “forget $x$” are left fibration and right fibration respectively. In particular, $C_{x/}$ and $C_{/x}$ are also quasicategories.

Proof. I claim that $\pi: C_{/x} \to C$ is a right fibration. Explicitly, this map sends the $n$-simplex $a: \Delta^n \to C_{/x}$, which corresponds to $\tilde{a}: \Delta^n \star \Delta^0 \to C$ such that $\tilde{a}|(\emptyset \star \Delta^0) = x$, to the $n$-simplex
\( \partial(\Delta^n \star \emptyset) \to C \). Using the join/slice adjunction, there is a bijective correspondence between lifting problems

\[
\begin{array}{ccc}
\Lambda^n_j \ar[r]^f \ar[d] & C_x \ar[d] & x \\
\Delta^n \ar[r]_g \ar[ru] & C \ar[u] \end{array}
\]

\[ \iff \]

\[
\emptyset \star \Delta^0 \to (\Lambda^n_j \star \Delta^0) \cup \Lambda^n \star \emptyset \to (f,g) \to C
\]

The right hand diagram is isomorphic to

\[
\begin{array}{ccc}
\{n+1\} \ar[r] \ar[d] & \Lambda^{n+1}_j \ar[r] \ar[ru] & C \\
\Delta^{n+1} \ar[ru] & &
\end{array}
\]

If \( C \) is a quasicategory, then an extension exists for \( 0 < j \leq n \).

Since right fibrations are inner fibrations, the composite \( C_x \to C \to \ast \) is an inner fibration, and thus \( C_x \) is a quasicategory.

The case of \( C_x/ \to C \) is similar, using the correspondence

\[
\begin{array}{ccc}
\Lambda^n_j \ar[r] \ar[d] & C_x/ \ar[d] & x \\
\Delta^n \ar[r] \ar[ru] & C \ar[u] \end{array}
\]

\[ \iff \]

\[
\{0\} \ar[r] \ar[d] & \Lambda^{n+1}_{j+1} \ar[r] \ar[ru] & C \\
\Delta^{n+1} \ar[ru] & &
\]

### 25.3. Initial and terminal objects

An initial object of a quasicategory \( C \) is an \( x \in C_0 \) such that every \( f: \partial \Delta^n \to C \) (for all \( n \geq 1 \)) such that \( f|\{0\} = x \), there exists an extension \( f': \Delta^n \to C \).

A terminal object of \( C \) is an initial object of \( C^{\text{op}} \). That is, a \( y \in C_0 \) such that every \( f: \partial \Delta^n \to C \) with \( f|\{n\} = y \) extends to \( \Delta^n \).

\[
\begin{array}{ccc}
\{0\} \ar[r] \ar[d] & \partial \Delta^n \ar[r] \ar[ru] & C \\
\Delta^n \ar[ru] & &
\end{array}
\]

\[
\begin{array}{ccc}
\{n\} \ar[r] \ar[d] & \partial \Delta^n \ar[r] \ar[ru] & C \\
\Delta^n \ar[ru] & &
\end{array}
\]

If \( C \) is the nerve of a category, then these definitions turn out to coincide with the usual notion of initial and terminal objects in a category. For instance, consider the definition of initial object applied to \( x \in C_0 \).

- The condition for \( n = 1 \) says that for every object \( y \) in \( C \) there exists \( f: x \to y \),
- The condition for \( n = 2 \) says that for every triple of maps \( f: x \to y, g: y \to z \), and \( h: x \to z \), we must have \( h = gf \). In particular (taking \( f = 1_x \)), we see there is at most one map from \( x \) to any object.
- The conditions for \( n \geq 3 \) are trivially satisfied.

We can characterize the property of being initial or terminal in terms of the “forgetful” functor from the slice.

### 25.4. Proposition

If \( C \) is a quasicategory, then \( x \in C_0 \) is initial if and only if \( C_x/ \to C \) is a trivial fibration, and terminal if and only if \( C_x/ \to C \) is a trivial fibration.
Proof. Left as an exercise using the join/slice adjunction. □

25.5. Remark. This implies that if \( x \) is initial, then \( C_{x/} \to C \) is a categorical equivalence. Later (\ref{??}) we’ll be able to show the converse: if \( C_{x/} \to C \) is a categorical equivalence, then \( x \) is initial.

A crucial fact about initial and terminal objects in an ordinary category is that they are unique up to unique isomorphism. One way to formulate this is as follows: given a category \( C \), let \( C^{\text{init}} \subseteq C \) be the full subcategory spanned by the initial objects. Then one of two cases applies: either there are no initial objects, so \( C^{\text{init}} \) is empty, or there is at least one initial object, and \( C^{\text{init}} \) is equivalent to the terminal category \([0]\).

This leads to an analogous formulation for quasicategories.

25.6. Proposition. Let \( C \) be a quasicategory. Let \( C^{\text{init}} \) and \( C^{\text{term}} \) denote respectively the full subcategory spanned by initial objects and terminal objects. Then (i) either \( C^{\text{init}} \) is empty or is categorically equivalent to the terminal quasicategory \( \Delta^0 \), and (ii) either \( C^{\text{term}} \) is empty or is categorically equivalent to the terminal quasicategory \( \Delta^0 \).

Proof. Since \( C^{\text{term}} = ((C^{\text{op}})^{\text{init}})^{\text{op}} \), we just need to consider the case of initial objects. By definition of initial object, any \( f: \partial \Delta^n \to C^{\text{init}} \) with \( n \geq 1 \) can be extended to \( g: \Delta^n \to C \), and the image of \( g \) must lie in the full subcategory \( C^{\text{init}} \) since all of its vertices do. If \( C^{\text{init}} \neq \emptyset \), then this extension condition also holds for \( n = 0 \), whence \( C \to \Delta^0 \) is a trivial fibration, and thus \( C \) is categorically equivalent to \( \Delta^0 \) by (20.1). □

There are some seemingly obvious facts about initial objects that we can’t prove just yet.

D. Deferred Proposition.

(1) Given an object \( x \) in a quasicategory \( C \), the object \( i_x \) in the slice \( C_{x/} \) corresponding to the edge \( 1_x \in C_1 \) is an initial object of \( C_{x/} \).

(2) Any object in a quasicategory isomorphic to an initial object is also an initial object.

Proofs will be given in (??).

25.7. Initial and terminal objects in functor categories. Here is a sample of a property of initial/terminal objects that we can now prove. A functor between ordinary categories whose values are all initial (or terminal) objects is itself initial (or terminal) as an object of the functor category. The same holds with categories replaced by quasicategories.

25.8. Proposition. Consider a map \( f: X \to C \) from a simplicial set to a quasicategory. Suppose that for every vertex \( x \in X_0 \) the object \( f(x) \in C_0 \) is initial (resp. terminal) in \( C \). Then the functor \( f \) is initial (resp. terminal) viewed as an object of \( \text{Fun}(X,C) \).

As a consequence, if \( C \) has an initial (or terminal) object \( c_0 \), then \( \text{Fun}(X,C) \) also has an initial (or terminal) object, e.g., the functor which is the composite of \( X \to \{c_0\} \to C \).

Proof. Assume \( f(x) \in C_0 \) is initial in \( C \) for all \( x \in X_0 \). Suppose given \( g: \partial \Delta^n \to \text{Fun}(X,C) \) with \( n \geq 1 \) and \( g\{0\} = f \). We want to show that there exists an extension \( g': \Delta^n \to \text{Fun}(X,C) \) of \( g \).

We convert this to the adjoint lifting problem:

\[
\begin{array}{ccc}
\{0\} \times X & \xrightarrow{f} & \partial \Delta^n \times X \\
\downarrow & & \downarrow g \\
\Delta^n \times X & \xrightarrow{g'} & C \\
\end{array}
\]

The strategy is to construct the extension by inductively constructing extensions \( \tilde{g}_k: F_k \to C \) where \( F_k = (\partial \Delta^n \times X) \cup \text{Sk}_{k}(\Delta^n \times X) \), \( k \geq 0 \) is the skeletal filtration (15.18) of the inclusion.
\(\partial \Delta^n \times X \to \Delta^n \times X\). Note that any \(k\)-simplex \(h = (a, b): \Delta^k \to \Delta^n \times X\) which is not contained in the subcomplex \(\partial \Delta^n \times X\) must necessarily be such that \(a: \Delta^k \to \Delta^n\) is surjective, whence \(a\) takes the vertex \(0 \in (\Delta^k)_0\) to \(0 \in (\Delta^n)_0\), and therefore \(g h(0) = \tilde g(0, b(0)) = f(b(0))\) is an initial object in \(C\) by hypothesis. Thus we can lift in

\[
\begin{array}{ccc}
\prod \partial \Delta^k(h|\partial \Delta^k) & \to & F_{k-1} \\
\downarrow & & \downarrow \tilde g_{k-1} \\
\prod \Delta^k(h) & \to & F_k
\end{array}
\]

since the square is a pushout and for each \(h: \Delta^k \to F_k\) we have that \((\tilde g_{k-1} h)|\partial \Delta^k\) sends the vertex 0 to an initial object of \(C\), so an extension of \((\tilde g_{k-1} h)|\partial \Delta^k\) to \(\Delta^k\) always exists. \(\square\)

26. Joins and slices in lifting problems

Recall that for an object \(x\) in a quasicategory \(C\), the slice objects \(C/x\) and \(C/x\) are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

26.1. Proposition. Let \(p: S \to C\) be a map of simplicial sets, and suppose \(C\) is a quasicategory. Then both \(C/p\) and \(C/p\) are quasicategories.

The proof is just like that of (25.2): we will show below (26.14) that \(C/p\) is a left fibration and \(C/p \to C\) is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

26.2. Pushout-joins. We define an analogue of the pushout-product for the join. Given maps \(i: A \to B\) and \(j: K \to L\) of simplicial sets, the pushout-join (or box-join) \(i \boxplus j\) is the map

\[
i \boxplus j: (A \star L) \amalg_{A \star K} (B \star K) \to B \star L.
\]

Warning: unlike the pushout-product, the pushout-join is not symmetric, since the join is not symmetric: \(i \boxplus j \neq j \boxplus i\).

26.3. Example. We have already observed examples of pushout-joins in the proof of (25.2), namely

\[
(\Lambda^n_j \subset \Delta^n) \amalg (\varnothing \subset \Delta^0) \approx (\Lambda^{n+1}_j \subset \Delta^{n+1}), \quad (\varnothing \subset \Delta^0) \amalg (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{1+n}_j \subset \Delta^{1+n}).
\]

A straightforward calculation shows that the pushout-join of a horn with a cell is always a horn:

\[
(\Lambda^n_j \subset \Delta^n) \amalg (\partial \Delta^k \subset \Delta^k) \approx (\Lambda^{n+1+k}_j \subset \Delta^{n+1+k}),
\]

\[
(\partial \Delta^k \subset \Delta^k) \amalg (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{k+1+n}_j \subset \Delta^{k+1+n}).
\]

Also, the pushout-join of a cell with a cell is always a cell:

\[
(\partial \Delta^n \subset \Delta^n) \amalg (\partial \Delta^k \subset \Delta^k) \approx (\partial \Delta^{n+1+k} \subset \Delta^{n+1+k}).
\]

26.4. Exercise. Prove the isomorphisms asserted in (26.3).

26.5. Remark. Both pushout-product and pushout-join are special cases of a general construction: given any functor \(F: sSet \times sSet \to sSet\) of two variables, you get a corresponding “pushout-F” functor:

\[
F: \text{Fun}([1], sSet) \times \text{Fun}([1], sSet) \to \text{Fun}([1], sSet).
\]

We will meet more examples later.
26.6. **Pullback-slices.** Just as the pushout-product is associated to the pullback-power, so the pushout-join is associated to two kinds of pullback-slices (or box-slices). Given maps \( K \xrightarrow{i} L \xrightarrow{q} X \xrightarrow{h} Y \), we define the map
\[
h^\oplus_{qj} : X_q X_{qj} Y_{h'/q}.
\]
where the components \( X_q X_{qj} \) and \( X_q Y_{h'/q} \) are the evident maps constructed by “restricting” along \( p : K \to L \) in the first case, and “composing” with \( h : X \to Y \) in the second case.

Similarly, given maps \( A \xrightarrow{i} B \xrightarrow{p} X \xrightarrow{h} Y \) we have
\[
h^\oplus_{ip} : X_{pij} X_{pij} Y_{hpj}.
\]

26.7. **Remark.** When \( Y = \ast \), these pullback-slice maps are just the restriction maps \( X_q \to X_{qj} \) and \( X_{pi} \to X_{pij} \).

26.8. **Remark.** Both pullback-hom and pullback-slices are special cases of a general construction: given any functor \( F : sSet^{tw} \to sSet \) from the twisted arrow category (24.14), you get a corresponding “pullback-F” functor \( F^\ominus : sSet^{tw} \to sSet \). In the case of pullback-hom, the \( F \) in question is a composite functor \( sSet^{tw} \xrightarrow{\text{Map}} sSet^{op} \times sSet \xrightarrow{\text{Map}} sSet \).

26.9. **Joins, slices, and lifting problems.** The pushout-join and pullback-slice interact with lifting problems in much the same way that pushout-product and pullback-power do.

26.10. **Proposition.** Given \( i : A \to B \), \( j : K \to L \), and \( h : X \to Y \), the following are equivalent.

1. \( (i \sqcup j) \sqsubseteq h \).
2. \( i \sqsubseteq (h^\oplus_{qj}) \) for all \( q : L \to X \).
3. \( j \sqsubseteq (h^\oplus_{ip}) \) for all \( p : B \to X \).

**Proof.** A straightforward exercise. The equivalence of (1) and (2) is
\[
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{i} & X_q \\
\downarrow & & \downarrow \ h^\oplus_{qj} \\
B & \xrightarrow{j} & X_{qj} \times_{Y_{h'/q}} Y_{h'q} & \leq \rightleftharpoons & (A * L) \cup_{A*K} (B*K) \xrightarrow{q} X \\
\end{array}
\end{array}
\]

26.11. **Proposition.** Let \( S \) and \( T \) be sets of maps in \( sSet \). Then \( S \boxdot T \subseteq S \boxdot T \).

**Proof.** This is formal and nearly identical to the proof of the weak saturation result for box-products (16.8).

26.12. **Proposition.** We have
\[
\text{Cell} \boxdot \text{Cell} \subseteq \text{Cell}, \quad \text{RHorn} \boxdot \text{Cell} \subseteq \text{InnHorn}, \quad \text{and} \quad \text{Cell} \boxdot \text{LHorn} \subseteq \text{InnHorn}.
\]

**Proof.** Immediate from (26.3) and (26.11).

26.13. **Corollary.** Given \( T \xrightarrow{i} S \xrightarrow{p} X \xrightarrow{f} Y \), consider the pullback-slice maps
\[
\ell : X_{pi} \to X_{pi} \times_{Y_{fpj}} Y_{fp}, \quad r : X_{pi} \to X_{pi} \times_{Y_{fpj}} Y_{fp}.
\]

We have the following.

1. \( j \in \text{Cell}, f \in \text{TrivFib} \) implies \( \ell, r \in \text{TrivFib} \).
2. \( j \in \text{Cell}, f \in \text{InnFib} \) implies \( \ell \in \text{L Fib}, r \in \text{RFib} \).
3. \( j \in \text{RHorn}, f \in \text{InnFib} \) implies \( \ell \in \text{TrivFib} \).
4. \( j \in \text{LHorn}, f \in \text{InnFib} \) implies \( r \in \text{TrivFib} \).

We are mostly interested in special cases when \( X = C \) is a quasicategory, and \( Y = * \).

26.14. **Corollary.** Given \( T \xrightarrow{j} S \xrightarrow{p} C \) with \( C \) a quasicategory and \( j \) a monomorphism, the induced map \( C/p \to C/pj \) is a right fibration, and \( C/p \to C/pj \) is a left fibration. In particular, \( C/p \to C \) is a right fibration and \( C/p \to C \) is a left fibration (case \( T = \emptyset \)).

26.15. **Corollary.** Given \( T \xrightarrow{j} S \xrightarrow{p} C \) with \( C \) a quasicategory, if \( j \) is right anodyne then \( C/p \to C/pj \) is a trivial fibration, while if \( j \) is left anodyne then \( C/p \to C/pj \) is a trivial fibration.

Another case we will need is when \( T = \emptyset \).

26.16. **Corollary.** Given \( S \xrightarrow{p} X \xrightarrow{f} Y \) where \( f \) is a trivial fibration, all of the maps in
\[
X_p/ \to X \times Y Y_{fp}/ \to Y_{fp} \quad \text{and} \quad X_p/ \to X \times Y Y_{fp}/ \to Y_{fp}
\]
are trivial fibrations.

**Proof.** The two pullback-slice maps are trivial fibrations by (26.13). The projections are each base changes of the trivial fibration \( f \), and so are trivial fibrations.

26.17. **Composition functors for slices.** Here is a nice consequence of the above. Let \( C \) be a quasicategory and \( f : x \to y \) a morphism in it; we represent \( f \) by a map \( \Delta^1 \to C \) of simplicial sets, which we also call \( f \). We obtain two functors
\[
C/x/ \leftarrow C/f/ \to C/y
\]
associated to the inclusions \( \{0\} \subset \Delta^1 \supset \{1\} \). The first inclusion \( \{0\} \subset \Delta^1 \) is a left-horn inclusion, and thus by (26.15) the restriction map \( p \) is a trivial fibration, and hence we can choose a section \( s : C/x/ \to C/f/ \) of \( p \).

The resulting composite \( qs : C/x/ \to C/y \) can be thought of as a functor realizing the operation which sends an object \( (c \xrightarrow{g} x) \) of \( C/x/ \) to “the object” \( (c \xrightarrow{fg} y) \) of \( C/y \) defined by “composing \( f \) and \( g \)” (but remember that such composition is not uniquely defined in a quasicategory \( C \); the choice of section \( s \) gives a collection of such choices for all \( g \).)

26.18. **Exercise.** Show that if \( C \) is a category, then \( p \) is an isomorphism, and that \( qs \) is precisely the functor \( C/x/ \to C/y \) described above.

27. Limits and colimits in quasicategories

27.1. **Definition of limits and colimits.** Now we can define the notion of a limit and colimit of a functor between quasicategories (and in fact of a map from a simplicial set to a quasicategory). Given a map \( p : K \to C \) where \( C \) is a quasicategory, a **colimit** of \( p \) is defined to be an initial object of the slice quasicategory \( C/p/ \). Explicitly, a colimit of \( p : K \to C \) is a map \( \tilde{p} : K \star \Delta^0 = K^\triangleright \to C \) extending \( p \), such that for \( n \geq 1 \) a lift exists in every diagram of the form
\[
\begin{array}{ccc}
K \star \{0\} & \xrightarrow{p} & K \star \partial \Delta^n \\
\downarrow & & \downarrow \\
K \star \Delta^n & \xrightarrow{\tilde{p}} & C
\end{array}
\]
Sometimes it is better to call \( \tilde{p} \) a **colimit cone**, in which case the restriction \( \tilde{p}|\emptyset \star \Delta^0 \) to the cone point is an object in \( C \) which can be called a “colimit of \( p \).”
Similarly, a limit of \( p \) is a terminal object of \( C/p \); explicitly, this is a map \( \tilde{p}: \Delta^0 \ast K = K^{\triangleleft} \to C \) extending \( p \) such that for \( n \geq 1 \) a lift exists in every

\[
\begin{array}{ccc}
\Delta^0 \ast K & \xrightarrow{\partial \Delta^n \ast K} & \Delta^n \ast K \\
\tilde{p} \downarrow & & \downarrow \\
\{n\} \ast K & \xrightarrow{\partial \Delta^n \ast K} & C
\end{array}
\]

27.2. Example. Consider the empty simplicial set \( K = \emptyset \). Then \( C_{/\emptyset} = C \), so a colimit of \( p: \emptyset \to C \) is precisely the same as an initial object of \( C \). Likewise, a limit of \( p \) is precisely the same as a terminal object of \( C \).

27.3. Example. Consider \( K = \Lambda_2^0 \), which is the nerve of a category which we can draw as the picture \((1 \leftarrow 0 \to 2)\). Then \( (\Lambda_2^0)^{\triangleleft} \approx \Delta^0 \times \Delta^1 \) is also an ordinary category; explicitly it has the form of a commutative square

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & 1 \\
\downarrow & & \downarrow \nu \\
2 & \xrightarrow{1} & \nu
\end{array}
\]

where \( \nu \) is the “cone vertex”. A colimit cone \( (\Lambda_2^0)^{\triangleleft} \to C \) is called a pushout diagram in \( C \).

Similar considerations give \( (\Lambda_2^0)^{\triangleright} \approx \Delta^1 \times \Delta^1 \); a limit cone \( (\Lambda_2^0)^{\triangleright} \to C \) is called a pullback diagram in \( C \).

27.4. Exercise. Let \( C' \subseteq C \) be an inclusion of a full subcategory. Show that if \( p: K \to C' \) has a colimit \( \tilde{p} \) in \( C \), and if the image of \( \tilde{p} \) is contained in \( C' \), then \( \tilde{p} \) is in fact a colimit of \( p \) in \( C' \).

27.5. Uniqueness of limits and colimits. Limits and colimits are unique if they exist.

27.6. Proposition. Let \( p: K \to C \) be a map to a quasicategory, and let \((C/p)^{\text{colim}} \subseteq C/p \) and \((C/p)^{\text{lim}} \subseteq C/p \) denote the full subcategories spanned by colimit cones and limit cones respectively. Then (i) either \((C/p)^{\text{colim}} \) is empty or is categorically equivalent to \( \Delta^0 \), and (ii) either \((C/p)^{\text{lim}} \) is empty or is categorically equivalent to \( \Delta^0 \).

Proof. This is just the uniqueness of initial and terminal objects (25.6).

We have noted above (25.4) that an object \( x \) in a quasicategory \( C \) is initial iff \( C_{/x} \to C \) is a trivial fibration, and terminal iff \( C_{/x} \to C \) is a trivial fibration. There is a similar characterization of limit and colimit cones.

27.7. Proposition. Let \( C \) be a quasicategory. Let \( \tilde{p}: K^{\triangleright} \to C \) be a map, and write \( p := \tilde{p}|K \). Then \( \tilde{p} \) is a colimit diagram if and only if \( C_{/p} \to C_{/p}^{\triangleright} \) is a trivial fibration.

Likewise, let \( \tilde{q}: K^{\triangleleft} \to C \) be a map, and write \( q := \tilde{q}|K \). Then \( \tilde{q} \) is a limit diagram if and only if \( C_{/q} \to C_{/q}^{\triangleleft} \) is a trivial fibration.

Proof. I’ll just do the case of colimits.

We make an elementary observation about iterated slices (see (27.8) below). There is an isomorphism \((C/p)^{\triangleright} \approx C_{/p}^{\triangleright}\), where the symbol \( \triangleright \) refers to both a morphism \( \tilde{p}: K^{\triangleright} \to C_{/p} \) (on the right-hand side of the isomorphism) and the corresponding object \( \tilde{p} \in (C/p)^0 \) (on the left-hand side of the isomorphism). The point is that in either simplicial set, a \( k \)-simplex corresponds to a map \( K \ast \Delta^0 \ast \Delta^k \to C \) which restricts to \( \tilde{p} \) on \( K \ast \Delta^0 \ast \emptyset \).

Using this, the statement amounts to the special case for initial and terminal objects (25.4).
27.8. **Exercise (Iterated slices).** Let \( f : A \star B \to C \) be a map of simplicial sets. Describe isomorphisms

\[
C_{f/} \approx (C_{f_A/})_{f_B/}, \quad C_{/f} \approx (C_{/f_B})_{f_A/},
\]

where \( f_A : A \to C \) and \( f_B : B \to C \) are the evident restrictions of \( f \) to subcomplexes, and \( \widetilde{f}_A : A \to C_{/f_B} \) and \( \widetilde{f}_B : B \to C_{f_A/} \) are the adjoints to \( f \).

27.9. **Limits and colimits in slices.** Given a map \( p : S \to C \) to a quasicategory, we have “forgetful functors” \( \pi : C_{/p} \to C \) and \( \pi : C_{p/} \to C \) from the slices to \( C \).

The following proposition says that an initial object of \( C \) implies a compatible initial object of \( C_{/p} \), and a terminal object of \( C \) implies a compatible terminal object of \( C_{p/} \).

27.10. **Proposition.** Let \( p : S \to C \) be a map from a simplicial set to a quasicategory.

1. Suppose \( C \) has an initial object. Then
   a. \( C_{/p} \) has an initial object, and
   b. if \( x \in (C_{/p})_0 \) is an object such that \( \pi(x) \in C_0 \) is initial in \( C \), then \( x \) is initial in \( C_{/p} \).

2. Suppose \( C \) has a terminal object. Then
   a. \( C_{p/} \) has an terminal object, and
   b. if \( x \in (C_{p/})_0 \) is an object such that \( \pi(x) \in C_0 \) is terminal in \( C \), then \( x \) is terminal in \( C_{p/} \).

**Proof.** (See [Lur09, 1.2.13.8].) I’ll only prove (1), as the proof of (2) is analogous.

I prove (1b) first. Let \( x \in (C_{/p})_0 \) and \( y = \pi(x) \in C_0 \); we need to show that if \( y \) is initial then so is \( x \). To show that \( x \) is initial we must show that a lift in any diagram of the form

\[
\begin{array}{ccc}
\Delta^0 \star \emptyset & \longrightarrow & (\Delta^0 \star \partial \Delta^n) \cup (\emptyset \star \Delta^n) \\
\downarrow & & \downarrow \\
\Delta^0 \star \Delta^n & \longrightarrow & C_{/p}
\end{array}
\]

for \( n \geq 0 \), using the identification \( (\Delta^0 \star \partial \Delta^n) \cup (\emptyset \star \Delta^n) \approx \partial \Delta^{n+1} \). This lifting problem is equivalent to one of the form

\[
\begin{array}{ccc}
\Delta^0 \star \emptyset \star S & \longrightarrow & (\Delta^0 \star \partial \Delta^n \star S) \cup (\emptyset \star \Delta^n \star S) \\
\downarrow & & \downarrow \\
\Delta^0 \star \Delta^n \star S & \longrightarrow & C
\end{array}
\]

which in turn is equivalent to one of the form

\[
\begin{array}{ccc}
S & \longrightarrow & \partial \Delta^n \star S \\
\downarrow & & \downarrow q \\
\Delta^n \star S & \longrightarrow & C
\end{array}
\]

(Here the maps marked \( x, x', x'' \) are all adjoints of each other.) Since \( y \) is initial, \( q \) is a trivial fibration (25.4), and therefore a lift exists since \( \partial \Delta^n \star S \to \Delta^n \star S \) is a monomorphism. We conclude that \( x \) is initial if \( y \) is.
Next we prove (1a). Suppose \( y \in C_0 \) is an initial object. This implies \( q: C_{y/} \to C \) is a trivial fibration (25.4). In particular, a lift exists in
\[
\begin{array}{ccc}
C_{y/} & \xrightarrow{q} & C \\
\downarrow & & \\
S & \xrightarrow{\pi} & C
\end{array}
\]
By an adjunction argument, \( x'' \) corresponds to a map \( x: \Delta^0 \to C_{y/} \) such that \( \pi(x) = y \). By what we have already proved, \( x \) must be initial since \( \pi(x) = y \) is initial. \( \square \)

27.11. \textbf{Remark.} In fact, the converses of (1b) and (2b) in (27.10) are also true. The proof requires (D), which we have not established yet.

We can now generalize the above to arbitrary limits in colimits. The following proposition says that colimits in \( C_{y/} \) or limits in \( C_{p/} \) can be “computed in the underlying quasicategory” \( C \) (if the corresponding colimit or limit in \( C \) exists).

27.12. \textbf{Proposition.} Let \( p: S \to C \) be a map from a simplicial set to a quasicategory.

(1) Let \( f: K \to C_{y/} \) be a map such that the composite map \( f_0 = \pi f: K \xrightarrow{f} C_{y/} \xrightarrow{\pi} C \) has a colimit cone in \( C \). Then
   (a) \( f \) admits a colimit cone, and
   (b) if \( \tilde{f}: K^{\triangleright} \to C_{y/} \) is such that the composite map \( K^{\triangleright} \xrightarrow{\tilde{f}} C_{y/} \to C \) is a colimit cone, then \( \tilde{f} \) is a colimit cone.

(2) Let \( f: K \to C_{p/} \) be a map such that the composite map \( f_0 = \pi f: K \xrightarrow{f} C_{p/} \xrightarrow{\pi} C \) has a limit cone in \( C \). Then
   (a) \( f \) admits a limit cone, and
   (b) if \( \tilde{f}: K^{\triangleleft} \to C_{p/} \) is such that the composite map \( K^{\triangleleft} \xrightarrow{\tilde{f}} C_{p/} \to C \) is a limit cone, then \( \tilde{f} \) is a limit cone.

The proof will make use an observation sketched in the following exercise: any composite of a slice-over followed by a slice-under can be reinterpreted as a slice-under followed by a slice-over.

27.13. \textbf{Exercise (Two-sided slice).} Fix a map \( p: A \star B \to X \) of simplicial sets. Describe a simplicial set \( X_{p/} \) which admits bijective correspondences
\[
\begin{cases}
A \star B \xrightarrow{p} X \\
A \star K \star B \xrightarrow{\sim} X_{p/}
\end{cases}
\]
natural in \( K \). Then construct natural isomorphisms
\[
(X_{pA}/)_{pB} \approx X_{p/} \approx (X_{pB})_{pA}/,
\]
where \( p_A: A \to X \) and \( p_B: B \to X \) are the evident restrictions of \( p \) to subcomplexes, and \( p_A: A \to X_{p/} \) and \( p_B: B \to X_{p/} \) are adjoints to \( p \).

\textbf{Proof of (27.12).} I prove (1), as (2) is analogous. Note that \( f: K \to C_{y/} \) is adjoint to a map \( g: K \star S \to C \), which in turn is adjoint to a map \( q: S \to C_{y/} \). Colimit cones of \( f_0 \) correspond precisely to initial objects of \( C_{f_0/} \); in particular, the hypothesis of (1) asserts that \( C_{f_0/} \) has an initial object. Likewise, colimit cones of \( f \) correspond exactly to initial objects of \( (C_{y/})_f \). As in (27.13) we have isomorphisms
\[
(C_{y/})_f \approx C_{y/} \approx (C_{f_0/})_q.
\]
To prove (1a) here it suffices to show that \((C_{f_0})/q\) has an initial object, which since \(C_{f_0}\) does using (27.10)(1a). To prove (1b) here it suffices to show that the projection \((C_{f_0})/q \to C_{f_0}\) has the property that objects sent to initial objects of \(C_{f_0}\) are initial in \((C_{f_0})/q\), which is immediate from (27.10)(1b).

\(\square\)

**Limits/colimits are invariant under categorical equivalence.** Where does this get proved?

28. **The Joyal extension and lifting theorems**

We are now at the point where we can state and prove Joyal’s theorems about extending or lifting maps along outer horns. This will allow us to prove many of the results whose proofs we have deferred up to now.

28.1. **Joyal extension theorem.** The following gives a condition for extending maps from *outer horns* into a quasicategory.

28.2. **Theorem** (Joyal extension). [Joy02, Thm. 1.3] Let \(C\) be a quasicategory, and fix a map \(f: \Delta^1 \to C\). The following are equivalent.

1. The edge represented by \(f\) is an isomorphism in \(C\).
2. Every \(a: \Lambda^0_n \to C\) with \(n \geq 2\) such that \(f = a|\Delta^{0,1}: \Delta^1 \to C\) admits an extension to a map \(\Delta^n \to C\).
3. Every \(b: \Lambda^0_n \to C\) with \(n \geq 2\) such that \(f = b|\Delta^{n-1,n}: \Delta^1 \to C\) admits an extension to a map \(\Delta^n \to C\).

I’ll call \(\langle 01 \rangle \in \Delta^n\) the **leading edge**, and \(\langle n-1,n \rangle \in \Delta^n\) the **trailing edge**. Thus, the implications \((1) \Rightarrow (2)\) and \((1) \Rightarrow (3)\) say that we can always extend \(\Lambda^0_n \to C\) to an \(n\)-simplex if the leading edge goes to an isomorphism in \(C\), and extend \(\Lambda^0_n \to C\) to an \(n\)-simplex if the trailing edge goes to an isomorphism in \(C\).

The implications \((2) \Rightarrow (1)\) and \((3) \Rightarrow (1)\) are easy, and are left as an exercise.

28.3. **Exercise** (Easy part of Joyal extension). Suppose \(C\) is a quasicategory with edge \(f \in C_1\), and suppose that every map \(a: \Lambda^0_3 \to C\) with \(n \in \{2,3\}\) and \(f = a|\Delta^{0,1}\) admits an extension along \(\Lambda^0_n \subset \Delta^n\). Prove that \(f\) is an isomorphism.

The non-trivial implications of Joyal extension will lead to proofs of the deferred propositions (A), (C), and (D).

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are conservative iso-fibrations.

28.4. **Conservative functors.** A functor \(p: C \to D\) between categories is conservative if whenever \(f\) is a morphism in \(C\) such that \(p(f)\) is an isomorphism in \(D\), then \(f\) is an isomorphism in \(C\).

The definition of a conservative functor between quasicategories is precisely the same.

28.5. **Proposition.** All left fibrations and right fibrations between quasicategories are conservative.

**Proof.** Consider a right fibration \(p: C \to D\), and a morphism \(f: x \to y\) in \(C\) such that \(p(f)\) is an isomorphism. We first show that \(f\) admits a preinverse.

Let \(a: \Lambda^2_2 \to C\) such that \(a_{12} = f\) and \(a_{02} = 1_y\). Let \(b: \Delta^2 \to C\) be any 2-simplex exhibiting a preinverse of \(p(f)\), i.e., such that \(b_{12} = p(f)\) and \(b_{02} = 1_{p(y)}\), so that \(b_{01}\) is a preinverse. Now have a diagram with a lift

\[
\begin{array}{ccc}
\Lambda^2_2 & \xrightarrow{a} & C \\
\downarrow & & \downarrow p \\
\Delta^2 & \xrightarrow{b} & D
\end{array}
\]

Thus, we have shown that every left fibration and right fibration between quasicategories is conservative. This completes our proof of the Joyal extension and lifting theorems.
which exhibits a preinverse of \( f \), which we will call \( g \).

Because \( p(f) \) was assumed to be an isomorphism in \( D \), its preinverse \( p(g) \) is also an isomorphism, and therefore by the above argument \( g \) admits a preinverse as well. We conclude that \( f \) is invertible by (10.4).

28.6. **Isofibrations.** We say that a functor \( p: C \to D \) of quasicategories is an **isofibration**\(^{21} \) if

1. \( p \) is an inner fibration, and
2. we have “isomorphism lifting” along \( p \). That is, for any \( c \in C_0 \) and isomorphism \( g: p(c) \to d' \), there exists a \( c' \in C_0 \) and isomorphism \( f: c \to c' \) such that \( p(f) = g \).

Recall that if \( C \) and \( D \) are nerves of ordinary categories, then any functor \( C \to D \) is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories we can replace (2) with the dual condition

(2') for any \( c \in C_0 \) and isomorphism \( g': d' \to p(c) \), there exists a \( c' \in C_0 \) and isomorphism \( f': c' \to c \) such that \( p(f') = g \).

To prove (2) from (2'), just apply condition (2') to the (unique) inverse of \( g \).

The symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

28.7. **Proposition.** An inner fibration \( p: C \to D \) between quasicategories is an isofibration if and only if \( h(p): h(C) \to h(D) \) is an isofibration of ordinary categories.

*Proof.* (\( \Rightarrow \)) Straightforward. (\( \Leftarrow \)) Suppose given an isomorphism \( g: p(c) \to d' \) in \( D \). If \( h(p) \) is an isofibration, there exists an isomorphism \( f': c \to c' \) in \( C \) such that \( p(f') \sim_r g \). Now choose a lift in

\[
\begin{array}{ccc}
\Lambda^2_1 & \xrightarrow{a} & C \\
\downarrow & & \downarrow p \\
\Delta^2_2 & \xrightarrow{b} & D
\end{array}
\]

where \( b \) exhibits \( p(f') \sim_r g \) and \( a((01)) = f' \) and \( a((12)) = 1_{c'} \). The edge \( f = s_{02} \) is a lift of \( g \), and

is an isomorphism since \( f' \sim_r f \). \( \square \)

28.8. **Exercise.** (i) Let \( \text{Group} \) denote the category of groups, whose objects are pairs \( G = (S, \mu) \) consisting of a set \( S \) and a function \( \mu: S \times S \to S \) satisfying a well-known list of axioms. Show that the functor \( U: \text{Group} \to \text{Set} \) which on objects sends \( (S, \mu) \mapsto S \) is an isofibration between ordinary categories.

(ii) Consider the functor \( U': \text{Group} \to \text{Set} \) defined on objects by \( G \mapsto \text{Hom}(\mathbb{Z}, G) \). Explain why, although \( U' \) is naturally isomorphic to \( U \), you don’t know how to show whether \( U' \) is an isofibration without explicit reference to the axioms of your set theory. The moral is that the property of being an isofibration is not “natural isomorphism invariant”.

28.9. **Left and right fibrations are isofibrations.**

28.10. **Proposition.** All left fibrations and right fibrations between quasicategories are isofibrations.

*Proof.* Suppose \( p: C \to D \) is a right fibration (and hence an inner fibration) between quasicategories, and consider

\[
\begin{array}{ccc}
\{1\} & \xrightarrow{f} & C \\
\downarrow & & \downarrow p \\
\Delta^1 & \xrightarrow{g} & D
\end{array}
\]

\(^{21}\)Joyal uses the term “quasifibration” in [Joy02]. Later in [Joy08a] this is called a “pseudofibration”. Lurie uses this notion, but never names it. The term “isofibration” is used by Riehl and Verity [RV15].
where $g$ represents an isomorphism. Because $p$ is a right fibration, there exists a lift $f$. Because right fibrations are conservative, $f$ represents an isomorphism. \qed

Note that the above proof checked explicitly isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary direct proof that right-fibrations satisfy (2).

28.11. Proof of the Joyal extension theorem.

Proof of (28.2). We prove (1) $\Rightarrow$ (2). Suppose given $a: \Lambda^n_0 \to C$ such that $f = a|\Delta^{0,1}$ represents an isomorphism. Observe (26.3) that $(\Lambda^n_0 \subset \Delta^n)$ is the pushout-join of a 1-horn with an $(n - 2)$-cell:

$$(\Lambda^n_0 \subset \Delta^n) \approx (\Delta^{0} \subset \Delta^{(0,1)} \boxdot (\partial \Delta^{2,\ldots,n} \subset \Delta^{2,\ldots,n})).$$

Using this, we get a correspondence of lifting problems

\[
\begin{array}{c}
\Delta^{0,1} \\
\downarrow \downarrow \\
\Lambda^n_0 \\
\downarrow f \\
\Delta^n
\end{array} \xrightarrow{\alpha} \begin{array}{c}
\Delta^{0,1} \\
\downarrow \downarrow \\
C \\
\downarrow \downarrow \\
\Delta^n
\end{array} \iff \begin{array}{c}
\Delta^{0} \\
\downarrow \downarrow \\
C/(a|\Delta^{(2,\ldots,n)}) \\
\downarrow q \\
C
\end{array}
\]

where $g$ is adjoint to $a|(\Delta^{0,1} \star \partial \Delta^{2,\ldots,n})$, and $h$ is adjoint to $a|(\Delta^{0} \star \Delta^{2,\ldots,n})$. Because $C$ is a quasicategory, both $p$ and $q$ are right fibrations (26.14), and therefore are conservative isofibrations (28.5), (28.10). Thus since $f$ represents an isomorphism, so does $g$ since $p$ is conservative, and therefore a lift exists since $q$ is an isofibration.

The proof of (2) $\iff$ (1) is left as an exercise (28.3). The proof of (1) $\iff$ (3) is similar. \qed

28.12. The Joyal lifting theorem. There is a relative generalization.

28.13. Theorem (Joyal lifting). Let $p: C \to D$ be an inner fibration between quasicategories, and let $f \in C_1$ be an edge such that $p(f)$ is an isomorphism in $D$. The following are equivalent.

1. The edge $f$ is an isomorphism in $C$.
2. For all $n \geq 2$, every diagram of the form

\[
\begin{array}{c}
\Delta^{0,1} \\
\downarrow \downarrow \\
\Lambda^n_0 \\
\downarrow f \\
\Delta^n
\end{array} \xrightarrow{\alpha} \begin{array}{c}
\Delta^{0,1} \\
\downarrow \downarrow \\
C \\
\downarrow \downarrow \\
\Delta^n
\end{array}
\]

admits a lift.

3. For all $n \geq 2$, every diagram of the form

\[
\begin{array}{c}
\Delta^{n-1,n} \\
\downarrow \downarrow \\
\Lambda^n \\
\downarrow f \\
\Delta^n
\end{array} \xrightarrow{\alpha} \begin{array}{c}
\Delta^{n-1,n} \\
\downarrow \downarrow \\
C \\
\downarrow \downarrow \\
\Delta^n
\end{array}
\]

admits a lift.
Proof. The implications \((2) \Rightarrow (1)\) and \((3) \Rightarrow (1)\) are elementary, as in (28.3).

For \((1) \Rightarrow (2)\), the first step is to prove that
\[
C_{/(a)Δ^{(2,\ldots,n)}} \xrightarrow{q} C_{/(pa)p'Δ^{(2,\ldots,n)}} \times D_{/(pa)p'Δ^2(2,\ldots,n)} \xrightarrow{p'} C
\]
are both right fibrations. For instance, the map \(q\) is the slice-power of the inner fibration \(p\) by a monomorphism, so is a right fibration by (26.13). The map \(p\) is the composite
\[
C_{/(a)∂Δ^{(2,\ldots,n)}} \times D_{/(pa)∂Δ^2(2,\ldots,n)} \xrightarrow{p'} C_{/(a)∂Δ^{(2,\ldots,n)}} \xrightarrow{p''} C,
\]
where \(p'\) is the base change of the right fibration \(D_{/(pa)Δ^2(2,\ldots,n)} \rightarrow D_{/(pa)Δ^2(2,\ldots,n)}\), and \(p''\) is a right fibration (in both cases by (26.14)) Then the proof of \((1) \Rightarrow (2)\) proceeds exactly as in (28.2). \qed

We note here that as a consequence of Joyal lifting we get a characterization of conservative inner fibrations between quasicategories in terms of a lifting property.

28.14. Corollary. Let \(p: C \rightarrow D\) be an inner fibration between quasicategories. The following are equivalent.

1. The functor \(p\) is conservative.
2. For all \(n \geq 2\), every diagram of the form
\[
\Delta^{(0,1)} \xrightarrow{f} \Lambda^n \xrightarrow{p} C
\]
such that \(p(f)\) represents an isomorphism in \(D\) admits a lift.
3. For all \(n \geq 2\), every diagram of the form
\[
\Delta^{[n-1,n]} \xrightarrow{f} \Lambda^n \xrightarrow{p} C
\]
such that \(p(f)\) represents an isomorphism in \(D\) admits a lift.

29. Applications of the Joyal extension theorem

We can now prove all the statements whose proofs we have deferred until now, as well as some others.

29.1. Quasigroupoids are Kan complexes. First we prove (A), the identification of quasigroupoids with Kan complexes.

29.2. Proposition. Every quasigroupoid is a Kan complex.

Proof. In a quasigroupoid, the Joyal extension property (28.2) applies to all maps from \(Λ^n_0\) and \(Λ^n_n\), since every edge is an isomorphism. \qed

From now on we will use terms “quasigroupoid” and “Kan complex” interchangeably.
29.3. **Invariance of slice categories.** Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

29.4. **Exercise** (Reformulation of Joyal extension). If \( f : x \to y \) is an edge in a quasicategory \( C \), then the following are equivalent: (1) \( f \) is an isomorphism; (2) \( C_{f/} \to C_{x/} \) is a trivial fibration; (3) \( C_f \to C_y \) is a trivial fibration.

29.5. **Corollary.** If \( f : x \to y \) is an isomorphism in a quasicategory \( C \), then \( C_{x/} \) and \( C_{y/} \) are categorically equivalent, and \( C_{x/} \) and \( C_{y/} \) are categorically equivalent.

**Proof.** Consider \( C_{x/} \xrightarrow{i} C_f \xrightarrow{\rho} C_{y/} \). We have already observed (26.15) that \( \rho \in \text{TrivFib} \), since \( \{1\} \subset \Delta^1 \) is right anodyne. The reformulation of Joyal extension (29.4) implies that \( \pi \in \text{TrivFib} \) when \( f \) is an isomorphism. \( \square \)

29.6. **Initial objects.** Now we prove (D) about initial objects.

29.7. **Proposition.** If \( x \) is an object in a quasicategory \( C \), then the vertex \( i_x \in (C_{x/})_0 \) corresponding to \( 1_x \in C_1 \) is an initial object of \( C_{x/} \).

**Proof.** The map \( 1_x : x \to x \) in \( C \) is obviously an isomorphism. We have a correspondence of lifting problems

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{i_x} & C_{x/} \\
\partial \Delta^n & \searrow & \Delta^0 \ast \{0\} \\
\Delta^n & \searrow & \Delta^0 \ast \Delta^n \\
\end{array}
\]

and \( (\Delta^0 \ast \partial \Delta^n \subset \Delta^0 \ast \Delta^n) \approx (\Lambda_{1+n}^1 \subset \Delta^{1+n}) \), so a lift exists by the Joyal extension theorem. \( \square \)

29.8. **Proposition.** Any object in a quasicategory isomorphic to an initial object is also initial.

**Proof.** Let \( x \) be an initial object in \( C \), and let \( c \) be an object isomorphic to \( x \). It is easy to see that \( x \) is initial in the homotopy category \( hC \), and therefore \( c \) is initial in \( hC \) also. This has a useful consequence: any map between \( x \) and \( c \) (in either direction) must be an isomorphism in \( C \).

We next note another fact: if \( x \) is initial, any map \( f : S \to C \) extends along \( S \subset \Delta^0 \ast S \) to a map \( f' : \Delta^0 \ast S \to C \) such that \( f'|\Delta^0 \) represents \( x \). This is a consequence of the fact (25.4) that \( p : C_{/x} \to C \) is a trivial fibration whence, there exists a map \( s : C \to C_{/x} \) such that \( ps = \text{id}_C \); set \( f' \) be the adjoint to \( sf : S \to C_{/x} \).

To show \( c \) is initial in \( C \), we need to extend any \( a : \partial \Delta^n \to C \) with \( a_0 = c \) to a map \( \widetilde{a} : \Delta^n : C \). This follows from a succession of two extension problems:

\[
\begin{array}{ccc}
(\Delta^0 \ast \emptyset) & \xrightarrow{\Pi (\emptyset \ast \partial \Delta^n)} & C \\
\downarrow & & \downarrow \\
\Delta^0 \ast \partial \Delta^n & \xrightarrow{h} & \Delta^0 \ast \Delta^n \\
\emptyset \ast \Delta^n & \to & \Delta^0 \ast \Delta^n
\end{array}
\]

The extension \( g \) exists by the remarks of the previous paragraph since \( x \) is initial. The extension \( h \) exists because the leading edge of \( g \) is a map \( x \to c \) in \( C \), which is an isomorphism by the remarks of the first paragraph. The desired extension \( \widetilde{a} \) is \( h|(\emptyset \ast \Delta^n) \). \( \square \)
29.9. **Proof of objectwise criterion for natural isomorphisms.** Next we prove (C), the fact that natural isomorphisms are the same as natural transformations which are isomorphisms objectwise. We can restate this as follows. Consider the map

\[
\text{Proposition.}
\]

\[
\text{Fun}(j, C): \text{Fun}(D, C) \to \text{Fun}(\text{Sk}_0 D, C) \approx \prod_{d \in D_0} C
\]

induced by restriction along \( j: \text{Sk}_0 D \to D \). The statement of (C) amounts to showing that this Fun\((j, C)\) is conservative. This is a special case of the following proposition.

29.10. **Proposition.** Let \( j: K \to L \) be a monomorphism of simplicial sets such that \( j: K_0 \xrightarrow{\sim} L_0 \) is a bijection. Then for every quasicategory \( C \) the restriction map \( \text{Fun}(j, C): \text{Fun}(L, C) \to \text{Fun}(K, C) \) is conservative.

The proof we give uses ideas from [Lur09, §3.1.1].

**Proof.** Note that \( p = \text{Fun}(j, C) \) is always an inner fibration between quasicategories (since \( j \) is a monomorphism and \( \text{InnHorn} \circ \text{Cell} \subseteq \text{InnHorn} \)). Therefore we can apply the criterion of (26.15) to show \( p \) is conservative: we show that for every \( n \geq 2 \) and every diagram of the form

\[
\begin{array}{ccc}
\Delta^{(0,1)} & \xrightarrow{f} & \Delta^n_0 \\
\downarrow & & \downarrow \alpha \\
\Delta^n & \xrightarrow{p=\text{Fun}(j, C)} & \Delta^n \\
\end{array}
\]

such that \( p(f) \) represents an isomorphism in \( \text{Fun}(K, C) \), a lift \( a \) exists.

We reduce this lifting problem to a different one using a retraction. For \( n \geq 2 \) we can define maps

\[
\Delta^n \xrightarrow{s} \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n
\]

uniquely characterized by their effect on vertices: \( s(x) = (1, x) \), and \( r(0, 1) = 0 \) and \( r(x, y) = y \) if \((x, y) \neq (0, 1)\). We check that

- \( rs = \text{id} \),
- \( s(\Delta^0_0) = \{1\} \times \Delta^n_0 \subseteq (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Delta^0_0) \),
- \( r(\{0\} \times \Delta^n) = \Delta^{[n] \setminus 1} \subseteq \Delta^0_0 \), and
- \( r(\Delta^1 \times \Delta^{[n] \setminus j}) = \Delta^{[n] \setminus j} \) if \( j \neq 0 \), whence \( r(\Delta^1 \times \Delta^0_0) = \Delta^0_0 \).

Therefore we can form the solid arrow diagram

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{s} & (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Delta^0_0) \\
\downarrow & & \downarrow r \\
\Delta^1 \times \Delta^n & \xrightarrow{r} & \Delta^n \\
\end{array}
\]

and observe that to produce a lift \( a \), it suffices to produce a map \( b \) which is a lift in its rectangle: given \( b \), take \( a = bs \).

Note that \( r \) sends each edge \( \Delta^1 \times \{k\} \) in \((\{0\} \times \Delta^n) \cup (\Delta^1 \times \Delta^0_0) \) to either: the degenerate edge \( \langle kk \rangle \) in \( \Delta^n_0 \) (if \( k \neq 1 \)), or the leading edge \( \langle 01 \rangle \) in \( \Delta^0_0 \) (if \( k = 1 \)). Thus for all \( k \in [n] \) the restriction of \( p\) to \( \Delta^1 \times \{k\} \) represents an isomorphism in \( \text{Fun}(K, C) \).

By adjunction producing a lift \( b \) amounts to showing that a lift exists in

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{a} & \text{Fun}(T, C) \\
\downarrow & & \downarrow \text{Fun}(j, C) \\
\Delta^1 & \xrightarrow{b} & \text{Fun}(S, C)
\end{array}
\]

and
where \( i : S \to T \) is the monomorphism \((\Lambda^n_0 \times L) \cup (\Delta^n \times K) \to \Delta^n \times L\), and the edge in \( \text{Fun}(S,C) \) represented by \( v \) is such that for every \( k \in [n] \) the restriction \( v|\{k\} \times K \) represents an isomorphism in \( \text{Fun}(\{k\} \times K) \).

Note that \( i : S \to T \) induces a bijection on vertices, and that the hypothesis on \( v \) implies that the restriction \( v|\{x\} \) of \( v \) to every vertex \( x \in S_0 \) represents an isomorphism in \( \text{Fun}(\{x\},C) \approx C \). Thus the proposition follows from the following lemma. □

29.11. Lemma. Let \( C \) be a quasicategory, and let \( i : S \to T \) be a monomorphism such that \( i : S_0 \to T_0 \) is a bijection. Then for every diagram

\[
\begin{array}{c}
\Delta^1 \downarrow^v \\
\downarrow^x \\
\text{Fun}(i,C) \quad \downarrow \text{Fun}(T,C)
\end{array}
\]

such that \( v \) represents an objectwise natural isomorphism of functors \( S \to C \) (i.e., such that the image of \( v \) under \( \text{Fun}(S,C) \to \text{Fun}(\{x\},C) \approx C \) represents an isomorphism in \( C \) for all vertices \( x \in S_0 \)), a lift \( t \) exists.

Proof. First note that in the case that a lift \( t \) does exist, it necessarily represents a natural transformation between functors \( T \to C \) which is an objectwise isomorphism, since \( i : S_0 \to T_0 \) is a bijection and we have assumed \( v \) is so.

Let \( C \) denote the class of monomorphisms \( i : S \to T \) such that \( i : S_0 \to T_0 \) is a bijection, and such that the conclusion of the lemma holds. A straightforward exercise shows that \( C \) is a weakly saturated class. (Note that for the cases of composition and transfinite composition, this requires the observation of the previous paragraph, which is why we need to assume here that elements of \( C \) induce a bijection on vertices.)

Let \( \text{Cell}_{\geq 1} = \{ \partial \Delta^n \subset \Delta^n \}_{n \geq 1} \). From the skeletal filtration (15.18) we see that \( \text{Cell}_{\geq 1} \) is precisely the class of monomorphisms which are bijections on vertices. Thus the proposition follows once we show that \( \text{Cell}_{\geq 1} \subseteq C \), which follows from the following lemma applied to the case of \( D = * \) and \((i,j) = (0,0)\). □

The following lemma is a kind of “pushout-product” version of Joyal lifting.

29.12. Lemma. Suppose \( p : C \to D \) is an inner fibration of quasicategories, and suppose \( n \geq 1 \), and either \((i,j) = (0,0)\) or \((i,j) = (1,n)\). For any diagram

\[
\begin{array}{c}
\Delta^1 \times \{j\} \ar[r]^-f \ar[d] & (\{i\} \times \Delta^n) \cup_{\{i\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) \ar[l] \ar[r] & C \\
\Delta^1 \times \Delta^n \ar[u] \ar[r] & D
\end{array}
\]

such that \( f \) represents an isomorphism in \( C \), a lift exists.

Proof. This is a calculation, given in the appendix (59.5). □

Now that we have proved the objectwise criterion for natural isomorphisms (C), we can reinterpret the conclusion of (29.11): the functor \( \text{Fun}(T,C) \to \text{Fun}(S,T) \) is an isofibration when \( S_0 \to T_0 \) is a bijection. Perhaps show here that the bijection condition can be dropped, and do a relative version here, rather than in a later section.
29.13. **Older proof.** (This material can probably be removed now.)

**Proof.** Let \( C \) be the class of monomorphisms \( j: K \to L \) for which the conclusion holds, i.e., such that \( \text{Fun}(j, C) \) is conservative for all quasicategories \( C \). To prove the claim, it suffices to show (a) that \( C \) is a weakly saturated class, and (b) that \( C \) contains the inclusions \( \partial \Delta^n \subset \Delta^n \) for all \( n \geq 1 \). Given this, the result follows using the skeletal filtration (15.18).

Note that \( \text{Fun}(j, C) \) is always an inner fibration between quasicategories, and recall the “lifting characterization” (28.14) of conservative inner fibrations between quasicategories which follows from the Joyal lifting theorem. Using this, the proof of (a) is a straightforward exercise.

To show (b) that \( \text{Fun}(\Delta^n, C) \to \text{Fun}(\partial \Delta^n, C) \) is conservative for \( n \geq 1 \), we again use (28.14), which we convert into its adjoint form. In fact, it suffices to prove that in any diagram of the form

\[
\Delta^{(0,1)} \times \{0\} \to (\Lambda_0^m \times \Delta^n) \cup (\Delta^m \times \partial \Delta^n) \to C
\]

\[
j \downarrow \quad \downarrow \quad \downarrow
\]

\[
\Delta^2 \times \Delta^n
\]

with \( m \geq 2 \) and \( n \geq 1 \) such that \( f_0 \) represents an isomorphism in \( C \), a lift exists. This reduction uses that \( f_0 \) is the composite

\[
\Delta^{(0,1)} \to \text{Fun}(\Delta^n, C) \xrightarrow{p} \text{Fun}(\partial \Delta^n, C) \to \text{Fun}(\{0\}, C)
\]

induced by restriction along \( \{0\} \subset \partial \Delta^n \subset \Delta^n \), so that if \( p(f) \) represents an isomorphism in \( \text{Fun}(\partial \Delta^n, C) \), then \( f_0 \) represents an isomorphism in \( C \).

This statement is a consequence of the following lemma (29.14). \( \Box \)

29.14. **Proposition** (Pushout-product version of Joyal lifting). Suppose \( p: C \to D \) is an inner fibration between quasicategories. Suppose \( m, n \geq 1 \). For any diagram

\[
\Delta^{(0,1)} \times \{0\} \to (\Lambda_0^m \times \Delta^n) \cup (\Delta^m \times \partial \Delta^n) \to C
\]

\[
j \downarrow \quad \downarrow \quad \downarrow
\]

\[
\Delta^m \times \Delta^n \quad \to \quad D
\]

such that \( f \) represents an isomorphism in \( C \), a lift exists.

Note: if we take instead \( m \geq 2 \) and \( n = 0 \), the above statement becomes the Joyal lifting theorem.

**Proof.** Proved as [Joy08a, 5.8] (though there it is the \( \Lambda_0^m \subset \Delta^m \) case that is proved). \( \Box \)

The idea is to produce the map \( j \) by iteratively attaching simplices along horns, which in each case are either:

1. an inner horn \( \Lambda_i^k \subset \Delta^k \) for \( 0 < i < k \), or
2. a horn \( \Lambda_0^k \subset \Delta^k \) with \( k \geq 2 \) along a map from \( \Lambda_0^k \) such that the restriction to its leading edge \( \Delta^{(0,1)} \subset \Lambda_0^k \) of the map to \( C \) is identical to \( f \).

The existence of the lifting follows, by applying the definition of inner fibration in cases of type (1), or the Joyal extension theorem in cases of type (2).
Here is a picture for the case $m = 1$, $n = 1$. The source of $(\Lambda^1_0 \subset \Delta^1) \square (\partial \Delta^1 \subset \Delta^1)$ looks like
\[
\begin{array}{c}
01 \\ \downarrow \\ 00 \sim 10
\end{array}
\]

First attach the 2-simplex $\langle 00, 01, 11 \rangle$, which intersects the source along the inner horn $\Lambda^2_0$. Then attach the 2-simplex $\langle 00, 10, 11 \rangle$, which intersects what we have already built along the horn $\Lambda^2_0$, whose leading edge $\langle 00 \rangle$ is sent to an isomorphism in $C$.

Here are pictures for the case $m = 2$, $n = 1$. Here is the source of $(\Lambda^2_0 \subset \Delta^2) \square (\partial \Delta^1 \subset \Delta^1)$.

The following chart lists all non-degenerate simplices in the complement, with “√” marking those in the source.

\[
\begin{array}{cccccccc}
\langle 10, 21 \rangle & \langle 10, 20, 21 \rangle & \langle 10, 11, 21 \rangle & \langle 00, 10, 21 \rangle & \langle 00, 11, 21 \rangle & \langle 00, 10, 20, 21 \rangle & \langle 00, 10, 11, 21 \rangle & \langle 00, 01, 11, 21 \rangle \\
\sqrt{(21)} & \sqrt{(20, 21)} & \sqrt{(11, 21)} & \sqrt{(10, 21)} & \sqrt{(11, 21)} & \sqrt{(10, 20, 21)} & \sqrt{(10, 11, 21)} & \sqrt{(01, 11, 21)} \\
\sqrt{(10)} & \sqrt{(10, 21)} & \sqrt{(00, 21)} & \sqrt{(00, 21)} & \sqrt{(00, 20, 21)} & \sqrt{(00, 11, 21)} & \sqrt{(00, 11, 21)} & \sqrt{(00, 01, 21)} \\
\sqrt{(10, 20)} & \sqrt{(10, 11)} & \sqrt{(00, 10)} & \sqrt{(00, 10)} & \sqrt{(00, 10, 21)} & \sqrt{(00, 10, 11)} & \sqrt{(00, 10, 11)} & \sqrt{(00, 0, 11)} \\
\end{array}
\]

Note that the simplices $\langle 10, 21 \rangle$, $\langle 00, 10, 21 \rangle$, and $\langle 00, 11, 21 \rangle$ of the complement appear multiple times as faces. We attach simplices to the domain in the following order:

1. $\langle 10, 11, 21 \rangle$, 2. $\langle 00, 01, 11, 21 \rangle$, 3. $\langle 00, 10, 11, 21 \rangle$, 4. $\langle 00, 10, 20, 21 \rangle$.

Only the final step involves attaching along a non-inner horn; in that case, the attaching map sends the leading edge of $\Lambda^2_0$ to $\langle 00, 10 \rangle$.

**Part 4. The fundamental theorem**

Recall that a functor $f: C \to D$ between quasicategories is said to be an equivalence there exists a $g: D \to C$ such that $gf$ and $fg$ are naturally isomorphic to the respective identity functors. When $C$ and $D$ are ordinary categories, there is a well-known criterion for the existence of such a $g$, namely: $f$ is an equivalence if and only if $f$ is fully faithful and essentially surjective. Here

- fully faithful means that $\text{Hom}_C(x, y) \to \text{Hom}_D(f(x), f(y))$ is a bijection of sets for every pair of objects $x, y \in \text{ob} C$, and
- essentially surjective means that for every object $d \in \text{ob} D$ there exists an object $c \in \text{ob} C$ such that $f(c)$ is isomorphic to $d$.

I like to call this fact the **Fundamental Theorem of Category Theory**. This is non-standard and frankly pretentious terminology (I am unaware of any standard descriptive name for this result). I want to give this fact a fancy name in order to signpost it, as it is quite nonconstructive: to prove it requires making a choice for each object $d$ in $D$ of an object $c$ of $C$ such that $f(c) \approx d$ (so it in fact relies on an appropriate form of the axiom of choice). (Exercise: prove the “Fundamental theorem” by first assuming an arbitrary choice of object $g(d) \in \text{ob} C$ and isomorphism $\alpha(d): f(g(d)) \to d$ for each object of $d$, and extending this to the data of a categorical inverse of $f$.)

**Example.** Fix a field $k$. Let $\text{Mat}$ be the category whose objects are non-negative integers $n \geq 0$, and whose morphisms $A: n \to m$ are $(m \times n)$-matrices with entries in $k$, so that composition is matrix multiplication. Let $\text{Vect}$ be the category of finite dimensional $k$-vector spaces and linear maps. Every basic class in linear algebra proves that the evident functor $F: \text{Mat} \to \text{Vect}$ is fully

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22I also don’t know when it was first formulated.
faithful and essentially surjective. Therefore $F$ is an equivalence of categories. However, there is no canonical choice of an inverse functor, whose construction relies on making an arbitrary choice of basis for each vector space.

We are going to state and then prove an analogue of this result for functors between quasicategories. This will first require an analogue of hom-sets, namely the quasigroupoid of maps between two objects.

### 30. Mapping spaces of a quasicategory

Given a quasicategory $C$ and objects $x, y \in C_0$, the mapping space (or mapping quasigroupoid) from $x$ to $y$ is the simplicial set defined by the pullback square

$$
\begin{array}{ccc}
\text{map}_C(x, y) & \longrightarrow & \text{Fun}(\Delta^1, C) \\
\downarrow & & \downarrow \\
\{ (x, y) \} & \longrightarrow & C \times C
\end{array}
$$

That is, $\text{map}_C(x, y)$ is the fiber of the restriction map $\text{Fun}(\Delta^1, C) \to \text{Fun}(\partial \Delta^1, C)$ over the point $(x, y) \in (C \times C)_0$, where we the isomorphism $\text{Fun}(\partial \Delta^1, C) \approx C \times C$ induced by the isomorphism $\partial \Delta^1 \approx \Delta^0 \sqcup \Delta^0$.

If $C = N(A)$ is the nerve of a category, then $\text{map}_C(x, y)$ is a discrete simplicial set (2.5) corresponding to the set $\text{Hom}_C(x, y)$.

### 30.1. Mapping spaces are Kan complexes.

The terminology “space” is justified by the following

### 30.2. Proposition.

The simplicial sets $\text{map}_C(x, y)$ are quasigroupoids (and hence Kan complexes by (A)).

This is a special case of the following, applied to $\text{Fun}(\Delta^1, C) \to \text{Fun}(\partial \Delta^1, C)$, the restriction along $j = (\partial \Delta^1 \subset \Delta^1)$.

### 30.3. Proposition.

Let $C$ be a quasicategory, and let $j : K \to L$ be a monomorphism which is a bijection on vertices. Then the fibers of the restriction map $p = \text{Fun}(j, C) : \text{Fun}(L, C) \to \text{Fun}(K, C)$ are quasigroupoids.

**Proof.** Consider a vertex $g \in \text{Fun}(K, C)_0$, and let $F$ denote the fiber of $p$ over $g$, i.e., the pullback of $p$ along the inclusion $\{ g \} \to \text{Fun}(K, C)$.

Note that $p : \text{Fun}(L, C) \to \text{Fun}(K, C)$ is an inner fibration, a case of enriched lifting using $\text{InnHorn} \sqcup \text{Cell} \subseteq \text{InnHorn}$ (16.7). Therefore $F \to \{ g \}$, which is a pullback of $p$, is also an inner fibration, so $F$ is a quasicategory.

To show that the quasicategory $F$ is a quasigroupoid, it suffices to show that every $\Lambda^2_0 \to F$ extends to a 2-simplex. Thus we must consider the lifting problem

$$
\begin{array}{ccc}
\Delta^{(0,1)} & \longrightarrow & \Lambda^2_0 \\
\downarrow & & \downarrow \\
\Delta^2 & \longrightarrow & \{ g \}
\end{array}
\xrightarrow{f}
\begin{array}{ccc}
\Lambda^2_0 & \longrightarrow & F \\
\downarrow & & \downarrow^p \\
\{ g \} & \longrightarrow & \text{Fun}(K, C)
\end{array}
$$

Since the right hand square is a pullback, it suffices to produce a lift $\Delta^2 \to \text{Fun}(L, C)$. Since $p(f)$ is an identity map and hence an isomorphism in $\text{Fun}(K, C)$, by the lifting criterion for conservative inner fibrations (28.14) it suffices to show that $p$ is conservative, which is the case by (29.10) since $j$ is a bijection on vertices. \qed
30.4. **Mapping spaces and homotopy classes.** The set of morphisms \( x \to y \) in a quasicategory is precisely the set of 0-simplices of \( \text{map}_C(x, y) \).

30.5. **Proposition.** Let \( C \) be a quasicategory. For any two maps \( f, g : x \to y \) in \( C \), we have that \( f \approx g \) (equivalence under the relation used to define the homotopy category \( hC \)) if and only if \( f \) and \( g \) are isomorphic as objects of the quasigroupoid \( \text{map}_C(x, y) \). That is,

\[
\text{Hom}_{hC}(x, y) \approx \pi_0 \text{map}_C(x, y)
\]

for every pair \( x, y \) of objects of \( C \).

**Proof.** Suppose \( f, g \in \text{map}_C(x, y)_0 \) are isomorphic, so that in particular there is a morphism \( f \to g \) in the quasigroupoid \( \text{map}_C(x, y) \). This amounts to a map \( \Delta^1 \times \Delta^1 \to C \) which can be represented by a diagram of simplices in \( C \) of the form:

\[
\begin{array}{c}
\xymatrix{ & y \\
1_x \ar[ur]^g & x \\
x \ar[ur]^h & b \\
\end{array}
\]

This explicitly exhibits a chain \( f \sim_r h \sim_l g \) of homotopies, so \( f \approx g \) as desired.

Conversely, if \( f \approx g \), we can explicitly construct a map \( H : f \to g \) in \( \text{map}_C(x, y) \): in terms of the above picture, let \( h = g \), let \( b \) be an explicit choice of right-homotopy \( f \sim_r g \), and let \( a = g_{001} \). \( \square \)

30.6. **Extended mapping spaces and composition.** Given a finite list \( x_0, \ldots, x_n \in C_0 \) of objects in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

\[
\begin{array}{ccc}
\text{map}_C(x_0, \ldots, x_n) & \longrightarrow & \text{Fun}(\Delta^n, C) \\
\downarrow & & \downarrow \\
\{(x_0, \ldots, x_n)\} & \longrightarrow & C^{\times (n+1)}
\end{array}
\]

where the right-hand vertical arrow is induced by restriction along \( \text{Sk}_0 \Delta^n \to \Delta^n \), using the isomorphism \( \text{Sk}_0 \Delta^n \approx (\Delta^0)^{\Pi(n+1)} \). By (30.3) the extended mapping spaces are quasigroupoids.

We can compare the extended mapping spaces to the fibers of \( \text{Fun}(I^n, C) \to C^{\times (n+1)} \), which is seen to be an \( n \)-fold product of mapping spaces.

30.7. **Lemma.** The map

\[
g_n : \text{map}_C(x_0, \ldots, x_n) \to \text{map}_C(x_{n-1}, x_n) \times \cdots \times \text{map}_C(x_0, x_1)
\]

induced by restriction along the spine inclusion \( I^n \subseteq \Delta^n \) is a trivial fibration. In particular, this map is a categorical equivalence between Kan complexes.

**Proof.** The map \( g_n \) is a base change of \( p : \text{Fun}(\Delta^n, C) \to \text{Fun}(I^n, C) \). Since \( I^n \subseteq \Delta^n \) is inner anodyne (12.12), and \( C \) is a quasicategory, the map \( p \) is a trivial fibration by enriched lifting using \( \text{InnHorn} \square \text{Cell} \subseteq \text{InnHorn} \) (16.7). \( \square \)

The inclusions \( I^2 \subseteq \Delta^2 \supseteq \Delta^{0.2} \) induce restriction maps

\[
\text{Fun}(I^2, C) \leftarrow \text{Fun}(\Delta^n, C) \to \text{Fun}(\Delta^{0.2}, C).
\]

Restricting to the fibers over a triple \( (x_0, x_1, x_2) \) of objects, we obtain a zig-zag of maps of Kan complexes

\[
\text{map}_C(x_1, x_2) \times \text{map}_C(x_0, x_1) \xleftarrow{g_2} \text{map}_C(x_0, x_1, x_2) \to \text{map}_C(x_0, x_2),
\]
where the second map is induced by restriction along $\Delta^{(0,2)} \subset \Delta^2$, and the first map $g_2$ is a categorical equivalence, and in fact a trivial fibration. After choosing a categorical inverse to $g_2$ (e.g., a section of $g_2$ using (20.10)), we obtain a “composition” map

$$\text{comp}: \text{map}_C(x_1, x_2) \times \text{map}_C(x_0, x_1) \to \text{map}_C(x_0, x_2).$$

This map is not uniquely determined, since it depends on a choice of categorical inverse to $g_2$. However, any two categorical inverses to $g_2$ are naturally isomorphic (proof?), and therefore $\text{comp}$ is defined up to natural isomorphism. That is, it is a well-defined map in $h\text{Kan}$, the homotopy category of Kan complexes (defined as a full subcategory of $h\text{QCat}$).

**30.9. Proposition.** The two maps obtained by composing the sides of the square

$$\text{map}_C(x_2, x_3) \times \text{map}_C(x_2, x_1) \times \text{map}_C(x_0, x_1) \xrightarrow{\text{id} \times \text{comp}} \text{map}_C(x_2, x_3) \times \text{map}_C(x_0, x_2)$$

are naturally isomorphic. That is, the diagram commutes in $h\text{Kan} \subset h\text{QCat}$.

**Proof.** Here is a diagram of simplicial sets which actually commutes on the nose, i.e., not merely in the homotopy category, but actually commutes in $\text{QCat}$. I use “$(x, y, z)$” as shorthand for “$\text{map}_C(x, y, z)$”, etc.

$$(x_2, x_3) \times (x_1, x_2) \times (x_0, x_1) \xrightarrow{\sim} (x_2, x_3) \times (x_0, x_1, x_2) \to (x_2, x_3) \times (x_0, x_2, x_3)$$

The maps labelled “$\sim$” are categorical equivalences, and in fact are trivial fibrations. All the maps in the above diagram are obtained via restriction along inclusions in

$$\Delta^{[2,3]} \cup \Delta^{[1,2]} \cup \Delta^{[0,1]} \xrightarrow{\sim} \Delta^{[2,3]} \cup \Delta^{[0,1,2]} \leftarrow \Delta^{[2,3]} \cup \Delta^{[0,2]}$$

where the maps labelled “$\sim$” are inner anodyne, and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define $\text{comp}$. After passing to $h\text{Kan}$ the categorical equivalences become isomorphisms, and the result follows.

**30.10. Segal categories.** Thus, a quasicategory does not quite give rise to a category “enriched over quasigroupoids”. Although we can define a composition law, it is not uniquely determined, and is only associative “up to homotopy”.

What we do get is a Segal category. A **Segal category** is a functor

$$M: \Delta^{\text{op}} \to \text{sSet}$$

such that

1. The simplicial set $M([0])$ is discrete, i.e., $M([0]) = \text{Sk}_0 M([0])$, and
2. For each $n \geq 1$ the “Segal map”

$$M([n]) \xrightarrow{(n-1, n)^*, \ldots, (0,1)^*} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$
is a “weak equivalence” of simplicial sets.

We will define “weak equivalence” of simplicial sets below. For now, we note that a map between Kan complexes is a weak equivalence if and only if it is a categorical equivalence, and that if each $M([n])$ is a Kan complex, then so are the fiber products which appear in the above definition.

Given a quasicategory $C$, we obtain a functor $M_C : \Delta^{op} \to sSet$ by

$$
M_C([0]) := \text{Sk}_0 C,
$$

$$
M_C([n]) := \prod_{x_0, \ldots, x_n \in C_0} \text{map}_C(x_0, \ldots, x_n).
$$

This object encodes all the structure we used above. For instance, the zig-zag

$$
M_C([1]) \times_{M_C([0])} M_C([1]) \overset{\langle (12)^*, (01)^* \rangle}{\longrightarrow} M_C([2]) \overset{(02)^*}{\longrightarrow} M_C([1])
$$

is a coproduct over all triples $x_0, x_1, x_2 \in C_0$ of the zig-zag (30.8) used to define “composition”.

You also get a Segal category from any “simplicially enriched” category. Suppose $C$ is a (small) category which is enriched over the category of simplicial sets, with object set $\text{ob}\, C$, and function objects $C(x, x') \in sSet$ for each $x, x'$. Then we can define $M_C : \Delta^{op} \to sSet$ by

$$
M_C([0]) := \text{ob}\, C,
$$

$$
M_C([n]) := \prod_{x_0, \ldots, x_n \in \text{ob}\, C} C(x_{n-1}, x_n) \times \cdots \times C(x_0, x_1).
$$

We thus obtain functors

$$
\text{QCat} \to \text{SeCat} \leftarrow \text{sCat}
$$

relating quasicategories, Segal categories, and simplicially enriched categories. Simplicially enriched categories were proposed as a model for $\infty$-categories by Dwyer and Kan\textsuperscript{23}, while Segal categories were proposed as a model for $\infty$-categories by Hirschowitz and Simpson [HS01]\textsuperscript{24}. All of these models are known to be equivalent in a suitable sense; see [Ber10] for more about these models and their comparison.

30.11. The enriched homotopy category of a quasicategory. Given a quasicategory $C$ we can produce a vestigial version of a category enriched over quasigroupoids, called the enriched homotopy category of $C$ and denoted $\mathcal{HC}$.\textsuperscript{25} This object will be a category enriched over $h\text{Kan}$, the homotopy category of Kan complexes, whose underlying category is $hC$.

We now define $\mathcal{HC}$. The objects of $\mathcal{HC}$ are just the objects of $C$. For any two objects $x, y \in C_0$, we have the quasigroupoid

$$
\mathcal{HC}(x, y) := \text{map}_C(x, y)
$$

which we will regard as an object of the homotopy category $h\text{Kan}$ of Kan complexes. Composition $\mathcal{HC}(x_1, x_2) \times \mathcal{HC}(x_0, x_1) \to \mathcal{HC}(x_0, x_2)$ is the composition map defined above. Composition is associative as shown above (30.9).

The underlying ordinary category of $\mathcal{HC}$ is just the ordinary homotopy category $hC$, since

$$
\text{Hom}_{h\text{Kan}}(\Delta^0, \text{map}_C(x, y)) \approx \pi_0 \text{map}_C(x, y) \approx \text{Hom}_{hC}(x, y).
$$

\textsuperscript{23}They called them “homotopy theories” instead of “$\infty$-categories; see [DS95, §11.6].

\textsuperscript{24}In fact, they generalize this to “Segal $n$-categories”, which were the first effective model for $(\infty, n)$-categories.

\textsuperscript{25}Lurie usually calls this “$hC$”, though he also uses that notation for the ordinary homotopy category of $C$ that we have already discussed. I prefer to have two separate notations.
30.12. **Warning.** A quasicategory $C$ cannot be recovered from its enriched homotopy category $\mathcal{H}C$, not even up to equivalence. In fact, there exist $h$Kan-enriched categories which do not arise as $\mathcal{H}C$ for any quasicategory $C$. A proof is outside the scope of these notes; however, we note that counterexamples may be produced from associative $H$-spaces which are not loop spaces.

30.13. **Exercise.** Let $C$ and $D$ be quasicategories. Show that there is an isomorphism $\mathcal{H}(C \times D) \approx \mathcal{H}C \times \mathcal{H}D$ of $h$Kan-enriched categories.

30.14. **Exercise.** Let $C$ be a quasicategory. Describe how to use the enriched homotopy category $\mathcal{H}C$ to define, for each morphism $f : x \to y$ in $C$ and object $c$ in $C$, maps $f^* : \text{map}_C(y,c) \to \text{map}_C(x,c)$, $f_* : \text{map}_C(c,x) \to \text{map}_C(c,y)$ in $h$Kan corresponding to pre- and post-composition with $f$. Show that these fit together to give a well-defined functor $\text{map}(-,-) : \mathcal{H}C^{\text{op}} \times \mathcal{H}C \to h\text{Kan}$.

31. **The fundamental theorem of quasicategory theory**

31.1. **Fully faithful and essentially surjective functors between quasicategories.** Note that any functor $f : C \to D$ of quasicategories induces functors $\text{map}_C(x,y) \to \text{map}_D(f(x),f(y))$ for every pair of objects $x,y$ in $C$. We say that a functor $f : C \to D$ between quasicategories is

- **fully faithful** if for every pair $c,c' \in C_0$, the resulting map $\text{map}_C(c,c') \to \text{map}_D(fc,fc')$ is a categorical equivalence, and
- **essentially surjective** if the induced functor $hf : \mathcal{H}C \to \mathcal{H}D$ is essentially surjective; i.e., if for every $d \in D_0$ there exists $c \in C_0$ and an isomorphism $fc \to d$ in $D_1$.

Another way to say this: $f : C \to D$ is fully faithful and essentially surjective iff the induced $h$Kan-enriched functor $Hf : \mathcal{H}C \to \mathcal{H}D$ is an equivalence of enriched categories.

31.2. **Proposition.** If $f : C \to D$ is a categorical equivalence between quasicategories, then $f$ is fully faithful and essentially surjective.

**Proof.** We already know that $hf : \mathcal{H}C \to \mathcal{H}D$ is an equivalence of ordinary categories, which implies essential surjectivity.

To show that $f$ is fully faithful, choose a categorical inverse $g$ of $f$. Given $x,y \in C_0$, consider the induced diagram of quasigroupoids

$$
\begin{array}{ccc}
\text{map}_C(x,y) & \xrightarrow{f} & \text{map}_D(fx,fy) \\
\downarrow{gf} & & \downarrow{g} \\
\text{map}_C(gfx,gfy) & \xrightarrow{f} & \text{map}_D(fgfx,fgfy) \\
\downarrow{fg} & & \end{array}
$$

We will need the following lemma, which assert that being fully faithful is a natural isomorphism invariant of functors. Given this, and since $gf \approx \text{id}_C$ and $fg \approx \text{id}_D$, we see that the maps in the diagram marked $gf$ and $fg$ are categorical equivalences, i.e., they are isomorphisms in the ordinary category $h\text{QCat}$. Therefore the map marked $g$ and both maps marked $f$ are also isomorphisms in $h\text{QCat}$, i.e., are categorical equivalences.

31.3. **Lemma.** If $f, f' : C \to D$ are functors which are naturally isomorphic, then $f$ is fully faithful if and only if $f'$ is.

**Proof.** **Proof needed.** Use lemma below?

The converse is true, but not as straightforward.
E. Deferred Proposition (Fundamental Theorem of Quasicategory Theory). A map $f: C \rightarrow D$ between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.

This is a non-trivial result. It gives a necessary and sufficient condition for $f: C \rightarrow D$ to admit a categorical inverse, but it does not spell out how to construct such an inverse. After many preliminaries, we will give the proof in §39.

31.4. 2-out-of-3 for fully faithful essentially surjective functors. The following result will be useful in the proof of the fundamental theorem. Recall the 2-out-of-3 property of a class of morphisms (22.5), and that the class of categorical equivalences has this property (22.9).

31.5. Proposition. The class $C$ of fully faithful and essentially surjective functors between quasicategories satisfies the 2-out-of-3 property.

Proof. Any identity functor $\text{id}: C \rightarrow C$ is manifestly fully faithful and essentially surjective.

Next note that if a functor $f: C \rightarrow D$ between quasicategories is fully faithful and essentially surjective, then the induced $hf: hC \rightarrow hD$ is an equivalence of ordinary categories. Conversely, if $hf$ is an equivalence, then $f$ is essentially surjective.

Therefore if $C \xrightarrow{f} D \xrightarrow{g} E$ are functors of quasicategories such that two of $\{f, g, gf\}$ are fully faithful and essentially surjective, then the third is also essentially surjective. Thus, we only need to deal with the fully faithful property, for which there are 3 cases: showing that $f$, $g$, or $gf$ is fully faithful assuming the other two are categorical equivalences.

Given objects $x, x' \in C_0$, we have induced maps

$$\text{map}_C(x, x') \xrightarrow{f} \text{map}_D(fx, fx') \xrightarrow{g} \text{map}_E(gfx, gfx').$$

Proofs of two of the three cases (for $f$ and for $gf$) follow immediately using (22.9).

For the case of $g$, note that if $f$ and $gf$ are categorical equivalences, it follows by the same argument that $\text{map}_D(y, y') \xrightarrow{g} \text{map}_E(gy, gy')$ is a categorical equivalence for any $y, y'$ in the image of $f: C_0 \rightarrow D_0$. This is good enough, which follows from the following lemma. \qed

31.6. Lemma. Let $f: C \rightarrow D$ be a functor between quasicategories, and let $a: x \rightarrow y$ and $b: y' \rightarrow x'$ be isomorphisms in $C$. Then there is a commutative square

$$\begin{array}{ccc}
\text{map}_C(x, x') & \xrightarrow{f} & \text{map}_D(fx, fx') \\
\sim & & \sim \\
\text{map}_C(y, y') & \xrightarrow{f} & \text{map}_D(fy, fy')
\end{array}$$

in $h\text{Kan}$, where the vertical maps are categorical equivalences (i.e., isomorphisms in $h\text{Kan}$) induced by composition with $a$ and $b$.

Proof. Use (30.14). \qed

31.7. Fully faithful and essentially surjective functors between quasigroupoids. The special case for quasigroupoids is already interesting. Here, (E) specializes to the “Fundamental Theorem of Quasigroupoid Theory”, which says that a map $f: X \rightarrow Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\text{map}_X(x_0, x_1) \rightarrow \text{map}_Y(fx_0, fx_1)$ on “path spaces”, and
- induces a surjection $\pi_0 X \rightarrow \pi_0 Y$.

31.8. Remark. A mild variant of this says that a map $f: X \rightarrow Y$ between Kan complexes is a categorical equivalence if and only if it
• induces categorical equivalences $\Omega_x X \to \Omega_{fx} Y$ on “loop spaces”, and
• induces an isomorphism $\pi_0 X \to \pi_0 Y$.

Here $\Omega_x X := \text{map}_X(x, x)$. Under the correspondence between quasigroupoids and classical homotopy theory, this turns out to be an analogue of the Whitehead theorem\textsuperscript{26}, which says that a map between CW-complexes is a homotopy equivalence iff it induces an isomorphism on all homotopy groups.

We will prove (E) after first considering the special case of quasigroupoids=Kan complexes.

### 32. Anodyne maps and Kan fibrations

In the next few sections, we will develop some properties related to Kan complexes. As a byproduct, we’ll obtain the proof of the specialization of (E) to Kan complexes.

#### 32.1. Weak equivalence

Say that a map $f : X \to Y$ is a weak equivalence of simplicial sets if and only if $\text{Map}(f, G) : \text{Map}(Y, G) \to \text{Map}(X, G)$ is a categorical equivalence for every quasigroupoid (i.e., every Kan complex) $G$.

Every categorical equivalence is a weak equivalence, but not conversely. For maps between Kan complexes, weak equivalences and categorical equivalences coincide, as is straightforward to show via the same proof as (19.5).

#### 32.2. Proposition

Weak equivalences of simplicial sets satisfy the 2-out-of-3 property.

**Proof.** Proved just as for categorical equivalences (22.9). $\square$

#### 32.3. Remark

Given the analogy to categorical equivalence, a more sensible name might be “groupoidal equivalence”. However, the term “weak equivalence” is historically well-established.

#### 32.4. Simplicial homotopy equivalence

A simplicial homotopy inverse to a map $f : X \to Y$ of simplicial sets is a map $g : Y \to X$ such that there exists a chain of edges in $\text{Map}(X, X)$ connecting $\text{id}_X$ with $gf$, and a chain of edges in $\text{Map}(Y, Y)$ connecting $\text{id}_Y$ with $fg$. Such an $f$ is called a simplicial homotopy equivalence, and of course any homotopy inverse is also a simplicial homotopy equivalence.

#### 32.5. Proposition

Any simplicial homotopy equivalence is a weak equivalence.

**Proof.** First, if $f : X \to Y$ is a simplicial homotopy equivalence between Kan complexes, then it is clearly a categorical equivalence, because $\text{Map}(X, X)$ and $\text{Map}(Y, Y)$ are quasigroupoids.

In general, suppose $K$ is a Kan complex and consider $f^* : \text{Map}(Y, K) \to \text{Map}(X, K)$. By the same reasoning as used in the proof of (19.5), we see that $f^*$ is a simplicial homotopy equivalence between Kan complexes, so a categorical equivalence. $\square$

#### 32.6. Anodyne maps and Kan fibrations

Let

$$\text{Horn} = \{ \Lambda^n_j \subset \Delta^n \mid n \geq 1, 0 \leq j \leq n \} = \text{RHorn} \cup \text{LHorn}$$

denote the set of all horn inclusions. A map is anodyne if it is in $\overline{\text{Horn}}$, and is a Kan fibration if it is in $\text{KFib} := \text{Horn}^\square$.

Since $\text{Horn}$ is a set, the small object argument (13.10) applies to it: any map can be factored $f = pj$ with $j \in \overline{\text{Horn}}$ and $p \in \text{KFib}$.

#### 32.7. Proposition

We have that $\text{Horn} \sqcap \text{Cell} \subseteq \text{Horn}$.

**Proof.** This amounts to showing $\text{Horn} \sqcap \text{Cell} \subseteq \overline{\text{Horn}}$. See [JT08, Theorem 3.2.2], or [GZ67], or appendix. $\square$

\textsuperscript{26}The “Fundamental Theorem of Classical Homotopy Theory”?
Thus, we have that
\[ \text{Map}(L, X) \to \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y) \]
is a Kan fibration whenever \( K \subseteq L \) and \( X \to Y \) is a Kan fibration, and is a trivial fibration if \( K \subseteq L \) is also anodyne.

As a special case, we learn that if \( X \) is a Kan complex and \( K \subseteq L \), then \( \text{Fun}(L, X) \to \text{Fun}(K, X) \) is a Kan fibration. Here is another consequence.

32.8. **Proposition.** Every anodyne map is a weak equivalence.

*Proof.* If \( f : A \to B \) is anodyne, then \( \text{Map}(f, G) \) is a trivial fibration for every Kan complex \( G \), and hence a categorical equivalence (20.9).

32.9. **Exercise.** Show that the inclusion \( \{ j \} \subseteq \Delta^n \) of any vertex into any standard \( n \)-simplex is anodyne.

32.10. **Exercise.** Let \( f : X \to Y \) be any map between Kan complexes. Show that \( f \) is a Kan fibration if and only if it is an isofibration. (Hint: Joyal lifting.)

32.11. **Exercise.** Let \( f : X \to Y \) be any map between Kan complexes. Show that \( f \) is a Kan fibration if and only if it is a Kan fibration. (Hint: Joyal lifting.)

32.12. **Exercise.** Let \( f : X \to Y \) be any map between Kan complexes. Show that \( f \) is a Kan fibration if and only if it is an isofibration. (Hint: Joyal lifting.)

32.13. **The universal isomorphism.** Let Iso be the “walking isomorphism”, i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Let \( u : \Delta^1 \to N\text{Iso} \) be the inclusion representing the unique map \( 0 \to 1 \) in Iso.

32.14. **Proposition.** The map \( u : \Delta^1 \to N\text{Iso} \) is anodyne, and hence a weak equivalence.

*Proof.* The \( k \)-simplices of \( N\text{Iso} \) are in one-to-one correspondence with sequences \( x_0 x_1 \cdots x_k \) with \( x_i \in \{0, 1\} \). For each \( k \geq 0 \) there are exactly two non-degenerate \( k \)-simplices, corresponding to the alternating sequences \( 0101 \cdots \) and \( 1010 \cdots \).

Let \( u_k : \Delta^k \to N\text{Iso} \) be the non-degenerate simplex \( 0101 \cdots \), and let \( F_k \subseteq N\text{Iso} \) be the smallest subcomplex containing \( u_k \). Note that \( N\text{Iso} = \bigcup F_k \) and \( F_1 = u(\Delta^1) \). The commutative square

\[
\begin{array}{ccc}
\Lambda_0^k & \longrightarrow & F_{k-1} \\
\downarrow & & \downarrow \\
\Delta^k & \underset{u_k}{\longrightarrow} & F_k
\end{array}
\]
is a pushout square for all \( k \geq 1 \). This is by (21.3), since (1) it is a pullback, and (2) any simplex in the complement of \( F_{k-1} \subseteq F_k \) is the image of a unique simplex under \( u_k \).

It follows that \( u \) is anodyne. ☐

We obtain as a consequence the following criterion for an edge to be an isomorphism, which we will use later.

32.15. **Proposition.** Let \( C \) be a quasicategory, and \( f : \Delta^1 \to C \) a map representing morphism in \( C \). Then there exists \( f' : N\text{Iso} \to C \) such that \( f' u = f \) if and only if \( f \) represents an isomorphism in \( C \).

*Proof.* (\( \Rightarrow \)) Clear: consider induced maps on homotopy categories. (\( \Leftarrow \)) If \( f \) represents an isomorphism then it factors through \( \Delta^1 \to C^{\text{core}} \subseteq C \). Since the core is a quasigroupoid, and hence a Kan complex, an extension along the anodyne map \( u \) to a map \( N\text{Iso} \to C^{\text{core}} \subseteq C \) exists. ☐
32.16. Remark. Let $X \subset N_{\text{Iso}}$ be the subcomplex which is the union of the images of 2-simplices $010$ and $101$. The inclusion $v: \Delta^1 \to X$ representing the edge $01$ has the same property described in (32.15): $f: \Delta^1 \to C$ represents an isomorphism if and only if it extends along $v$. The proof is easy: an extension of $f$ to a map $f': X \to C$ exactly encodes a choice of morphism $g$ in $C$ (i.e., $f'((10))$) together with explicit homotopies $gf \sim 1$ and $fg \sim 1$, (i.e., $f'((010))$ and $f'((101))$).

However, it turns out that $\Delta^1 \to X$ is not a weak equivalence (and therefore that $X \to N_{\text{Iso}}$ is not a categorical equivalence). In particular, a map $X \to C$ to a quasicategory can fail to extend along $X \subset N_{\text{Iso}}$.

32.17. Exercise. Show that $X \to N_{\text{Iso}}$ is not anodyne, by constructing a map $X \to K(Z, 2)$ which does not extend over $N_{\text{Iso}}$. (See (8.12).)

32.18. Covering homotopy extension property. This is the special case of Horn $\Box \subseteq \text{Cell}$ which we will use several times in the next few sections: for any inclusion $K \subseteq L$, the map

$$(K \times \Delta^1) \cup_{K \times \{j\}} (L \times \{j\}) \to L \times \Delta^1$$

with either $j = 0$ or $j = 1$ is anodyne. This amounts to saying that

$$
\begin{array}{ccc}
(K \times \Delta^1) \cup (L \times \{j\}) & \to & X \\
\downarrow & & \downarrow p \\
L \times \Delta^1 & \to & Y
\end{array}
$$

has a lift whenever $p$ is a Kan fibration. This is sometimes called the “covering homotopy extension property”.

It may be helpful to think of this in the equivalent form, which asserts a lifting in

$$
\begin{array}{ccc}
\{j\} & \to & \text{Map}(L, X) \\
\downarrow & & \downarrow \\
\Delta^1 & \to & \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)
\end{array}
$$

when $K \subseteq L$ and $p: X \to Y$ a Kan fibration. This gets used the following way: to demonstrate $(K \subseteq L) \Box p$ for a given Kan fibration $p$ and inclusion $K \subseteq L$, we “deform” a lifting problem of this type along a “path” in the space $\text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$ of commutative squares to a lifting problem which we know has a lift.

32.19. Fundamental theorem for Kan complexes: reduction to Kan fibrations. We are going to show the following

32.20. Theorem. A map $f: X \to Y$ between Kan complexes is a weak equivalence if and only if it is fully faithful and essentially surjective.

We will prove this by reducing to Kan fibrations.

32.21. Lemma. To prove (32.20), it suffices to prove it for the case when $f$ is a Kan fibration.

Proof. $(\Rightarrow)$ Weak equivalences between Kan complexes are categorical equivalences, and we have already shown that these are fully faithful and essentially surjective (31.2).

$(\Leftarrow)$ Given $f$ between Kan complexes which is fully faithful and essentially surjective, use the small object argument (13.10) applied to Horn to factor it as $X \xrightarrow{j} V \xrightarrow{p} Y$.

---

27 This is isomorphic to the complex discussed in (20.6).
where \( j \) is anodyne and \( p \) is a Kan fibration. It follows that \( V \) is a Kan complex, \( j \) is a weak equivalence (32.8) between Kan complexes, and so is fully faithful and essentially surjective. Since the class of fully faithful and essentially surjective maps satisfy 2-out-of-3 (31.5), it follows that \( p \) also has this property. If we can use this to show \( p \) is a weak equivalence, it follows that \( f \) is a weak equivalence as desired. \( \square \)

We will prove the needed special case (that for Kan fibrations between Kan complexes, fully faithful and essentially surjective implies weak equivalence) in the next couple of sections, after analyzing Kan fibrations in more detail.

33. **Kan fibrations between Kan complexes**

In the next few sections, we are going to be considering various properties of Kan fibrations, with particular interest in Kan fibrations between Kan complexes. In particular, we are going to show that for a Kan fibration \( p: X \to Y \) where \( X \) and \( Y \) are Kan complexes, all of the following are equivalent.

1. \( p \) is a trivial fibration;
2. \( p \) is a weak equivalence;
3. \( p \) is a fiberwise deformation retraction;
4. \( p \) has contractible fibers;
5. \( p \) is fully faithful and essentially surjective.

The equivalence of (2) and (5) will complete the proof of the fundamental theorem for Kan complexes (32.20). (In fact, (1)–(4) are equivalent without the hypothesis that the objects are Kan complexes, though we will not prove that in all cases.)

33.1. **Fiberwise deformation retraction.** A map \( p: X \to Y \) is said to be a **fiberwise deformation retraction** if there exists

- \( s: Y \to X \) such that \( ps = \text{id}_Y \), and
- \( k: X \times \Delta^1 \to X \) such that \( k|X \times \{0\} = \text{id}_X \), \( k|X \times \{1\} = sp \), and \( pk = p\pi \), where \( \pi: X \times \Delta^1 \to X \) is projection; that is, the diagram

\[
\begin{array}{ccc}
X \times \{0,1\} & \xrightarrow{(id_X, sp)} & X \\
\downarrow{k} & & \downarrow{p} \\
X \times \Delta^1 & \xrightarrow{\pi} & X \xrightarrow{p} Y
\end{array}
\]

commutes.

Any fiberwise deformation retraction is a weak equivalence: \( s \) is a simplicial homotopy inverse to \( p \) (32.5).

33.2. **Exercise.** Show that the term “fiberwise” is justified: for each \( y \in Y_0 \), the projection \( p^{-1}(y) \to \{y\} \) of a fiber to its image is a simplicial homotopy equivalence.

33.3. **Exercise.** Show that if \( p: X \to Y \) is part of a fiberwise deformation retraction as above, then any base change of \( p \) is also part of a fiberwise deformation retraction.

Fiberwise deformation retractions of Kan fibrations are always trivial fibrations, as can be shown with the covering homotopy extension property.

33.4. **Lemma.** Let \( p: X \to Y \) be a Kan fibration between simplicial sets. Then \( p \) is part of a fiberwise deformation retraction if and only \( p \) is a trivial fibration.
Proof. [JT08, Prop. 3.2.5]. (⇒) We need to solve the lifting problem

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow^{i} & \searrow^{u} & \downarrow^{p} \\
B & \xrightarrow{b} & Y
\end{array}
\]

where \(i\) is a monomorphism. Using the data \(s\) and \(k\) of the fiberwise deformation retraction we obtain a commutative square

\[
\begin{array}{ccc}
(A \times \Delta^1) \cup (B \times \{1\}) & \xrightarrow{(k(a \times id_{\Delta^1}), sb)} & X \\
\downarrow^{j} & \downarrow^{p} \\
B \times \Delta^1 & \xrightarrow{\pi} & B \xrightarrow{b} Y
\end{array}
\]

Because \(p\) is a Kan fibration and \(j\) is anodyne by (32.7), a lift \(t\) exists. Then \(u := t|B \times \{0\}\) is the desired lift.

(⇐) Left as an exercise. □

33.5. Exercise. Show that any trivial fibration is a fiberwise deformation retraction.

33.6. Trivial Kan fibrations between Kan complexes. We know that trivial fibrations are always categorical equivalences (20.9). We now show that any Kan fibration between Kan complexes which is also a categorical equivalence is a trivial fibration.

33.7. Proposition. A Kan fibration \(p: X \to Y\) between Kan complexes is a trivial fibration if and only if it is a weak equivalence.

Proof. [JT08, Prop. 3.2.6] (⇒) We have already shown that trivial fibrations between quasicategories are always categorical equivalences, which implies they are weak equivalences if between Kan complexes.

(⇐) On the other hand, suppose \(p\) is a Kan fibration and a weak equivalence. Being a categorical equivalence between Kan complexes, \(p\) admits a categorical inverse: there exists \(f: Y \to X\) and maps \(u: X \times \Delta^1 \to X\) and \(v: Y \times \Delta^1 \to Y\) which give natural isomorphisms \(u: fp \to id_X\) and \(v: pf \to id_Y\). We will “deform” this data to a fiberwise deformation retraction.

Step 1. Since \(Y \times \{0\} \subset Y \times \Delta^1\) is anodyne by (32.7), a lift \(\alpha\) exists in

\[
\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{f} & X \\
\downarrow^{\alpha} & \downarrow^{p} \\
Y \times \Delta^1 & \xrightarrow{v} & Y
\end{array}
\]

Let \(s := \alpha|Y \times \{1\}\), so \(ps = id_Y\). The map \(\alpha\) exhibits a natural isomorphism \(\alpha: f \to s\) of functors \(Y \to X\). Since \(fp\) is naturally isomorphic to \(id_X\), we have \(id_X \approx fp \approx sp\), i.e., there exists a natural isomorphism \(w: sp \to id_X\).

Step 2. Consider the natural isomorphism \((sp)w: sp = spsp \to sp\) of functors \(X \to X\). We have a commutative diagram

\[
\begin{array}{ccc}
\Delta^2 & \xrightarrow{a} & \text{Map}(X, X) \\
\downarrow^{t} & \nearrow^{w} & \downarrow^{\text{Map}(X, p)} \\
\Delta^2 & \xrightarrow{b} & \text{Map}(X, Y)
\end{array}
\]
where $a_{01} = w$, $a_{02} = spw$, and $b$ is the degeneracy $b = (pw)_{011}$; this commutes since $pspw = pw$. Since $p$ is a Kan fibration, so is $\text{Map}(X, p)$ by (32.7), and therefore a lift $t$ exists. Let $k = t|\Delta^{(1,2)}: \Delta^1 \to \text{Map}(X, X)$. It is clear that this is a natural isomorphism $k$: $\text{id}_Y \to sp$, and that this is “fiberwise”, i.e., $pk = b_{12} = p\pi$ as maps $X \times \Delta^1 \to Y$.

Thus, we have exhibited $p$ as a fiberwise deformation retraction, so $p$ is a trivial fibration by (33.4).


33.9. Corollary. Let $X$ be a simplicial set. The following are equivalent.

1. $X$ is a quasicategory which is categorically equivalent to $\Delta^0$.
2. $X \to \Delta^0$ is a trivial fibration.
3. Every $\partial\Delta^n \to X$ extends over $\partial\Delta^n \subset \Delta^n$.

Such an $X$ is necessarily a Kan complex.

Proof. We have (2) $\iff$ (3) by definition, and we know that (2) $\implies$ (1). Given (1), we have that $X$ is a quasicroupoid, and hence a Kan complex (29.2), and (2) follows by the previous proposition (33.7).

We say that an $X$ satisfying these conditions is a contractible Kan complex.

33.10. Monomorphisms which are weak equivalences. We can now characterize the monomorphisms which are weak equivalences, in terms of maps into Kan complexes.

33.11. Proposition. Let $j: A \to B$ be a monomorphism of simplicial sets. Then $j$ is a weak equivalence if and only if $\text{Map}(j, X): \text{Map}(B, X) \to \text{Map}(A, X)$ is a trivial fibration for all Kan complexes $X$.

Proof. Assume $X$ is an arbitrary Kan complex. We know that $\text{Map}(j, X)$ is always a Kan fibration between Kan complexes by (32.7). We have by definition that $j$ is a weak equivalence iff all $\text{Map}(j, X)$ are weak equivalences, which holds iff all $\text{Map}(j, X)$ are trivial fibrations by (33.7).

33.12. Remark. The class $\text{WkEq} \cap \text{Cell}$ of monomorphisms which are weak equivalences is a weakly saturated class. In fact, using (33.11) it is easy to show that it is the left complement of the class of maps of the form $p^{\Box j}$, where $p: X \to \Delta^0$ is a projection from a Kan complex $X$, and $j: \partial\Delta^n \to \Delta^n$ is a cell inclusion. Furthermore, $\text{WkEq} \cap \text{Cell}$ contains the weakly saturated class $\text{Horn}$ of anodyne maps.

It turns out that $\text{Horn} = \text{WkEq} \cap \text{Cell}$, i.e., the injective weak equivalences are precisely the same as the anodyne maps. This is a fairly non-trivial fact, and we will address it again later. Maybe.

As a consequence, it will follow that (33.7) and (33.15) hold without the condition that the objects be Kan complexes.


33.14. Proposition. If $j: A \to B$ is a monomorphism and a weak equivalence of simplicial sets, and $p: X \to Y$ is a Kan fibration between Kan complexes, then $p^{\Box j}: \text{Map}(B, X) \to \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$ is a trivial fibration.
Proof. The map $p^\Box_i$ is a Kan fibration between Kan complexes, using $\text{Horn} \Box \text{Cell} \subseteq \text{Horn}$ (32.7). Consider the diagram

$$
\begin{array}{ccc}
\text{Map}(B, X) & \xrightarrow{p^\Box_i} & \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y) \\
\downarrow & & \downarrow \\
\text{Map}(B, Y) & \xrightarrow{q} & \text{Map}(A, Y)
\end{array}
$$

in which the square is a pullback. By (33.11) the maps $q$ and $q'(p^\Box_i)$ are trivial fibrations. The pullback $q'$ of $q$ is also a trivial fibration, and so $p^\Box_i$ is a weak equivalence by 2-out-of-3 (32.2), and therefore a trivial fibration since it is a Kan fibration between Kan complexes (33.7). \qed

We also obtain another characterization of Kan fibrations between Kan complexes.

33.15. Corollary. A map $p: X \to Y$ between Kan complexes is a Kan fibration if and only if $j \Box p$ for all $j$ which are monomorphisms and weak equivalences.

Proof. ($\Leftarrow$) Straightforward, since inner horn inclusions are monomorphisms and weak equivalences. ($\Rightarrow$) Immediate from the previous proposition. \qed

34. The fiberwise criterion for trivial fibrations

We give another criterion for Kan fibration to be a trivial fibration, in terms of its fibers.

34.1. Fiberwise criterion for trivial fibrations. The fiber $p^{-1}(y)$ of a map $p: X \to Y$ over a vertex $y \in Y_0$ is the pullback of $p$ along $\{y\} \to Y$.

If $p: X \to Y$ is a trivial fibration, then since $\text{TrivFib} = \text{Horn}^{\Box}$ we see immediately that every projection $p^{-1}(y) \to \ast$ from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes.

34.2. Proposition. Let $p: X \to Y$ be a Kan fibration. Then $p$ is a trivial fibration if and only if every fiber of $p$ is a contractible Kan complex.

Proof. We have just observed ($\Rightarrow$). So suppose $p$ is a Kan fibration whose fibers are contractible Kan complexes, and consider a lifting problem

$$
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{a} & X \\
\downarrow & & \downarrow p \\
\Delta^n & \xrightarrow{\gamma} & Y
\end{array}
$$

We will construct a lift $t$ by using the covering homotopy extension property to “deform” this to a lifting problem involving a single fiber of $p$, which admits a solution by the hypothesis on the fibers of $p$.

Let $\gamma: \Delta^n \times \Delta^1 \to \Delta^n$ be the unique map given on vertices by $\gamma(i, 0) = i$ and $\gamma(i, 1) = n$. Thus $\gamma|\Delta^n \times \{0\} = \text{id}$, while $\gamma|\Delta^n \times \{1\}$ factors through the vertex $\{n\} \subseteq \Delta^n$. Since $\partial \Delta^n \times \{0\} \subseteq \partial \Delta^n \times \Delta^1$ is anodyne by $\text{Horn} \Box \text{Cell} \subseteq \text{Horn}$, a lift exists in

$$
\begin{array}{ccc}
\partial \Delta^n \times \{0\} & \xrightarrow{a} & X \\
\downarrow & & \downarrow p \\
\partial \Delta^n \times \Delta^1 & \xrightarrow{\gamma} & \Delta^n \times \Delta^1 \\
& & \downarrow \gamma \\
& & \Delta^n \\
& & \text{TrivFib}
\end{array}
$$
Since \( b\gamma|\Delta^n \times \{1\} \) factors through a vertex \( y \in Y_0 \), we have a lift \( d \) in
\[
\begin{array}{ccc}
\partial \Delta^n \times \{1\} & \xrightarrow{c|\partial \Delta^n \times \{1\}} & p^{-1}(y) \\
\downarrow & j \downarrow & \downarrow p \\
\Delta^n \times \{1\} & \xrightarrow{y} & Y
\end{array}
\]
since \( p^{-1}(y) \) is a contractible Kan complex. Putting this together we obtain a commutative square
\[
\begin{array}{ccc}
(\partial \Delta^n \times \Delta^1) \cup_{\partial \Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{(c, jd)} & X \\
\downarrow & \downarrow & \downarrow p \\
\Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & Y
\end{array}
\]
which admits a lift \( s \) since \( (\partial \Delta^n \subset \Delta^n) \square \{1\} \subset \Delta^1 \) is anodyne. The restriction \( t := s|\Delta^n \times \{0\} \) provides a solution to the original lifting problem, since \( t|\partial \Delta^n \times \{0\} = c|\partial \Delta^n \times \{0\} = a \) and \( pt = (b\gamma)|\Delta^n \times \{0\} = b \). □

We often apply the above result in the following way.

34.3. **Corollary.** Suppose
\[
\begin{array}{ccc}
X' & \xrightarrow{p'} & X \\
\downarrow p & & \downarrow p \\
Y' & \xrightarrow{g} & Y
\end{array}
\]
is a pullback square such that (1) \( p \) is a Kan fibration and (2) \( g_0 : Y'_0 \to Y_0 \) is surjective. Then \( p \) is a trivial fibration if and only if \( p' \) is a trivial fibration. Furthermore, if all objects are Kan complexes, then \( p \) is a weak equivalence if and only if \( p' \) is a weak equivalence.

**Proof.** The fibers of \( p \) all appear as fibers of \( p' \) by (2). Use the fiberwise criterion (34.2) and (33.7). □

34.4. **Criterion for fully faithful map between Kan complexes.** As an application of the fiberwise criterion, we have the following criterion for a Kan fibration between Kan complexes to be fully faithful.

34.5. **Proposition.** A Kan fibration \( p : X \to Y \) between Kan complexes is fully faithful iff the induced map
\[
q = p^\square(\partial \Delta^1 \subset \Delta^1) : \operatorname{Map}(\Delta^1, X) \to \operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial \Delta^1, Y)} \operatorname{Map}(\partial \Delta^1, X)
\]
is a trivial fibration.

**Proof.** Consider the pullback square
\[
\begin{array}{ccc}
\prod_{x,x'} \operatorname{map}_X(x, x') & \xrightarrow{q} & \operatorname{Map}(\Delta^1, X) \\
\downarrow & & \downarrow q \\
\prod_{x,x'} \operatorname{map}_Y(fx, fx') & \xrightarrow{\operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial \Delta^1, Y)} \operatorname{Map}(\partial \Delta^1, X)} & \operatorname{Map}(\Delta^1, X)
\end{array}
\]
where the products are over all pairs of elements of $X_0$. The map along the bottom is a bijection on vertices. The map $q$ is a Kan fibration by enriched lifting (32.7), and all the objects are Kan complexes, again using (32.7). The result follows using (34.3). \qed

35. Fundamental theorem for Kan complexes

In this section, we will prove quasigroupoid version of the fundamental theorem (32.20), i.e., that fully faithful and essentially surjective maps between quasigroupoids (=Kan complexes) are weak equivalences. We note that we have already reduced (32.21) to the case of Kan fibrations.

35.1. Proposition. A Kan fibration $p: X \to Y$ between Kan complexes which is fully faithful and essentially surjective is a weak equivalence.

Proof. Consider

$$X \xrightarrow{j=(p, \text{id}_X)} \text{Map}(\Delta^1, Y) \times_Y X \xrightarrow{\pi_1} \text{Map}(\Delta^1, Y) \xrightarrow{(1)^*} Y$$

where the fiber product is constructed from $\text{Map}(\Delta^1, Y) \xrightarrow{(0)^*} Y \xrightarrow{p} X$, and $i: Y \to \text{Map}(\Delta^1, Y)$ is adjoint to the projection $Y \times \Delta^1 \to \Delta^1$. Since $(1)^* i = \text{id}_Y$ we see that $(1)^* \pi_1 j = p$.

A straightforward argument shows that the map $(0)^* : \text{Map}(\Delta^1, Y) \to Y$ is the projection part of a fiberwise deformation retraction, with section $i$; the homotopy is the adjoint to restriction along a suitable map $\Delta^1 \times \Delta^1 \to \Delta^1$. The projection map $\pi_2 : \text{Map}(\Delta^1, Y) \times_Y X \to X$ is the base change of $(0)^* \times p$, and so is also a fiberwise deformation retraction (33.3), and so is a simplicial homotopy equivalence, and hence a weak equivalence.

Since $\pi_2 j = \text{id}_X$, it follows that $j$ is a weak equivalence by 2-out-of-3 (32.2). Again using 2-out-of-3, we see that to show $p$ is a weak equivalence it suffices to show that $r_1 := (1)^* \pi_1$ is a weak equivalence.

Now consider the diagram

$$\begin{array}{ccc}
\text{Map}(\Delta^1, Y) \times_Y (X \times X) & \xrightarrow{\pi_2} & X \times X \\
\text{id} \times (\text{id} \times p) & & \pi_2 \\
\text{Map}(\Delta^1, Y) \times_Y (X \times Y) & \xrightarrow{\pi_2} & X \times Y \\
\pi_1 & & \pi_2 \\
\end{array}$$

in which the composite along the bottom is also $r_1$, and the rectangle is a pullback. We complete the proof using the fiberwise criterion (34.3) to show that $r_1$ (which is a Kan fibration) is a trivial fibration; i.e., we need that $p$ is surjective on 0-simplices and that $r_1'$ is a trivial fibration.

First, note that since $p$ is an essentially surjective Kan fibration, $p$ must actually be surjective on 0-simplices, by a straightforward argument using the fact that $\{0\} \subset \Delta^1 \sqcup p$.

Both projections on the top row are Kan fibrations, since $X \to \Delta^0$ and $\text{Map}(\Delta^1, Y) \to \text{Map}(\partial \Delta^1, Y)$ are Kan fibrations. Thus $r_1'$ is a Kan fibration between Kan complexes, so we only need to show that $r_1'$ is a weak equivalence, since then (33.7) will give that $r_1'$ is a trivial fibration.

Now consider

$$\text{Map}(\Delta^1, X) \xrightarrow{q} \text{Map}(\Delta^1, Y) \times_{\text{Map}(\partial \Delta^1, Y)} \text{Map}(\partial \Delta^1, X) \xrightarrow{r_1'} X$$

where $q$ is the evident box-power map. The composite $r_1' q$ is equal to the evident restriction map along \{1\} \subset \Delta^1. Since \{1\} \subset \Delta^1 is anodyne it follows that $r_1' q$ is a trivial fibration. Finally, since $p$ is fully faithful, (34.5) gives that $q$ is a trivial fibration, and hence it follows that $r_1'$ is a weak equivalence by 2-out-of-3. The proof is complete. \qed
36. Isofibrations

In this section, we return to isofibrations, which were defined in (28.6). The moral is that isofibrations between quasicategories play a role analogous to Kan fibrations between Kan complexes.

36.1. Characterizations of isofibrations. Recall that a functor \( f : C \to D \) between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every

\[
\begin{array}{ccc}
\Delta^1 & \rightarrow & C \\
\downarrow & & \downarrow p \\
\{j\} & \rightarrow & D
\end{array}
\]

with \( j = 0 \) such that \( f \) represents an isomorphism admits a lift \( g \) which is also represents isomorphism. Furthermore, it is equivalent to require \((2')\) instead of (2), where \((2')\) is the same statement with \( j = 1 \).

Note that \( C \to \ast \) is an isofibration for any quasicategory \( C \) (because identity maps are isomorphisms).

We have the following “lifting criterion” for isofibrations.

36.2. Proposition. An map \( p \) between quasicategories is an isofibration iff (1) it is an inner fibration and (2) \( \{\{0\} \subset N(Iso)\} \sqcup p \).

Proof. \((\Leftarrow)\) Straightforward, using the fact (32.15) that every \( f : \Delta^1 \to D \) representing an isomorphism factors through a map \( N(Iso) \to D \).

\((\Rightarrow)\) Let \( p \) be an isofibration. To solve the lifting problem

\[
\begin{array}{ccc}
\Delta^1 & \rightarrow & C \\
\downarrow & & \downarrow p \\
\{0\} & \rightarrow & D
\end{array}
\]

recall from the proof of (32.14) that \( N(Iso) = \bigcup F_k \) where \( F_k \) is obtained from \( F_{k-1} \) by gluing along \( \Lambda^0_k \subset \Delta^k \); we construct lifts \( s_k : F_k \to C \) inductively. A lift \( s_1 : F_1 = \Delta^1 \to C \) exists by the definition of isofibration, and we may assume that \( s_1 \) is an isomorphism in \( C \). Then the Joyal lifting theorem (28.13) provides lifts \( s_k \) for \( k \geq 2 \).

In other words, the isofibrations are precisely the maps between quasicategories which are contained in \( (\text{InnHorn} \cup \{\{0\} \subset N(Iso)\}) \sqcup \). In particular, the pullback of an isofibration along a map from a quasicategory is also a quasicategory.

36.3. Remark. We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called “categorical fibration”, and will be discussed later.

Here is another characterization of isofibrations in terms of cores (=maximal sub-quasigroupoids, defined in (10.7)). Remember that any functor \( p : C \to D \) restricts to a functor \( p^{\text{core}} : C^{\text{core}} \to D^{\text{core}} \) between Kan complexes.

36.4. Proposition. A map \( p : C \to D \) between quasicategories is a isofibration if and only if (1) it is an inner fibration, and (2) \( p^{\text{core}} : C^{\text{core}} \to D^{\text{core}} \) is a Kan fibration.

Proof. \((\Rightarrow)\) Let \( p \) be an isofibration. Then \( p^{\text{core}} \) is also an isofibration, by an elementary argument. (The point is that the relevant lifting problems for \( p^{\text{core}} \) clearly have lifts \( s \) with target \( C \), since \( p \) is an isofibration; an easy argument shows that the image of such lifts \( s \) must actually land in \( C^{\text{core}} \).)
Thus have reduced to showing that any isofibration between Kan complexes is a Kan fibration, which is a straightforward exercise using Joyal lifting (32.11).

(⇐) If \(p^\text{core}\) is a Kan fibration, then it is immediate that property (2) of an isofibration holds. \(\square\)

In particular, isofibrations between Kan complexes are precisely Kan fibrations. (This can be proved directly using Joyal lifting, as in (32.11).)

36.5. **Lifting properties for isofibrations.** We are now ready to prove the following proposition, which will be the key tool in what follows.

36.6. **Proposition.** Let \(p: C \to D\) be an isofibration between quasicategories, and \(i: K \to L\) any monomorphism of simplicial sets. Then the induced pullback-power map

\[
p^{\square i}: \text{Fun}(L, C) \to \text{Fun}(K, C) \times_{\text{Fun}(K,D)} \text{Fun}(L, D)
\]

is an isofibration.

**Proof.** Fix a map \(p: C \to D\) between quasicategories. We first note that the class maps \(i: K \to L\) such that \(p^{\square i}\) is an isofibration is weakly saturated. In fact, let \(S := \text{InnHorn} \cup \{ \{0\} \subset N(\text{Iso}) \} \).

Two applications of (16.4) show that \(S \sqcup (p^{\square i})\) iff \((S \sqcup i) \sqcup p\) iff \((p^{\text{SS}})\). Therefore, (36.2) implies that the class \(C := \{ i \mid p^{\square i} \in \text{IsoFib} \}\) is the left complement of \(p^{\text{JS}}\), and thus weakly saturated.

Therefore, to show that \(C\) contains all monomorphisms, it suffices show that it contains \(i = (\partial \Delta^n \subset \Delta^n)\) for \(n \geq 0\).

Note that we will certainly have that \(p^{\square i}\) is an inner fibration, using \(\text{InnHorn} \cap \text{Cell} \subset \text{InnHorn}(16.7)\).

If \(n = 0\), then \(p^{\square i} = p\) so the claim is trivial.\(^{28}\)

Now assume \(n \geq 1\). Since \(p^{\square i}\) is an inner fibration between quasicategories, it suffices to solve the lifting problem

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & \text{Fun}(\Delta^n, C) \\
\downarrow & & \downarrow ^{p^{\square i}} \\
\Delta^1 & \underset{f}{\longrightarrow} & \text{Fun}(\partial \Delta^n, C) \times_{\text{Fun}(\partial \Delta^n, D)} \text{Fun}(\Delta^n, D)
\end{array}
\iff
\begin{array}{ccc}
(\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial \Delta^n) & \longrightarrow & C \\
\downarrow & & \downarrow ^{p} \\
\Delta^1 \times \Delta^n & \longrightarrow & D
\end{array}
\]

where \(f\) represents an isomorphism in the target.

The edge \(f' := (g|\Delta^1 \times \Delta^0)\) is the same as the composite

\[
\Delta^1 \underset{f}{\longrightarrow} \text{Fun}(\partial \Delta^n, C) \times_{\text{Fun}(\partial \Delta^n, D)} \text{Fun}(\Delta^n, C) \longrightarrow \text{Fun}(\partial \Delta^n, C) \longrightarrow \text{Fun}(\{0\}, C).
\]

Since \(f\) is an isomorphism in the fiber product, it follows that \(f'\) is an isomorphism in \(C\). Therefore a lift exists by the pushout-product version of Joyal lifting (29.12), since \(n \geq 1\). \(\square\)

One consequence of the above is that if \(p\) is an isofibration and \(i\) is a monomorphisms, then

\[
(p^{\square i})^{\text{core}}: \text{Fun}(L, C)^{\text{core}} \to (\text{Fun}(K, C) \times_{\text{Fun}(K,D)} \text{Fun}(L, D))^{\text{core}}
\]

is a Kan fibration (36.4). It turns out we can replace the target (the core of a pullback) with a pullback of cores.

36.7. **Corollary.** If \(p: C \to D\) is any functor between quasicategories and \(K \subseteq L\), then

\[
(\text{Fun}(K, C) \times_{\text{Fun}(K,D)} \text{Fun}(L, D))^{\text{core}} = \text{Fun}(K, C)^{\text{core}} \times_{\text{Fun}(K,D)^{\text{core}}} \text{Fun}(L, D)^{\text{core}}.
\]

\(^{28}\)This step is the only place in the proof where we actually use the fact that \(p\) is an isofibration, and not merely an inner fibration! In fact, if \(p\) is merely an inner fibration, but \(K_0 = L_0\), then \(p^{\square i}\) is an isofibration. This proof is closely related to that of (29.10).
Proof. Both sides of (36.8) can be regarded as subobjects of $\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$, which we note is a quasicategory since $r: \text{Fun}(L, D) \to \text{Fun}(K, D)$ is an inner fibration and $\text{Fun}(K, C)$ is a quasicategory. The left-hand side is clearly contained in the right-hand side, since any functor between quasicategories takes isomorphisms to isomorphisms. By (36.6), $r: \text{Fun}(L, D) \to \text{Fun}(K, D)$ is actually a isofibration, and so $r^\text{core}$ is a Kan fibration (36.4). Thus the right-hand side of (36.8) is a pullback of Kan complexes along a Kan fibration, and thus is a Kan complex, which is necessarily contained in the left-hand side of (36.8). □

36.9. Trivial fibrations between quasicategories. Now we can prove a generalization of (33.7), which identified trivial fibrations between Kan complexes as the Kan fibrations which are weak equivalences.

36.10. Proposition. Let $p: C \to D$ be a map between quasicategories. Then $p$ is a trivial fibration if and only if it is an isofibration and a categorical equivalence.

Proof. [Joy08a, Theorem 5.15]. (⇒) If $p$ is a trivial fibration, it is an inner fibration and $(\{0\} \subset N(\text{Iso})) \sqsubset p$, so it is an isofibration (36.2). We have already shown that $p$ is a categorical equivalence (20.9).

(⇐) Conversely, suppose $p$ is isofibration and categorical equivalence. It suffices to show that for any inclusion $i: K \subseteq L$, the map

$$g := (p^\Box)^\text{core}: \text{Fun}(L, C)^\text{core} \to \text{Fun}(K, C)^\text{core} \times_{\text{Fun}(K, D)^\text{core}} \text{Fun}(L, D)^\text{core}$$

is surjective on 0-simplicies. In fact, we will show that $g$ is a trivial fibration, which implies what we want. By (36.6), the map $p^\Box: \text{Fun}(L, C) \to \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$ is an isofibration; by (36.4), the restriction $(p^\Box)^\text{core}$ of this map to cores is a Kan fibration; by (36.7), this restriction is precisely $g$. Thus $g$ is a Kan fibration between Kan complexes. It will therefore suffice by (33.7) to show that $g$ is also a categorical equivalence.

The maps

$$\text{Fun}(L, C) \to \text{Fun}(L, D), \quad \text{Fun}(K, C) \to \text{Fun}(K, D)$$

are categorical equivalences since $p$ is, and so induce categorical equivalences on cores. These maps are also isofibrations by (36.6), and therefore the restrictions to cores are Kan fibrations. Thus $\text{Fun}(L, C)^\text{core} \to \text{Fun}(L, D)^\text{core}$ and $\text{Fun}(K, C)^\text{core} \to \text{Fun}(K, D)^\text{core}$ are Kan fibrations and weak equivalences between Kan complexes, and thus are trivial fibrations (33.7). Thus in

$$\text{Fun}(L, C)^\text{core} \xrightarrow{g} \text{Fun}(K, C)^\text{core} \times_{\text{Fun}(K, D)^\text{core}} \text{Fun}(L, D)^\text{core} \to \text{Fun}(L, D)^\text{core}$$

the second map is a trivial fibration being a pullback of $\text{Fun}(K, p)^\text{core}$, and the composite is a weak equivalence. It follows that $g$ is a weak equivalence by the 2-out-of-3 property for weak equivalences (32.2), and the result is proved.

This proof made essential use of our characterization of trivial fibrations between Kan complexes, which is why we had to prove that special case first.

Now we show that isofibrations have liftings with respect to all monomorphisms which are categorical equivalences.

36.11. Proposition. A map $p: C \to D$ with $D$ a quasicategory is an isofibration if and only if $j \sqsubset p$ for every $j: K \to L$ which is both a monomorphism and a categorical equivalence.

Proof. (⇐) Immediate from the the characterization of isofibrations as maps between quasicategories in the right complement of $\text{InnHorn} \cup \{\{0\} \subset N\text{Iso}\}$ (36.2).

(⇒) Suppose $p$ is an isofibration. It suffices to show that for $j \in \text{Cell} \cap \text{CatEq}$,

$$q: \text{Fun}(K, C) \to \text{Fun}(L, C) \times_{\text{Fun}(L, D)} \text{Fun}(K, D)$$

is a trivial fibration since 36.10. Proposition.

□
is a trivial fibration, whence it is surjective on 0-simplices and thus $j \square p$. Since $j$ is a monomorphism, (36.6) says $q$ is an isofibration between quasicategories, so by the criterion for an isofibration to be a trivial fibration (36.10) it suffices to show that $q$ is a categorical equivalence. Since $j$ is a categorical equivalence both $\text{Fun}(j, C)$ and $\text{Fun}(j, D)$ are categorical equivalences. The map $\text{Fun}(j, D)$ is also an isofibration by (36.6), so it is a trivial fibration by (36.10). Therefore the basechange $\text{Fun}(L, C) \times_{\text{Fun}(L, D)} \text{Fun}(K, D) \to \text{Fun}(L, C)$ of $\text{Fun}(j, D)$ is a trivial fibration, and the result follows using the 2-out-of-3 property of categorical equivalences (22.9).

\section*{36.12 \textbf{Monomorphisms which are categorical equivalences}}

We can now prove a generalization of (33.11), which characterized the injective weak equivalences.

\section*{36.13 \textbf{Proposition}}

Let $j : K \to L$ be a monomorphism of simplicial sets. Then $j$ is a categorical equivalence if and only if $\text{Map}(j, C) : \text{Map}(L, C) \to \text{Map}(K, C)$ is a trivial fibration for all quasicategories $C$.

\textit{Proof.} Straightforward using the fact that $\text{Map}(j, C)$ is an isofibration for any inclusion (36.6), and that isofibrations which are categorical equivalences are trivial fibrations (36.10). \hfill $\Box$

\section*{36.14 \textbf{Remark}}

As a consequence of (36.13), we see that the class $\text{Cell} \cap \text{CatEq}$ of injective categorical equivalences between simplicial sets is a weakly saturated class: it is the left complement of the class of maps of the form $p^\square$ where $p : C \to \Delta^0$ is a projection from a quasicategory, and $i$ is a cell inclusion. Furthermore, this class contains the class of inner anodyne maps.

This class is not the same as $\text{InnHorn}$. For instance, every inner anodyne map is a bijection on vertices, but $\{0\} \to N\text{Iso}$ which is not bijective on vertices is an injective categorical equivalence. Neither is it the same as the weak saturation of $\text{InnHorn} \cup \{\{0\} \subset N\text{Iso}\}$. (\textbf{Proof?}) This is a significant way in which the theory of quasicategories is not entirely parallel with the theory of Kan complexes.

\section*{37. Localization of Quasicategories}

\subsection*{37.1 \textbf{Quasigroupoidification}}

Let $C$ be a quasicategory, and $X$ a simplicial set. Let

\[ \text{Fun}(X)(X, C) \subseteq \text{Fun}(X, C) \]

denote the full subcategory spanned by objects which are functors $f : X \to C$ with the property that $f(X) \subseteq C^{\text{core}}$.

Note that $\text{Fun}(X)(X, C)$ is a quasicategory, but not necessarily a quasigroupoid: for instance, morphisms are functors $f : X \times \Delta^1 \to C$ such that $f(X \times \{0\}) \subseteq C^{\text{core}}$ and $f(X \times \{1\}) \subseteq C^{\text{core}}$, but need not satisfy $f(X \times \Delta^1) \subseteq C^{\text{core}}$.

\subsection*{37.2 \textbf{Lemma}}

Consider $\Delta^1 \subset N(\text{Iso})$ representing the map $0 \to 1$ in $\text{Iso}$. The restriction map $\text{Fun}(N(\text{Iso}), C) \to \text{Fun}(\Delta^1, C)$ factors through a trivial fibration

\[ \text{Fun}(N(\text{Iso}), C) \to \text{Fun}(\Delta^1)(\Delta^1, C). \]

\textit{Proof.} We’ve already proved this is surjective on zero-simplices (32.15). In that proof, we noted that $N(\text{Iso}) = \bigcup F_k$ with $F_k = F_{k-1} \cup_{\Lambda^k_0} \Delta^k$ and $F_1 = \Delta^1$, and the result was immediate using the Joyal extension theorem.

To prove this lemma, it suffices to show that each $\text{Fun}(\Delta^1)(F_k, C) \to \text{Fun}(\Delta^1)(F_{k-1}, C)$ is a trivial fibration for $k \geq 2$. There are pullback squares

\[
\begin{array}{ccc}
\text{Fun}(\Delta^1)(F_k, C) & \longrightarrow & \text{Fun}(\Delta^1)(\Delta^k, C) \\
\downarrow & & \downarrow \\
\text{Fun}(\Delta^1)(F_{k-1}, C) & \longrightarrow & \text{Fun}(\Delta^1)(\Lambda^k_0, C)
\end{array}
\]
so it suffices to solve the lifting problems \((\partial \Delta^n \subset \Delta^n) \sqsubset (\Fun(\Delta^{(0,1)})(\Delta^k, C) \to \Fun(\Delta^{(0,1)})(\Lambda^n_0, C))\) when \(n \geq 0\) and \(k \geq 0\). This adjoints to the lifting problem

\[
\begin{array}{ccc}
\Delta^{(0,1)} & \xrightarrow{f} & \Lambda^n_0 \\
\downarrow & & \downarrow \\
\Delta^k & \xrightarrow{g} & \Fun(\partial \Delta^n, C)
\end{array}
\]

where \(f\) is such that for each \(j \in [n]\) the composite \(\Delta^{(0,1)} \to \Fun(\Delta^n, C) \to \Fun(\{j\}, C) \approx C\) represents an isomorphism in \(C\). By the objectwise criterion for natural isomorphisms \((C)\) we see that \(f\) itself represents an isomorphism in \(\Fun(\Delta^n, C)\), so the desired lift exists by Joyal lifting \((28.13)\).

37.3. **Proposition.** Let \(i : X \to X'\) be any anodyne map to a Kan complex \(X'\). Then the restriction map \(\Fun(i, C) : \Fun(X', C) \to \Fun(X, C)\) factors through a trivial fibration

\[q : \Fun(X', C) \to \Fun^{(X)}(X, C)\]

**Proof.** Every edge in \(X\) maps to an isomorphism in \(X'\). Therefore \(\Fun(i, C)\) must factor through a map \(q\) into the subcomplex \(\Fun^{(X)}(X, C) \subseteq \Fun(X, C)\). We also know that \(\Fun(i, C)\) is an isofibration by \((36.6)\), and therefore the map \(q\) is also an isofibration. Therefore by \((36.10)\) it is enough to show that for each such \(i\) the map \(q\) is a categorical equivalence.

Given any two anodyne maps \(i : X \to X'\) and \(i' : X \to X''\) to Kan complexes, there exists a categorical equivalence \(f : X' \to X''\) such that \(fi = i'\) (because \(\Fun(X', X'') \to \Fun(X, X'')\) is a trivial fibration by \((32.7)\)). Therefore in the commutative diagram

\[
\begin{array}{ccc}
\Fun(X', C) & \xrightarrow{\Fun(f, C)} & \Fun(X'', C) \\
\downarrow q & & \downarrow q' \\
\Fun^{(X)}(X, C) & & 
\end{array}
\]

the map \(\Fun(f, C)\) is a categorical equivalence, so \(q\) is a categorical equivalence iff \(q'\) is.

Thus, given a simplicial set \(X\), to show that any anodyne \(i : X \to X'\) to a Kan complex induces a categorical equivalence \(q : \Fun(X', C) \to \Fun^{(X)}(X, C)\), it suffices to do so for a single choice of such an \(i\).

Now given any simplicial sets \(X\), construct maps \(f\) and \(g\) of the form

\[X \xrightarrow{f} Y := X \cup \bigcup \Delta^1 \prod N(\text{Iso}) \xrightarrow{g} X',\]

where the coproducts defining \(Y\) are taken over all maps \(\Delta^1 \to X\), and \(g\) is any inner anodyne map to a quasicategory. By construction \(f\) and \(g\) are anodyne, and therefore \(i = gf\) is anodyne.

I claim that the quasicategory \(X'\) is actually a Kan complex, i.e., a quasigrouploid, i.e., a quasicategory such that \(h(X')\) is a groupoid. To see this, note that every edge in \(Y\) factors through a map \(\text{Iso} \to Y\) by construction; from this, we see that every edge in the simplicial set \(Y\) must be a preisomorphism, whence its fundamental category \(hY\) is a groupoid. Since \(g\) is inner anodyne, \(hY \to hX'\) is an equivalence, so \(hX'\) is a groupoid as desired.

Now \(\Fun(i, C)\) is the composite of

\[\Fun(X', C) \xrightarrow{\Fun(g, C)} \Fun(Y, C) \xrightarrow{g} \Fun^{(X)}(X, C) \subseteq \Fun(X, C)\]
Since $g$ is inner anodyne, $\text{Fun}(g,C)$ is a trivial fibration by (36.6). The map $p$ is a pullback of maps $\text{Fun}(N(\text{Iso}),C) \to \text{Fun}(\Delta^1)(\Delta^1,C)$, which are also trivial fibrations by (37.2). The result is proved.

We write $X \to X_{\text{Kan}}$ for any choice of anodyne map to a Kan complex, and call it a quasi-groupoidification of $X$. What we have shown is that for any quasicategory $C$ we get a categorical equivalence

$$\text{Fun}(X_{\text{Kan}},C) \approx \text{Fun}^{(X)}(X,C).$$

We can apply this construction when $X$ is a quasicategory, or even when $X$ is the nerve of an ordinary category, and obtain interesting new Kan complexes.

37.4. Example. It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact the nerve of some poset [Tho80]. Thus, for every Kan complex $K$, there exists an ordinary category $A$ and a weak equivalence $NA \to K$, and hence a categorical equivalence $(NA)_{\text{Kan}} \to K$ between Kan complexes.

We note that there is also a classical groupoidification construction, which given an ordinary category $A$ produces an ordinary groupoid $A_{\text{Gpd}}$ by “formally inverting all maps”. We have that $h((NA)_{\text{Kan}}) \approx N(A_{\text{Gpd}})$, but in general $(NA)_{\text{Kan}}$ is not weakly equivalent to $N(A_{\text{Gpd}})$.

37.5. Exercise. Let $A$ be the poset of proper and non-empty subsets of $\{0,1,2,3\}$. Show that $A_{\text{Gpd}}$ is equivalent to the one-object category, but that $(NA)_{\text{Kan}}$ is not equivalent to the one-object category. (In the second case, the idea is that the geometric realization of $NA$ is a 2-sphere. Explicitly, you can prove non-equivalence by showing $\pi_0\text{Fun}(NA,K(\mathbb{Z},2)) \approx \mathbb{Z}$, using the Eilenberg-MacLane object of §8.9.)

37.6. Localization of quasicategories. There is a more general construction, which applies to a simplicial set $X$ equipped with a subcomplex $W \subseteq X$. Let

$$\text{Fun}^{(W)}(X,C) \subseteq \text{Fun}(X,C)$$

denote the full subcategory spanned by objects $f : X \to C$ such that $f(W) \subseteq C^{\text{core}}$. Note that this really only depends on the 1-simplices in $W$.

37.7. Proposition. Given an inclusion $W \subseteq X$, choose an anodyne map $W \to W_{\text{Kan}}$ to a Kan complex, and then choose an inner anodyne map $g : Y := X \cup_W W_{\text{Kan}} \to X'$ to a quasicategory. Then for any quasicategory $C$, the restriction map $\text{Fun}(X',C) \to \text{Fun}(X,C)$ factors through a trivial fibration

$$\text{Fun}(X',C) \to \text{Fun}^{(W)}(X,C).$$

Proof. We have

$$\begin{array}{ccc}
\text{Fun}(Y,C) & \overset{\text{Fun}(g,C)}{\longrightarrow} & \text{Fun}(X',C) \\
\downarrow & & \downarrow \\
\text{Fun}(W_{\text{Kan}},C) & \overset{p}{\longrightarrow} & \text{Fun}^{(W)}(W,C) \\
\end{array}$$

in which both squares are pullbacks. The map $\text{Fun}(g,C)$ is a trivial fibration since $g$ is inner anodyne, while $p$ is a trivial fibration as we have shown (37.3).

We sometimes write $X \to X_{\{W\}}$ for any map $X \to X'$ obtained as in the proposition. Note that any $X \to X_{\{X\}}$ is an example of $X \to X_{\text{Kan}}$. The observation is that for any quasicategory $C$, we have a categorical equivalence

$$\text{Fun}^{(W)}(X,C) \approx \text{Fun}(X_{\{W\}},C).$$
37.8. **Quasicategories from relative categories.** A relative category is a pair \( W \subseteq C \) consisting of an ordinary category \( C \) and a subcategory \( W \) containing all the objects of \( C \). The above construction gives, for any relative category, a map

\[
C \to C(W),
\]
unique up to categorical equivalence. We may call \( C(W) \) the localization of \( C \) with respect to \( W \).

It turns out that many quasicategories of interest arise as such localizations.

38. **The path fibration**

To prove the fundamental theorem of quasicategories for a general map between quasicategories, we will reduce to the special case of isofibrations. We do this by means of the “path fibration” construction, which provides a factorization of a map into a categorical fibration followed by an isofibration.

38.1. **The path fibration for quasicategories.** Let \( f : C \to D \) be a functor between quasicategories. We define a factorization \( C \xrightarrow{j} P(f) \xrightarrow{p} D \) by means of the following diagram.

\[
\begin{array}{ccc}
C & \xrightarrow{j} & P(f) \\
\downarrow{s_0} & & \downarrow{p} \\
C & \xrightarrow{f} & D
\end{array}
\]

Here the square is a pullback square. The map \( j \) is the unique one so that \( s_0 j = \text{id}_C \), and \( t j \) is induced by \( C \xrightarrow{j} D \xrightarrow{\pi} \text{Fun}(\Delta^1, D) \) where \( \pi \) is adjoint to the projection \( D \times \Delta^1 \to D \). The maps \( r_1 : \text{Fun}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D) \to D \) are induced by restriction along \( \{i\} \subset \Delta^1 \).

In particular, the objects of \( P(f) \) are pairs \( (c, \alpha) \) consisting of an object \( c \in C_0 \) and an isomorphism \( \alpha : f(c) \to d \) in \( D \). The map \( j \) sends an object \( c \) to \( (c, 1_{f(c)}) \), while \( p \) sends \( (c, \alpha) \) to \( d \).

The factorization \( C \xrightarrow{j} P(f) \xrightarrow{p} D \) is called the path fibration of \( f \), because of the following.

38.2. **Lemma.** The map \( j \) is a categorical equivalence and \( p \) is an isofibration.

**Proof.** For \( i \in \{0,1\} \) the inclusion \( \{i\} \subset N(\text{Iso}) \) is an equivalence of ordinary categories, hence a categorical equivalence. Therefore the composite map

\[
\text{Fun}(N(\text{Iso}), D) \xrightarrow{q} \text{Fun}(\Delta^1, D) \xrightarrow{r_1} D
\]

induced by restriction along \( \{i\} \subset N(\text{Iso}) \) is a categorical equivalence. We have shown (37.2) that \( q \) is a trivial fibration, and therefore the \( r_1 \) are categorical equivalences by 2-out-of-3 (22.9).

The restriction map \( \text{Fun}(\Delta^1, D) \to \text{Fun}(\partial \Delta^1, D) = D \times D \) is an isofibration (36.6). We claim that \( r : \text{Fun}(\Delta^1, D) \to D \times D \) is also an isofibration. It easily seen that \( r \) is an inner fibration, since \( \text{Fun}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D) \) is full. To show that \( r \) is also isofibration is then straightforward (e.g., if \( \alpha : \alpha_0 \to \alpha_1 \) is a natural isomorphism of functors \( \Delta^1 \to D \), and \( \alpha_0(\Delta^1) \subseteq D^{\text{core}} \), then \( \alpha_1(\Delta^1) \subseteq D^{\text{core}} \)).

Since the projections \( D \times D \to D \) are isofibrations, it follows that the \( r_1 \) are isofibrations, and hence trivial fibrations since they are categorical equivalences (36.10).

Therefore \( s_0 \), being a pullback of \( r_0 \), is a trivial fibration. Therefore \( P(f) \) is a quasicategory and \( j \) is a categorical equivalence.
In the commutative diagram

\[
\begin{array}{ccc}
P(f) & \rightarrow & \text{Fun}(\Delta^1, D) \\
\downarrow^p & & \downarrow^{r=(r_0,r_1)} \\
C \times D & \rightarrow & D \times D \\
\downarrow^s & & \\
D & \rightarrow & F \times \text{id}_D
\end{array}
\]

the square is a pullback. We have shown that \( r \) is an isofibration, and hence so is its pullback \( s \). Therefore \( p = \pi s \) is an isofibration, as desired.

\[\square\]

38.3. The path fibration for Kan complexes. If \( f: C \rightarrow D \) is a functor between quasicategories, and \( D \) is a Kan complex, then \( \text{Fun}(\Delta^1, D) = \text{Fun}(\Delta^1, D) \), and the diagram defining the path fibration takes the form

\[
\begin{array}{ccc}
C & \rightarrow & \text{Fun}(\Delta^1, D) \\
\downarrow^j & & \downarrow^{r_0} \\
C & \rightarrow & D \\
\downarrow^{so} & & \\
D & \rightarrow & D
\end{array}
\]

38.4. Proposition. If \( f: C \rightarrow D \) is a functor between Kan complexes, then \( j \) is a weak equivalence and \( p \) is a Kan fibration.

Proof. Immediate. \[\square\]

39. Proof of the fundamental theorem

We are ready to finish the proof of (E), The Fundamental Theorem of Quasicategories.

39.1. Proposition. If \( f: C \rightarrow D \) is a fully faithful and essentially surjective functor between quasicategories, then \( f \) is a categorical equivalence.

We will prove this below. First note that the quasigroupoid version of this result (which we proved earlier) gives us the following.

39.2. Lemma. If \( f: C \rightarrow D \) is a fully faithful and essentially surjective functor between quasicategories, then \( f: C^{\text{core}} \rightarrow D^{\text{core}} \) is a categorical equivalence (in fact, a weak equivalence).

Proof. Note that if \( f \) is fully faithful and essentially surjective, so is \( f^{\text{core}} \). Apply (32.20). \[\square\]

Note that the path fibration construction gives us a factorization of \( f \) into \( C \xrightarrow{j} P(f) \xrightarrow{p} D \), where \( j \) is a categorical equivalence and \( p \) is an isofibration. Because both categorical equivalences and (fully faithful + essentially surjective) are classes which satisfy 2-out-of-3 (22.9), (31.5), proving the fundamental theorem reduces to the case that \( f \) is itself an isofibration.

To prove the isofibration case of (39.1), we will deduce it from the following.

39.3. Proposition. If \( p: C \rightarrow D \) is an isofibration which is fully faithful and essentially surjective, then \( q = (p^{\square})^{\text{core}}: \text{Fun}(L,C)^{\text{core}} \rightarrow \text{Fun}(K,C)^{\text{core}} \times_{\text{Fun}(K,D)^{\text{core}}} \text{Fun}(L,D)^{\text{core}} \) is a trivial fibration for every monomorphism \( i: K \rightarrow L \).

Proof that (39.3) implies (39.1). As noted above, it is enough to consider isofibrations \( p: C \rightarrow D \) which are fully faithful and essentially surjective. By (39.3), for any monomorphism \( i: K \rightarrow L \) the map \( q = (p^{\square})^{\text{core}} \) is a trivial fibration, and therefore surjective on vertices. The core of a quasicategory has all its objects, and thus the box power map \( p^{\square}: \text{Fun}(L,C) \rightarrow \text{Fun}(K,C) \times_{\text{Fun}(K,D)} \text{Fun}(L,D) \) is surjective on vertices. Thus, we have \( i \square p \) for every monomorphism \( i \), whence \( p \) is a trivial fibration, and thus a categorical equivalence. \[\square\]
39.4. **Proof of** (39.3). We start with the following lemma, which says that isofibrations between quasicategories which are trivial fibrations on cores are characterized by a lifting property.

39.5. **Lemma.** There exists a set of maps $S$ such that for any isofibration $q: C \to D$ between quasicategories, we have $S \varshoveleft q \in \text{TrivFib}$.  

**Proof.** Given an inclusion $K \subseteq L$ of simplicial sets, we can use two applications of the small object argument (13.10) to construct

\[
\begin{array}{ccc}
K & \longrightarrow & K_{\text{Kan}} \\
\downarrow & & \downarrow \nwarrow \\
L & \longrightarrow & L' & \longrightarrow & L_{\text{Kan}}
\end{array}
\]

in which the square is pushout, the horizontal maps are anodyne, and the objects $K_{\text{Kan}}$ and $L_{\text{Kan}}$ are Kan complexes. (So $K \to K_{\text{Kan}}$ and $L \to L_{\text{Kan}}$ are examples of quasigroupoidification as in (37.1).

We will show that $(K \subseteq L) \otimes q_{\text{core}}$ iff $(K_{\text{Kan}} \subseteq L_{\text{Kan}}) \otimes q$. The lemma will follow immediately by taking $S = \{ (\partial \Delta^n)_{\text{Kan}} \subseteq (\Delta^n)_{\text{Kan}} \mid n \geq 0 \}$.

$(\Rightarrow)$ Suppose $(K \subseteq L) \otimes q_{\text{core}}$. Since $K_{\text{Kan}}$ and $L_{\text{Kan}}$ are Kan complexes, any maps from them to quasicategories must factor through cores. Thus it suffices to find a lift in the right-hand square of

\[
\begin{array}{ccc}
K & \longrightarrow & K_{\text{Kan}} \\
\downarrow & & \downarrow \nwarrow \\
L & \longrightarrow & L' & \longrightarrow & L_{\text{Kan}} \\
\downarrow & & & & \downarrow \nwarrow \\
C_{\text{core}} & \longrightarrow & C_{\text{core}} \\
\downarrow & & \downarrow \nwarrow \\
D_{\text{core}} & \longrightarrow & D_{\text{core}}
\end{array}
\]

By hypothesis, a lift $s$ exists, and therefore a lift $s'$ since the left-hand square is a pushout. Because $L' \to L_{\text{Kan}}$ is anodyne and $q_{\text{core}}$ is a Kan fibration (36.4), an extension to a lift $s''$ exists.

$(\Leftarrow)$ Suppose $(K_{\text{Kan}} \subseteq L_{\text{Kan}}) \otimes q$. Consider a lifting problem

\[
\begin{array}{ccc}
K & \longrightarrow & C_{\text{core}} \\
\downarrow & & \downarrow \nwarrow \\
L & \longrightarrow & D_{\text{core}}
\end{array}
\]

Because $C_{\text{core}}$ is a Kan complex and $K \to K_{\text{Kan}}$ is anodyne, the map $a$ factors through some $a': K_{\text{Kan}} \to C_{\text{core}}$, and there is a unique compatible map $b': L' \to D_{\text{core}}$ from the pushout along $K \subseteq L$. Again, $b'$ factors through $b'': L_{\text{Kan}} \to D_{\text{core}}$. Thus we have extended the original square to a diagram

\[
\begin{array}{ccc}
K & \longrightarrow & K_{\text{Kan}} \\
\downarrow & & \downarrow \nwarrow \\
L & \longrightarrow & L' & \longrightarrow & L_{\text{Kan}} \\
\downarrow & & & & \downarrow \nwarrow \\
C_{\text{core}} & \longrightarrow & C_{\text{core}} & \longrightarrow & C \\
\downarrow & & \downarrow \nwarrow & & \downarrow \nwarrow \\
D_{\text{core}} & \longrightarrow & D_{\text{core}} & \longrightarrow & D
\end{array}
\]

A lift $t$ exists by hypothesis, and since $L_{\text{Kan}}$ is a Kan complex it factors through a unique lift $t'$ (using that $C_{\text{core}} \to C$ and $D_{\text{core}} \to D$ are monomorphisms). The composite $L \to L_{\text{Kan}} \to C_{\text{core}}$ is the desired lift. \hfill $\square$

Fix an isofibration $p: C \to D$ between quasicategories which is fully faithful and essentially surjective. Consider the class

\[\mathcal{C}_p := \{ i \mid i \text{ is a monomorphism and } (p^{\square i})_{\text{core}} \in \text{TrivFib} \} .\]

The statement of (39.3) amounts to showing that $\mathcal{C}_p$ contains every monomorphism.
39.6. **Lemma.** The class $C_p$ is weakly saturated.

**Proof.** First note that for any monomorphism $i$, the map $p^{i!}$ is an isofibration since $p$ is (36.6). Using the set of maps $S$ provided by the previous lemma (39.5), for a monomorphism $i$ we have that $(p^{i!})^\text{core} \subseteq \text{TrivFib}$ if $S \sqcap (p^{i!})$ if $i \sqcap (p^{\boxtimes S})$. Thus $C_p$ is the intersection of $(p^{\boxtimes S})$ with $\text{Cell}$, and so is weakly saturated.

39.7. **Lemma.** Let $p : C \to D$ be an isofibration between quasicategories. If $K \subseteq L$, and if $(\emptyset \subseteq K)$ and $(\emptyset \subseteq L)$ are elements of $C_p$, then $(K \subseteq L) \in C_p$.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Map}(L,p)^\text{core} & \to & (\text{Map}(K,C) \times_{\text{Map}(K,D)} \text{Map}(L,D))^\text{core} \\
\downarrow & & \downarrow q \\
\text{Map}(K,C)^\text{core} & \to & \text{Map}(L,D)^\text{core}
\end{array}
$$

where the square is a pullback (36.7). By hypothesis, both $\text{Map}(K,p)^\text{core}$ and $\text{Map}(L,p)^\text{core}$ are trivial fibrations, whence $q$ is also a trivial fibration. By 2-out-of-3, we have that $(p^{\boxtimes i})^\text{core}$ is a categorical equivalence. Since $p$ is an isofibration, we have that $(p^{\boxtimes i})^\text{core}$ is a Kan fibration (36.6), (36.7) between Kan complexes, and therefore is a trivial fibration (33.7) as desired.

Next we observe that $(\emptyset \subseteq \Delta^n) \in C_p$ if $p$ is fully faithful and essentially surjective.

39.8. **Proposition.** If $p : C \to D$ is an isofibration which is fully faithful and essentially surjective, then $\text{Fun}(\Delta^n, C)^\text{core} \to \text{Fun}(\Delta^n, D)^\text{core}$ is a trivial fibration for all $n \geq 0$.

**Proof.** In the case $n = 0$, this means showing that $p^\text{core} : C^\text{core} \to D^\text{core}$ is a trivial fibration, which we have already observed (39.2).

For $n \geq 1$, consider the diagram

$$
\begin{array}{ccc}
\prod_{c_0, \ldots, c_n} \text{map}_C(c_0, \ldots, c_n) & \to & \text{Fun}(\Delta^n, C)^\text{core} \\
\downarrow & & \downarrow q \\
\prod_{c_0, \ldots, c_n} \text{map}_D(p(c_0, \ldots, c_n)) & \to & \text{Fun}(\Delta^n, D)^\text{core}
\end{array}
$$

in which all the squares are pullbacks. The map $r$ is a base change of $(p^{\times n+1})^\text{core} : (C^{\times n+1})^\text{core} \to (D^{\times n+1})^\text{core}$, which is isomorphic to the $(n+1)$-fold product of $p^\text{core} : C^\text{core} \to D^\text{core}$, which we have just noted is a trivial fibration. Thus, $r$ is a trivial fibration, so it will suffice to show that $q$ is a trivial fibration, for which we will use the fiberwise criterion (34.3).

We know that $p^{\boxtimes (\text{Sk}_0 \Delta^n \cap \Delta^n)}$ is an isofibration (36.6), and thus $q = (p^{\boxtimes (\text{Sk}_0 \Delta^n \cap \Delta^n)})^\text{core}$ is a Kan fibration (36.4). Because $p$ is an essentially surjective isofibration, $p^\text{core}$ is surjective on vertices, and thus $j$ is surjective on 0-simplices. Thus by the fiberwise criterion (34.3) we need to show that $\prod q_{c_0, \ldots, c_n}$ is a trivial fibration. Since coproducts of trivial fibrations are trivial fibrations (20.2) we thus reduce to showing that each $q_{c_0, \ldots, c_n}$ (which is a Kan fibration between Kan complexes being a pullback of $q$) is a weak equivalence, and hence a trivial fibration (33.7). This is immediate from the fact that $p$ is fully faithful and induces maps compatible with the weak
equivalences $\text{map}_C(c_0, \ldots, c_n) \to \text{map}_C(c_0, c_1) \times \cdots \times \text{map}_C(c_{n-1}, c_n)$ and $\text{map}_D(p c_0, \ldots, p c_n) \to \text{map}_D(p c_0, p c_1) \times \cdots \times \text{map}_D(p c_{n-1}, p c_n)$.

**Proof of (39.3)**. As we have already noted it suffices to show that
\[ \overline{\text{Cell}} \subseteq \mathcal{C}_p = \{ i \mid i \in \overline{\text{Cell}}, (p^{\square_1})^{\text{core}} \in \text{TrivFib} \} \]
for any fully faithful and essentially surjective isofibration $p: C \to D$. As $\mathcal{C}_p$ is weakly saturated (39.6), it is enough to show that $(\partial \Delta^n \subseteq \Delta^n) \in \mathcal{C}_p$ for all $n \geq 0$. We will do this by induction on $n$.

For the case $n = 0$, this is immediate from (39.8). For $n \geq 1$, suppose we have $(\partial \Delta^k \subseteq \Delta^k) \in \mathcal{C}_p$ for $k < n$. Since $\partial \Delta^n$ is equal to its own $(n-1)$-skeleton, skeletal filtration (15.18) gives that $(\emptyset \subseteq \partial \Delta^n) \in \mathcal{C}_p$. Then use (39.7) and (39.8) to conclude that $(\partial \Delta^n \subseteq \Delta^n) \in \mathcal{C}_p$ as desired. $\square$

**Part 5. Model categories**

**40. Categorical fibrations**

A map $p: X \to Y$ of simplicial sets is a **categorical fibration** if and only if $j \sqcap p$ for all $j$ which are monomorphisms and categorical equivalences. I’ll write $\text{CatFib}$ for the class of categorical fibrations.

Categorical fibrations generalize isofibrations. In fact, a map $p: C \to D$ with $D$ a quasicategory is a categorical fibration if and only if it is an isofibration, as we proved in (36.11).

**40.1. Proposition.** A map $p: X \to Y$ of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.

**Proof.** ($\Longrightarrow$) We have already proved this: (??) and (??). ($\Longleftarrow$) If $p$ is a categorical fibration and a categorical equivalence, factor $p$ as $X \overset{j}{\to} Z \overset{q}{\to} Y$ with $j$ a monomorphism and $q$ a trivial fibration. Then the usual argument shows that $p$ is a retract of $q$, using the fact that $j \sqcap p$ since $j$ is a categorical equivalence by $2$-of-$3$.

**40.2. Proposition.** If $p: X \to Y$ is a categorical fibration and $j: K \to L$ is a monomorphism, then $q: \text{Map}(L, X) \to \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$ is a categorical fibration. Furthermore, if either $j$ or $p$ is also a categorical equivalence, then so is $q$.

**Proof.** For the first, let $i: A \to B$ be a monomorphism which is a categorical equivalence. We have $i \sqcap q$ iff $(i \sqcap j) \sqcap p$. By definition of categorical fibration, it suffices to show that $i \sqcap j$ is a categorical equivalence, i.e., to show $\text{Map}(i \sqcap j, C)$ is a categorical equivalence for every quasicategory $C$. In fact, $\text{Map}(i, C)$ is an isofibration and a categorical equivalence, hence a trivial fibration, and therefore $j \sqcap \text{Map}(i, C)$.

If $p$ is also a categorical equivalence, then it is a trivial fibration, and the result follows.

If $j$ is also a categorical equivalence, then for any monomorphism $i$, we have $i \sqcap q$ iff $(i \sqcap j) \sqcap p$ iff $j \sqcap (p^{\sqcap_1})$. But $p^{\sqcap_1}$ is a categorical fibration by what we have just proved, so the result holds. $\square$

**40.3. Categorical fibrations and the small object argument.** Clearly, $\text{CatFib} = (\overline{\text{Cell}} \cap \text{CatEq})^{\square_1}$ is a right complement to a class of maps. We would like to know that $\text{CatFib}$ is the right complement to a set of maps; then we could use the small object argument to factor any map into an injective categorical equivalence followed by a categorical fibration.

Unfortunately, it’s apparently not known how to write down an explicit set of maps $S$ so that $S^{\square_1} = \text{CatFib}$. What is known is that such a set **exists**.

**40.4. Proposition.** There exists a set $S$ of maps of simplicial sets such that $\overline{S} = \overline{\text{Cell}} \cap \text{CatEq}$, whence $S^{\square_1} = \text{CatFib}$.
In the rest of this section we will sketch a proof. The idea is to show that CatFib is the right complement of the class of all injective categorical equivalences \( K \to L \) for which the number of simplices in \( K \) and \( L \) is bounded by some explicit cardinal \( \kappa \). We obtain \( S \) by choosing one representative for each isomorphism class in this class; then \( S \) is a set because of the cardinality bound.

We will define a detection functor \( F: \text{Fun}([1], \text{sSet}) \to \text{Fun}([1], \text{Set}) \) on categories of morphisms. This will have the following properties:

- For each map \( f: X \to Y \), the map \( F(f) \) is a monomorphism of sets.
- A map \( f: X \to Y \) is a categorical equivalence if and only if \( F(f) \) is a bijection.
- The functor \( F \) commutes with \( \kappa \)-filtered colimits for some regular cardinal \( \kappa \).
- The functor \( F \) takes \( \kappa \)-small simplicial sets to \( \kappa \)-small sets.

We define \( F \) as the composite of several intermediate steps.

Step 1: Recall that the small object argument gives a functorial way to factor a map \( f \) as \( f = p_i \), with \( i \in S \) and \( p \in S^\omega \). “Functorial factorization” means that we get a section of the functor \( \text{Fun}([2], \text{sSet}) \to \text{Fun}([1], \text{sSet}) \) defining composition.

We can apply this using \( S = \text{InnHorn} \). Thus, given any simplicial set, we functorially obtain an inner anodyne map \( X \to X_{\text{QCat}} \) to a quasicategory \( X_{\text{QCat}} \). As a result, we have a functor \( f \mapsto f_{\text{QCat}}: \text{Fun}([1], \text{sSet}) \to \text{Fun}([1], \text{sSet}) \), with the property that \( f \) is a categorical equivalence if and only if \( f_{\text{QCat}} \) is, and both source and target of \( f_{\text{QCat}} \) are quasicategories.

Step 2: Form the path fibration \( Q(f): P(f_{\text{QCat}}) \to Y_{\text{QCat}} \) of \( f_{\text{QCat}} \). The map \( Q(f) \) is thus an isofibration between quasicategories, and is a trivial fibration if and only if \( f \) is a categorical equivalence.

Step 3: Write \( Q(f): X' \to Y' \). Define \( E(f) \) to be the map of sets

\[
E(f): \prod_n \text{Hom}(\Delta^n, X') \to \prod_n \text{Hom}(\partial\Delta^n, X') \times_{\text{Hom}(\partial\Delta^n, Y')} \text{Hom}(\Delta^n, Y').
\]

Thus, \( f \) is a categorical equivalence if and only if \( E(f) \) is surjective.

Step 4: Write \( E(f): E_0(f) \to E_1(f) \), and define \( F(f) \) by

\[
F(f): \text{colim}[E_0(f) \times_{E_1(f)} E_0(f) \Rightarrow E_0(f)] \to E_1(f).
\]

In other words, \( F(f) \) is the map from the image of \( E(f) \) to \( E_1(f) \). Thus, \( F(f) \) is always a monomorphism, and \( f \) is a categorical equivalence if and only if \( F(f) \) is a bijection.

There exists a regular cardinal \( \kappa \) such that \( F \) commutes with \( \kappa \)-filtered colimits, and takes \( \kappa \)-small simplicial sets to \( \kappa \)-small sets. (In fact, we can take \( \kappa = \omega^+, \) the successor to the countable cardinal).

Using the detection functor, we can prove the following key lemma.

40.5. Lemma. Let \( f: X \subseteq Y \) be an inclusion which is a categorical equivalence. Every \( \kappa \)-small subcomplex \( A \subseteq Y \) is contained in a \( \kappa \)-small subcomplex \( B \subseteq Y \) with the property that \( B \cap X \subseteq B \) is a categorical equivalence.

Proof. For a subcomplex \( A \subseteq Y \) let \( f_A \) denote the inclusion \( A \cap X \subseteq A \). The collection of all \( \kappa \)-small subcomplexes of \( Y \) is \( \kappa \)-filtered. Thus

\[
\text{colim}_{\text{\(\kappa\)-small} A \subseteq Y} F(f_A) = F(f),
\]

which we have assumed is an isomorphism. Thus for any \( \kappa \)-small \( A \subseteq \) there must exist a \( \kappa \)-small \( A' \supset A \) such that a lift exists in

\[
\begin{array}{ccc}
F_0(f_A) & \longrightarrow & F_0(f_{A'}) \\
\downarrow & & \downarrow \\
F_1(f_A) & \longrightarrow & F_1(f_{A'})
\end{array}
\]
This is because $F_1(f_A)$ is a $\kappa$-small set, so any lift $F_1(f_A) \to F_0(F)$ factors through some stage of the $\kappa$-filtered colimit.

We use transfinite induction to obtain a sequence $\{A_i\}$ indexed by $i < \kappa$, where at limit ordinals we take a colimit. Set $B := \text{colim } A_i$. Because $\kappa$ is regular $|B| < \kappa$, and we have that $F(f_B)$ is an isomorphism by construction.

Consider the collection of monomorphisms $i: A \to B$ such that $i$ is a categorical equivalence and $|B| < \kappa$. Choose a set $S$ of such spanning all isomorphism classes of such maps; this is a set because of the cardinality bound. Clearly $S \subseteq \text{Cell} \cap \text{CatEq}$.

40.6. Proposition. We have $\mathcal{S} = \text{Cell} \cap \text{CatEq}$.

Proof. [Joy08a, D.2.16]. Given an injective categorical equivalence $X \subseteq Y$, we consider the following poset $\mathcal{P}$. The objects of $\mathcal{P}$ are subobjects $P \subseteq Y$ such that $X \subseteq P$ so that the inclusion $X \to P$ is contained in $\mathcal{S}$. The morphisms of $\mathcal{P}$ are inclusions $P \to Q$ of subobjects of $Y$ which are contained in $\mathcal{S}$. Because $\mathcal{S}$ is weakly saturated, the hypotheses of Zorn’s lemma apply to give a maximal element $M$ of $\mathcal{P}$. Since $X \subseteq Y$ is assumed to be a categorical equivalence, 2-out-of-3 gives that $M \subseteq Y$ is a categorical equivalence.

If $M = Y$ we are done, so suppose $M \neq Y$. Then there exists a $\kappa$-small $A \subseteq Y$ not contained in $M$, which by the above lemma can be chosen so that $A \cap M \subseteq A$ is a categorical equivalence, and thus an element of $S$. The pushout $M \subseteq A \cup M$ of this map is thus in $\mathcal{S}$ contradicting the maximality of $M$.

In particular, we learn that every map can be factored into an injective categorical equivalence followed by a categorical fibration.

41. The Joyal model structure on simplicial sets

41.1. Model categories. A model category (in the sense of Quillen) is a category $\mathcal{M}$ with three classes of maps: $W$, Cof, Fib, conventionally called weak equivalences, cofibrations, and fibrations, satisfying the following axioms.

- $\mathcal{M}$ has all small limits and colimits.
- $W$ satisfies the 2-out-of-3 property.
- $(\text{Cof} \cap W, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap W)$ are weak factorization systems (13.12).

Conventionally, an object $X$ is cofibrant if the map from the initial object is a cofibration, and fibrant if the map to the terminal object is a fibration.

41.2. Remark. The third axiom implies that Cof, Cof $\cap$ W, Fib, and Fib $\cap$ W are closed under retracts.

41.3. Exercise. Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. Hint: construct a factorization of $f$ which is itself a retract of a factorization of $g$\textsuperscript{29}.

41.4. Exercise (Slice model categories). Let $\mathcal{M}$ be a model category, and let $X$ be an object of $\mathcal{M}$. Show that the slice categories $\mathcal{M}_{/X}$ and $\mathcal{M}_{X/}$ admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under $\mathcal{M}_{/X} \to \mathcal{M}$ or $\mathcal{M}_{X/} \to \mathcal{M}$ are weak equivalences, cofibrations, and fibrations in $\mathcal{M}$.

\textsuperscript{29}In many formulations of model categories, weak equivalences being closed under retracts is taken as an axiom. The formulation we use is described in Riehl, “A concise definition of a model category” [Rie09], which gives a solution to this exercise.
41.5. **The Joyal model category.**

41.6. **Theorem (Joyal).** The category of simplicial sets admits a model structure, in which

- $W = \text{categorical equivalences } (\text{CatEq})$,
- $\text{Cof} = \text{monomorphisms } (\text{Cell})$,
- $\text{Fib} = \text{categorical fibrations } (\text{CatFib})$.

Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.

**Proof.** Categorical equivalences satisfy 2-out-of-3 by (22.9). We have that

- $\text{Cof} = \text{Cell}$ by definition,
- $\text{Fib} \cap W = \text{TFib} = \text{Cell}^{\mathbb{2}}$ by (40.1),
- $\text{Cof} \cap W = S$ for some set $S$ (40.4),
- $\text{Fib} = \text{CatFib} = (\text{Cof} \cap W)^{\mathbb{2}} = S^{\mathbb{2}}$ by definition,

so both $(\text{Cof} \cap W, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap W)$ are weak factorization systems via the small object argument (13.10). Thus, we get a model category.

We have shown (36.11) that the categorical fibrations $p: C \to D$ with $D$ a quasicategory are precisely the isofibrations. Applied when $D = \ast$, this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations. □

41.7. **Remark.** It is a standard fact that a model category structure is uniquely determined by its cofibrations and fibrant objects. Thus, the Joyal model structure is the unique model structure on simplicial sets with $\text{Cof} = \text{monomorphisms}$ and with fibrant objects the quasicategories.

41.8. **Cartesian model categories.** Recall that the category of simplicial sets is cartesian closed. A **cartesian model category** is a model category which is cartesian closed, with the following properties. Suppose $i: A \to B$ and $j: K \to L$ are cofibrations and $p: X \to Y$ is a fibration. Then

- $i \Box j: (A \times L) \cup_{A \times K} (B \times K) \to B \times L$
  is a cofibration, and is in addition a weak equivalence if either $i$ or $j$ is also a weak equivalence, and
- $p \Box j: \text{Map}(L, X) \to \text{Map}(K, X) \times_{\text{Map}(K, Y)} \text{Map}(L, Y)$
  is a fibration, and is in addition a weak equivalence if either $j$ or $p$ is also a weak equivalence.

In fact, we only need to specify one of the above two properties, as they imply each other.

41.9. **Proposition.** The Joyal model structure is cartesian.

**Proof.** This is just (40.2). □

42. **The Kan-Quillen model structure on simplicial sets**

A map $p: X \to Y$ is a **groupoidal fibration** if and only if $j \Box p$ for all $j$ which are monomorphisms and weak equivalences. I write GpdFib for the class of categorical fibrations.

42.1. **The Kan-Quillen model structure.**

42.2. **Theorem (Cisinski).** The category of simplicial sets admits a model structure, in which

- $W = \text{weak equivalences}$,
- $\text{Cof} = \text{monomorphisms}$,
- $\text{Fib} = \text{groupoidal fibrations}$.

Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.
Proof. This goes very much the same way as the Joyal model structure, and I won’t spell it out in detail. First build a detecting functor $F$ so that a map $f$ is a weak equivalence iff $F(f)$ is a bijection; this is just as in the categorical equivalence case, except that we make use of functorial replacement $X \mapsto X_{\text{Kan}}$ by Kan complexes, rather than by quasicategories. Using this, we can show that $\text{Cell} \cap \text{WkEq} = S$ and $\text{GpdFib} = S^\square$ for some set $S$, giving the factorization system $(\text{Cof} \cap \text{W}, \text{Fib})$.

We know that trivial fibrations are weak equivalences and are certainly groupoidal fibrations. The converse is proved just as in the categorical fibration case (40.1). This gives the other factorization system $(\text{Cof}, \text{Fib} \cap \text{W})$.

We have already proved that Kan fibrations between Kan complexes have the lifting property of groupoidal fibrations (33.15), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case. □

42.3. Proposition. The Quillen model structure is cartesian.

Proof. We must show that $p^{\square j}$ is a groupoidal fibration if $j$ is a monomorphism and $p$ a groupoidal fibration, and also that it is a weak equivalence if either $j$ or $p$ is. This is proved by an argument nearly identical to the proof of (40.2). □

42.4. Kan fibrations are groupoidal fibrations. The above model structure was actually first produced by Quillen. In Quillen’s formulation, the fibrations were taken to be the Kan fibrations. In fact, this is the same model structure, by

42.5. Proposition (Quillen). $\text{KFib} = \text{GpdFib}$.

We will not give a proof of this here. The non-trivial part is to show that $\text{KFib} \subseteq \text{GpdFib}$; note that we already know that a Kan fibration between Kan complexes is a groupoidal fibration by (33.15). This is usually done via an argument (due to Quillen) based on the theory of minimal fibrations. See for instance Quillen’s original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that $\text{KFib}$ is the weak cosaturation of the class of Kan fibrations between Kan complexes. In fact one can even show that every Kan fibration is a base change of a Kan fibration between Kan complexes, see [KLV12].

43. Model categories and homotopy colimits

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of weak equivalences.

43.1. Creating new model categories. Given a model category $\mathcal{M}$, many other categories related to it can also be equipped with model category structures, such as functor categories $\text{Fun}(C, \mathcal{M})$ where $C$ is a small category. We won’t consider general formulations of this here, but we will consider some special cases.

As an example, we consider the case of $C = [1] = \{0 \rightarrow 1\}$.

43.2. Proposition. There exists a model structure on $\mathcal{N} := \text{Fun}([1], \mathcal{M})$ in which a map $\alpha : X \to X'$ is

- a weak equivalence if $\alpha(i) : X(i) \to X'(i)$ is a weak equivalence in $\mathcal{M}$ for $i = 0, 1$
- a cofibration if $\alpha(i)$ is a cofibration in $\mathcal{M}$ for $i = 0, 1$, and
- a fibration in $\mathcal{M}$ if both $\alpha(1)$ and the map $(X(01), \alpha(0)) : X(0) \to X(1) \times_{X'(1)} X'(0)$ are fibrations in $\mathcal{M}$.

Proof. It is clear that $\mathcal{N}$ has small limits and colimits, and that weak equivalences in it have the 2-out-of-3 property. It remains to show that $(\text{Cof} \cap \text{W}, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap \text{W})$ are weak factorization systems.
Let $\alpha: X \to X'$ be a map in $\mathcal{N}$. Choose

- a factorization $X(1) \to Y(1) \to X'(1)$ of $\alpha(1)$ of the form $\text{Fib} \circ \text{Cof} \cap W$, and
- a factorization $X(0) \to Y(0) \to X'(0) \times_{X'(1)} Y(1) \ldots$

43.3. **Reedy lemma.**

43.4. **Proposition (Reedy lemma).** Let $F: \mathcal{M} \to \mathcal{N}$ be a functor between model categories which takes trivial cofibrations to weak equivalences. Then $F$ takes weak equivalences between cofibrant objects to weak equivalences.

**Proof.** Let $f: X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. Form the commutative diagram

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & Y \\
\downarrow & & \downarrow \text{id}_Y \\
X & \xrightarrow{a} & X \amalg Y \\
\downarrow \alpha & & \downarrow \beta \\
C & \xrightarrow{\gamma} & Y \\
\end{array}
\]

where the square is a pushout, and we have chosen a factorization $X \amalg Y \to C \to Y$ into a cofibration followed by a weak equivalence (e.g., a trivial fibration). Because $X$ and $Y$ are cofibrant, the maps $X \to X \amalg Y \leftarrow Y$ are cofibrations. Using this and the 2-of-3 property, we see that $a$ and $b$ are trivial cofibrations. Applying $F$ gives

\[
\begin{array}{ccc}
F(X) & \longrightarrow & F(C) \\
\downarrow F(b) & & \downarrow F(\gamma) \\
F(Y) & \longrightarrow & F(Y) \\
\end{array}
\]

in which $F(b)$ and $F(a)$ are weak equivalences by hypothesis, whence $F(b)$ is a weak equivalence by 2-of-3, and therefore $F(f) = F(p)F(b)$ is a weak equivalence, as desired.

The opposite of a model category is also a model category, by switching the roles of fibrations and cofibrations. Thus, there is a dual formulation of the Reedy lemma.

43.5. **Proposition (Reedy lemma, dual form).** Let $G: \mathcal{N} \to \mathcal{M}$ be a functor between model categories which takes trivial fibrations to weak equivalences. Then $G$ takes weak equivalences between fibrant objects to weak equivalences.

43.6. **Quillen pairs.** Given an adjoint pair of functors $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ between model categories, we see from the properties of weak factorization systems that

- $F$ preserves cofibrations if and only if $G$ preserves trivial fibrations, and
- $F$ preserves trivial cofibrations if and only if $G$ preserves fibrations.

If both of these are true, we say that $(F,G)$ is a **Quillen pair**.

Note that if $(F,G)$ is a Quillen pair, then the Reedy lemma (43.4) applies to $F$, and the dual form of the Reedy lemma (43.5) applies to $G$.

43.7. **Good colimits.** We can apply the above to certain examples of colimit functors.

43.8. **Exercise.** Let $S$ be a small discrete category (i.e., all maps are identities). Show that if $\mathcal{M}$ is a model category, then $\text{Fun}(S, \mathcal{M})$ is a model category in which $\alpha: X \to X'$ is

- a weak equivalence, cofibration, or fibration iff each $\alpha_s: X_s \to X'_s$ is one in $\mathcal{M}$.

These show that $\text{colim}: \text{Fun}(S, \mathcal{M}) \rightleftarrows \mathcal{M}: \text{const}$ is a Quillen pair, and use this to prove the next proposition.
43.9. Proposition. Given a collection $f_s: X_s \to X'_s$ of weak equivalences between cofibrant objects in $\mathcal{M}$, the induced map $\amalg f_s: \amalg X_s \to \amalg X'_s$ is a weak equivalence.

43.10. Exercise. Let $V$ be the category

$$
\begin{array}{c}
0 & \xleftarrow{10} & 1 & \xrightarrow{12} & 2 \\
\end{array}
$$

Show that if $\mathcal{M}$ is a model category, then $\text{Fun}(V, \mathcal{M})$ is a model category in which $\alpha: X \to X'$ is

- a weak equivalence if $\alpha(i): X(i) \to X'(i)$ is a weak equivalence for $i = 0, 1, 2$ (i.e., an objectwise weak equivalence), and is
- a cofibration if $\alpha(0)$ and $\alpha(1)$ are cofibrations, and the evident map $X(2) \cup_{X(1)} X'(1) \to X'(2)$ is a cofibration.

Then show that $\text{colim}: \text{Fun}(V, \mathcal{M}) \rightleftarrows \mathcal{M}: \text{const}$ is a Quillen pair, and use this to prove the next proposition. (Hint: determine what the fibrations in $\text{Fun}(V, \mathcal{M})$ must be according to the lifting property.)

43.11. Proposition. Given a natural transformation $\alpha: X \to X'$ of functors $V \to \mathcal{M}$, i.e., a diagram

$$
\begin{array}{ccc}
X(0) & \xleftarrow{X(1)} & X(2) \\
\sim & & \sim \\
X'(0) & \xleftarrow{X'(1)} & X'(2)
\end{array}
$$

in which the vertical maps are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(12)$ and $X'(12)$ are cofibrations, the induced map $\text{colim}_V X \to \text{colim}_V X'$ is a weak equivalence.

43.12. Exercise. Let $\omega$ be the category

$$
\begin{array}{c}
0 \to 1 \to 2 \to \cdots
\end{array}
$$

with objects indexed by natural numbers. Show that if $\mathcal{M}$ is a model category, then $\text{Fun}(\omega, \mathcal{M})$ is a model category in which $\alpha: X \to X'$ is

- a weak equivalence if each $\alpha(i)$ is a weak equivalence,
- a cofibration if $\alpha(0)$ is a cofibration, and $X'(i) \cup_{X(i)} X(i + 1) \to X'(i + 1)$ is a cofibration for all $i \geq 0$.

Then show that $\text{colim}: \text{Fun}(\omega, \mathcal{M}) \rightleftarrows \mathcal{M}: \text{const}$ is a Quillen pair, and use this to prove the next proposition.

43.13. Proposition. Give a natural transformation $\alpha: X \to X'$ of functors $\omega \to \mathcal{M}$ such that all maps $\alpha(i): X(i) \to X'(i)$ are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(i) \to X(i+1)$ and $X'(i) \to X'(i+1)$ are cofibrations, the induced map $\text{colim}_\omega X \to \text{colim}_\omega X'$ is a weak equivalence.

In the Joyal and Quillen model structures, all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (43.9), (43.11), (43.13) a good colimit. Thus, we see that good colimits are homotopy invariant. These “good colimits” are examples of what are called homotopy colimits.

Since the opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that good limits are homotopy invariant.

43.14. Exercise. State and prove the dual versions of all the results in this section.
44. A simplicial set is weakly equivalent to its opposite

Recall the opposite construction $X \mapsto X^{op}$ on general simplicial sets. We will now prove the following, which is a kind of a generalization of the fact that every groupoid is isomorphic to its own opposite.

44.1. Proposition. Every simplicial set $X$ is weakly equivalent to its opposite, in the sense that there exists a zig-zag of weak equivalences connecting $X$ and $X^{op}$.

Note that for any ordinary groupoid $G$, it is straightforward to construct an isomorphism $G \to G^{op}$, by sending each map to its inverse. Unfortunately, this proof cannot be replicated in our setting, even if we assume that $X$ is a quasigroupoid.

44.2. Remark. Here is one possible proof (in some sense, the most natural proof). Note that there is a homeomorphism of geometric realizations $|X| \approx |X^{op}|$. Then use the fact that geometric realization induces an equivalence $h(sSet, WkEq) \approx h(Top, WkEq)$. Of course, we haven’t actually proved this fact about homotopy categories yet.

I’ll give some different proofs which are internal to simplicial sets. I do this in part because it’s interesting to see how this is done, but also because it allows me to set up some technology which will be useful later.

44.3. General “singular” and “realization” functors. Consider a functor $C : \Delta \to A$ into some category $A$, i.e., a cosimplicial object in $A$. Often we write $C^n \in A$ instead of $C([n])$ for the values of this functor, and so write $C^\bullet$ for the whole functor. In most of our examples, we will actually have $A = sSet$, so $C^\bullet$ will be a cosimplicial simplicial set.

Given such a $C^\bullet$, realization and singular functors are an adjoint pair

$$Re = Re_{C^\bullet} : sSet \rightleftarrows A : Si_{C^\bullet} = Si$$

associated to $C^\bullet$. The singular functor $Si$ is always defined, and is defined by

$$(Si_X)_n := Hom_{sSet}(C([n]), X),$$

with simplicial operators induced by the fact that $C$ is a functor on $\Delta$. The left adjoint $Re$ is defined if $A$ is cocomplete\(^ {30} \).

44.4. Exercise. Show that if $A$ is cocomplete, and $X \in sSet$, then

$$Re X \approx \text{Cok} \left[ \coprod_{f : [n] \to [n]} \prod_{x \in X_n} C^n \Rightarrow \prod_{[p]} \coprod_{x \in X_p} C^{op} \right].$$

(Part of the exercise is to figure out what the two maps are.)

44.5. Example. For the cosimplicial space $\Delta^{\bullet}_{\text{top}} : \Delta \to \text{Top}$ taking $[n]$ to the topological $n$-simplex, $Re_{\Delta^{\bullet}_{\text{top}}}$ and $Si_{\Delta^{\bullet}_{\text{top}}}$ are just the usual geometric realization and singular complex functors.

44.6. Example. For the tautological functor $\Delta^\bullet : \Delta \to sSet$ sending $[n] \mapsto \Delta^n$, the functors $Re_{\Delta^\bullet}$ and $Si_{\Delta^\bullet}$ are isomorphic to the identity functor on simplicial sets.

44.7. Example. For the functor $(\Delta^\bullet)^{op} = \Delta^\bullet \circ \text{op} : \Delta \to sSet$, the functors $Re_{\Delta^\bullet_{\text{op}}}$ and $Si_{\Delta^\bullet_{\text{op}}}$ are both isomorphic to the opposite functor $X \mapsto X^{op}$.

\(^ {30} \)This means that $A$ has colimits for all functors $F : B \to A$ from small categories $B$.\)
44.8. The cosimplicial object $Iso^\bullet$. Consider $Iso^\bullet: \Delta \to s\text{Set}$, where $Iso^n$ is the (nerve of the) groupoid with objects $\{0,1,\ldots,n\}$ and unique isomorphisms between every object. This has corresponding realization and singular functors

$$Re = Re_{Iso^\bullet}: s\text{Set} \to s\text{Set}: Si_{Iso^\bullet} = Si.$$

There is an evident natural transformation $\alpha: \Delta^\bullet \to Iso^\bullet$, by the unique functors $[k] \to Iso^k$ sending object $j$ to $j$. This gives an adjoint-related pair of natural transformations

$$\eta_X: X \to Re_{Iso^\bullet} X, \quad \epsilon_X: Si_{Iso^\bullet} X \to X.$$

If we precompose with $op$, we obtain a cosimplicial object $Iso^\bullet \circ op = (Iso^\bullet)^{op}$. One sees that there are natural isomorphisms

$$(44.9) \quad Re_{Iso^{op}} \approx (Re_{Iso^\bullet})^{op}, \quad Si_{Iso^{op}} \approx (Si_{Iso^\bullet})^{op},$$

and that the evident natural transformation $\alpha \circ op: \Delta^\bullet \circ op \to Iso^\bullet \circ op$ gives rise to an adjoint-related pair of natural transformations

$$\eta'_X: X^{op} \to Re_{Iso^{op}} X, \quad \epsilon'_X: Si_{Iso^{op}} X \to X^{op}.$$

Furthermore, under the isomorphisms of (44.9) the maps $\eta'_X$ and $\epsilon'_X$ are identified with $\eta_{X^{op}}$ and $\epsilon_{X^{op}}$.

44.10. Exercise. Prove the statements of the previous paragraph.

Although $\Delta^\bullet$ and $\Delta^\bullet \circ op$ are not isomorphic as functors $\Delta \to s\text{Set}$, it is the case that $Iso^\bullet \approx Iso^\bullet \circ op$, using the isomorphisms of categories $Iso^n \to Iso^n$ given on objects by $x \mapsto n - x$. Putting all this together, we obtain natural transformations

$$X \xrightarrow{\eta_X} Re_{Iso^\bullet} X \approx Re_{Iso^\bullet} X^{op} \xrightarrow{\eta_{X^{op}}} X^{op}, \quad X \xleftarrow{\epsilon_X} Si_{Iso^\bullet} X \approx Si_{Iso^\bullet} X^{op} \xleftarrow{\epsilon_{X^{op}}} X^{op}.$$

We’ll show that that $\eta_X$, and hence $\eta_{X^{op}}$, are always weak equivalences, and that $\epsilon_X$, and hence $\epsilon_{X^{op}}$, are weak equivalences whenever $X$ is a Kan complex. In the following, $Re = Re_{Iso^\bullet}$ and $Si = Si_{Iso^\bullet}$.

44.11. Lemma. For each monomorphism $K \to L$, the induced map $(Re K) \amalg_K L \to Re L$ is a monomorphism. In particular,

- $Re$ preserves monomorphisms and $Si$ preserves trivial fibrations, and
- $\eta_L: L \to Re L$ is always a monomorphism.

Proof. Formally, it is enough to check the case of $\partial \Delta^n \subset \Delta^n$. To see this, check that the lifting problems

$$\begin{array}{ccc}
(Re K) \cup_K L & \xrightarrow{\eta_K} & X \\
\downarrow & & \downarrow \\
Re L & \xleftarrow{\epsilon_K} & Si X
\end{array} \iff \begin{array}{ccc}
K & \xrightarrow{\eta_K} & Si X \\
\downarrow & & \downarrow \\
L & \xleftarrow{\epsilon_K} & (Si Y) \times_Y X
\end{array}$$

are equivalent. This means we need to show that all monomorphisms are contained in the weakly saturated class $\mathcal{C}$, where $\mathcal{C}$ is the class of all the maps $(Si p, \epsilon_X): Si X \to (Si Y) \times_Y X$ such that $p \in \text{TrivFib}$, which means we only need to show that Cell is contained in it.

This is a calculation: $Re(\partial \Delta^n) \to Re(\Delta^n) = N(Iso^n)$ is isomorphic to inclusion of the subcomplex $K \subset N(Iso^n)$ whose $k$-simplices are sequences $x_0 \to \cdots \to x_k$ such that $\{x_0,\ldots,x_k\} \neq \{0,\ldots,n\}$. To show this, use the fact that $\Delta^n$ is a colimit of its $(n-1)$-dimensional faces along their intersections, and that $Re$ preserves colimits. The image of the simplex $(0,1,\ldots,n)$ in $Re(\Delta^n)$ intersects $K$ exactly in its boundary, so $(Re \partial \Delta^n) \cup_{\partial \Delta^n} \Delta^n \to Re \Delta^n$ is a monomorphism as desired. \hfill $\square$
44.12. Remark. Given any natural transformation $\lambda: F \to G$ of functors, and map $f: X \to Y$, we get induced maps

$$F(Y) \cup_{F(X)} G(X) \to G(Y), \quad F(X) \to F(Y) \times_{G(Y)} G(X).$$

These can be thought of as a variant of the “box” construction we’ve considered elsewhere (26.5), but associated to the “evaluation pairing” $\text{Fun}(s\text{Set}, s\text{Set}) \times s\text{Set} \to s\text{Set}$ rather than a functor $s\text{Set} \times s\text{Set} \to s\text{Set}$.

44.13. Skeletal induction. The next step is to show that $K \to \text{Re}K$ is a weak equivalence for every simplicial set $K$. To do this, we will use the following bit of machinery.

44.14. Proposition (Skeletal induction). Let $C$ be a class of simplicial sets with the following properties.

1. If $X \in C$, then every object isomorphic to $X$ is in $C$.
2. Every $\Delta^n \in C$.
3. The class $C$ is closed under good colimits. That is:
   a. any coproduct of objects of $C$ is in $C$;
   b. any pushout of a diagram $X_0 \leftarrow X_1 \to X_2$ of objects in $C$ along a monomorphism $X_1 \to X_2$ is in $C$;
   c. any colimit of a countable sequence $X_0 \to X_1 \to X_2 \to \cdots$ of objects in $C$, such that each $X_k \to X_{k+1}$ is a monomorphism, is in $C$.

Then $C$ is the class of all simplicial sets.

Proof. This is a straightforward consequence of the skeletal filtration (15.18). To show $X \in C$, it suffices to show each $\text{Sk}_n X \in C$ by (3c). So we show that all $n$-skeleta are in $C$ by induction on $n$, with base case $n = -1$ (the empty simplicial set), which is really a special case of (3a). Since $\text{Sk}_{n-1} X \subseteq \text{Sk}_n X$ is a pushout along a coproduct of maps $\partial \Delta^n = \text{Sk}_{n-1} \Delta^n \to \Delta^n$, this follows using (2), (3a), (3b), and the inductive hypothesis. □

44.15. Proposition. For every simplicial set $X$, the map $X \to \text{Re}X$ is a weak equivalence.

Proof. Let $C$ be the class of $X$ such that $\eta: X \to \text{Re}X$ is a weak equivalence. We verify the hypotheses of the above proposition. Property (1) is obvious.

To prove property (2) recall that $\eta_\Delta^1: \Delta^1 \to \text{Iso}^1$ is anodyne (32.14). We can identify $\eta_\Delta^n$ as a retract of $(\eta_\Delta^1)^{\times n}: (\Delta^1)^{\times n} \to (\text{Iso}^1)^{\times n}$, which is necessarily anodyne since a product of an anodyne map with any identity map is anodyne (similar to (21.9)). The retraction $\Delta^n \xrightarrow{f} (\Delta^1)^{\times n} \xrightarrow{g} \Delta^n$ and $\text{Iso}^n \xrightarrow{f} (\text{Iso}^1)^{\times n} \xrightarrow{g} \text{Iso}^n$ are maps which are given on vertices by

$$f(k) = \left(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0\right), \quad g(k_1, \ldots, k_n) = \max \{ j \mid k_j = 1 \}.$$

Property (3) involves colimits, which in every case are good colimits. In each case, we need to show that a map colim $X_i \to \text{Re} \text{colim} X_i$ is a weak equivalence when each $X_i \to \text{Re} X_i$ is. The functor $\text{Re}$ preserves colimits and monomorphisms (44.11), so in every case we are comparing good colimits, so the result follows from (43.9), (43.11), (43.13).

We thus obtain the desired result.

44.16. Corollary. Every simplicial set is weakly equivalent to its opposite $X^{\text{op}}$.

Proof. Both maps in $X \xrightarrow{\eta_X} \text{Re} X \approx \text{Re} X^{\text{op}} \xleftarrow{\eta_{X^{\text{op}}}} X^{\text{op}}$ are weak equivalences (44.15). □

44.17. Proposition. For each monomorphism $K \to L$, the induced map $(\text{Re} K) \amalg_K L \to \text{Re} L$ is a monomorphism and a weak equivalence.
Proof. Both squares

\[
\begin{array}{ccc}
K & \xrightarrow{\eta} & \Re K \\
\downarrow & & \downarrow \\
\Re K & \xrightarrow{id} & \Re K \\
\end{array}
\quad
\begin{array}{ccc}
\Re K & \xrightarrow{id} & \Re K \\
\downarrow & & \downarrow \\
\Re L & \xrightarrow{id} & \Re L \\
\end{array}
\]

are good pushouts, using (44.11). The evident map from the left square to the right square is a weak equivalence at the upper left, upper right, and lower left corners (44.15), so the result follows from the invariance of good pushouts (43.11). □

44.18. **Corollary.** If \( p : X \to Y \) is a Kan fibration, then \( \Si X \to \Si Y \times_Y X \) is a trivial fibration. In particular, if \( X \) is a Kan complex, then \( \Si X \to X \) is a trivial fibration.

In particular, for any Kan complex \( X \), both maps in \( X \xleftarrow{\epsilon_X} \Si X \approx \Si X^{\text{op}} \xrightarrow{\epsilon_X^{\text{op}}} X^{\text{op}} \) are trivial fibrations.

Proof. Straightforward, using (44.17). □

44.19. **Remark.** The object \( \Re X \) is not generally categorically equivalent to \( X \).

It can be shown that if \( C \) is a quasicategory, then \( \Si C \) is categorically equivalent to \( C^{\text{core}} \).

45. **Initial and terminal objects, revisited**

Recall the definition of initial and terminal objects in a quasicategory. One characterization was: \( x \) is an initial object of \( C \) iff the left fibration \( p : C_{x/} \to C \) is a trivial fibration, and a terminal object iff the right fibration \( p' : C_{/x} \to C \) is a trivial fibration.

When \( C \) is the nerve of an ordinary category, these reduce to the usual definitions of initial and terminal object. In this case, there is an equivalent characterization: \( x \) is initial if and only if \( \text{Hom}_C(x, y) \) is a singleton set for all objects \( y \) of \( C \), and terminal if and only if \( \text{Hom}_C(y, x) \) is a singleton set for all \( y \).

We would like to generalize this to the case of quasicategories.

F. **Deferred Proposition.** An object \( x \) of a quasicategory is initial if and only if \( \text{map}_C(x, c) \) is contractible for all objects \( c \) of \( C \), and terminal if and only if \( \text{map}_C(c, x) \) is contractible for all objects \( c \) of \( C \).

To prove this, you need to be able to relate mapping spaces of a quasicategory to the join/slice constructions that we used to define initial and terminal. We will establish such a relation in the next few sections.

45.1. **Right and left mapping spaces.** Let \( x, y \) be objects of a quasicategory \( C \). We define the **right mapping space** \( \text{map}_C^R(x, y) \) and **left mapping space** \( \text{map}_C^L(x, y) \) by pullback diagrams

\[
\begin{array}{ccc}
\text{map}_C^R(x, y) & \longrightarrow & C_{x/} \\
\downarrow & & \downarrow \pi \\
\Delta^0 & \xrightarrow{y} & C \\
\end{array}
\quad
\begin{array}{ccc}
\text{map}_C^L(x, y) & \longrightarrow & C_{/y} \\
\downarrow & & \downarrow \pi \\
\Delta^0 & \xrightarrow{x} & C \\
\end{array}
\]

where the maps labelled \( \pi \) are the evident forgetful maps.

For instance, an \( n \)-simplex of \( \text{map}_C^R(x, y) \) is precisely a map \( a : \Delta^{n+1} \to C \) such that \( a|\Delta^{\{0,\ldots,n\}} \) is the constant simplex at \( x \), and \( a|\Delta^{\{n+1\}} = y \). In particular, a 0-simplex in \( \text{map}_C^R(x, y) \) is a morphism \( x \to y \) in \( C \), while a 1-simplex in \( \text{map}_C^R(x, y) \) is a 2-simplex in \( C \) exhibiting the \( \sim_r \) relation between two maps, which we used to define the homotopy category in §9.
Recall (26.15) that when $C$ is a quasicategory, the maps $C_{x/} \to C$ and $C_{/y} \to C$ are left fibrations and right fibrations respectively. Thus both $\text{map}_C^R(x, y)$ and $\text{map}_C^L(x, y)$ are Kan complexes, by the following.

45.2. Exercise. Show that if $X \to \Delta^0$ is a left fibration or a right fibration, then $X$ is a Kan complex. (Hint: Joyal lifting.)

Furthermore, by the above remarks relating 1-simplices in the right and left mapping spaces to the homotopy relation, we have that $\pi_0 \text{map}_C^R(x, y) \approx \pi_0 \text{map}_C^L(x, y) \approx \text{Hom}_{\mathcal{C}}(x, y)$.

We will show below that both $\text{map}_C^R(x, y)$ and $\text{map}_C^L(x, y)$ are actually weakly equivalent to the standard mapping space $\text{map}_C(x, y)$.

45.3. Box products and right and left anodyne maps. Recall that $\text{InnHorn} \boxtimes \text{Cell} \subseteq \text{InnHorn}$ (16.7) and $\text{Horn} \boxtimes \text{Cell} \subseteq \text{Horn}$. We have an analogous fact for left or right anodyne maps.

45.4. Proposition. We have that $L\text{Horn} \boxtimes \text{Cell} \subseteq L\text{Horn}$ and $R\text{Horn} \boxtimes \text{Cell} \subseteq R\text{Horn}$.

Proof. See appendix. □

45.5. Fiberwise criterion for trivial fibrations, revisited. Recall the fiberwise criterion for trivial fibrations (34.2): a Kan fibration $p$ is a trivial fibration if and only if the fibers of $p$ are contractible Kan complexes. In fact, this still holds if we only know $p$ is a left or right fibration.

45.6. Proposition. Suppose $p: X \to Y$ is a right fibration or left fibration of simplicial sets. Then $p$ is a trivial fibration if and only if it has contractible fibers.

Proof. [Lur09, 2.1.3.4]. Let’s consider the case of $p: X \to Y$ a left fibration. The direction ($\Rightarrow$) is immediate, so we only need to prove ($\Leftarrow$).

We attempt to carry out the argument of the proof of (34.2), and show that $((\partial \Delta^n \subseteq \Delta^n) \boxtimes \{1 \subseteq \Delta^1\}) \subseteq p$ for all $n \geq 0$. The case of $n \geq 0$ is immediate, since the fibers of $p$ must be non-empty, since they are contractible, so we can assume $n \geq 1$.

Examining that proof of (34.2), we see that we used only the hypothesis that $p$ is a Kan fibration in order to have that

$$(\partial \Delta^n \times \{0\} \subseteq \partial \Delta^n \times \Delta^1) \subseteq p, \quad ((\partial \Delta^n \subseteq \Delta^n) \times \{1 \subseteq \Delta^1\}) \subseteq p.$$ 

In the first case, the inclusion $(\partial \Delta^n \times \{0\} \subseteq \partial \Delta^n \times \Delta^1)$ is left anodyne by (45.4), so the lifting problem still has a solution when $p$ is only a left fibration.

In the second case, we need to argue a little differently. In the proof of (34.2) this lifting problem appears when producing a lifting (for $n \geq 1$) in a diagram of the form

$$
\begin{array}{ccc}
(\partial \Delta^n \times \Delta^1) \cup_{\partial \Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{(c,jd)} & X \\
\Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & Y \\
\end{array}
$$

Pulling back along the factorization of the bottom map $b\gamma$, we obtain a diagram

$$
\begin{array}{ccc}
(\partial \Delta^n \times \Delta^1) \cup_{\partial \Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{p'} & C \xrightarrow{p} X \\
\Delta^n \times \Delta^1 & \xrightarrow{\gamma} & \Delta^n \xrightarrow{b} Y \\
\end{array}
$$

where the right-hand square is a pullback. Observe that (i) $p'$ is a left fibration, and hence an inner fibration, between quasicategories, and that (ii) the map $\gamma$ (as defined in the proof of (34.2)) sends...
the edge \( \{ n \} \times \Delta^1 \) to the degenerate edge \( \langle nn \rangle \) in \( \Delta^n \). Therefore we can apply the pushout-product version of Joyal lifting (29.12) to produce a lift \( s' \).

45.7. **Corollary.** An object \( x \) of a quasicategory \( C \) is initial if and only if \( \text{map}^R_C(x, c) \) is contractible for all objects \( c \) of \( C \), and is final if and only if \( \text{map}^L_C(c, x) \) is contractible for all objects \( c \) of \( C \).

**Proof.** The fibers of the left fibration \( C_{x/} \to C \) are precisely the right mapping spaces \( \text{map}^R_C(x, c) \). By what we just proved (45.6) these fibers are all contractible if and only if \( C_{x/} \to C \) is a trivial fibration, which we have noted (25.4) is equivalent to \( x \) being initial in \( C \). □

46. **The alternate join and slice**

We now want to compare the right and left mapping spaces, which are fibers of the projections \( C_{x/} \to C \) and \( C_{/x} \to C \), to the ordinary mapping spaces, which are fibers of \( \text{Fun}(\Delta^1, C) \to \text{Fun}(\partial \Delta^1, C) \). We do this using constructions called the “alternate join” and “alternate slice” [Lur09, §4.2.1].

Given an object \( x \) in \( C \), consider the map

\[
q: \text{Fun}(\Delta^1, C) \times_{\text{Fun}(\{0\}, C)} \{ x \} \to \text{Fun}(\{1\}, C) = C
\]

induced by restriction along \( \{0\} \subset \Delta^1 \). Note that the fiber of \( q \) over some object \( c \) of \( C \) is precisely the quasigroupoid map \( C(x, c) \). The domain of \( q \) is an example of what we will call the “alternate slice” construction, for which we will use the (unmemorable) notation \( C^{x/} \).

46.1. **Exercise.** Show that if \( C \) is an ordinary category, then \( C^{x/} \) is isomorphic to the usual slice category \( C_{x/} \), and \( q \) is isomorphic to the usual projection \( p: C_{x/} \to C \).

For a general quasicategory, \( q \) is not isomorphic \( p \). What is true is that there is a commutative diagram

\[
\begin{array}{ccc}
C_{x/} & \xrightarrow{f} & C^{x/} = \text{Fun}(\Delta^1, C) \times_{\text{Fun}(\{0\}, C)} \{ x \} \\
p \downarrow & & \downarrow q \\
C & \xleftarrow{p} & C
\end{array}
\]

The map \( f \) sends a simplex \( a: \Delta^k \to C_{x/} \), which corresponds to \( \overline{a}: \Delta^{k+1} \to C \) such that \( \overline{a}_0 = x \), to a simplex in \( C^{x/} \) corresponding to \( \overline{ar}: \Delta^k \times \Delta^1 \to C \), where \( r: \Delta^k \times \Delta^1 \to \Delta^{k+1} \) is the unique map given on vertices by \( r(i, 0) = 0, r(i, 1) = i + 1 \).

The characterization (F) of initial objects in terms of contractible mapping spaces thus amounts to the claim that \( p \) is a trivial fibration if and only if \( q \) has contractible fibers. In fact, we’ll prove that

- both \( p \) and \( q \) are left fibrations,
- \( f \) is a categorical equivalence.

Because \( p \) and \( q \) are left fibrations, they are trivial fibrations iff their fibers are contractible (45.6). Because \( f \) is a categorical equivalence, \( p \) is a categorical equivalence if and only if \( q \) is by 2-out-of-3 (22.9). The result follows because \( p \) and \( q \) are in particular isofibrations (28.10), and an isofibration is a trivial fibration if and only if it is a categorical equivalence (36.10).

In other words, we can regard \( C^{x/} \) as an alternate version of the slice construction, so we call it the “alternate slice”. It is related to an alternate version of the join, denoted \( X \diamond Y \) and called the “alternate join”, which we define first.
46.2. **The alternate join.** Given simplicial sets $X$ and $Y$, define the **alternate join** by the pushout diagram

$$(X \times \{0\} \times Y) \coprod (X \times \{1\} \times Y) \to X \times \Delta^1 \times Y \to X \circ Y$$

where the maps on top and left are induced by the evident inclusion and projection maps.

The alternate join comes with a natural comparison map

$$X \circ Y \to X \star Y,$$

defined as follows. Using the recipe of (23.11) for constructing maps to a join, we get a map $X \times \Delta^1 \times Y \to X \star Y$ corresponding to the triple $(g, f^{(0)}, f^{(1)})$, where $g: X \times \Delta^1 \times Y \to \Delta^1$, $f^{(0)}: X \times \{0\} \times Y \to X$, and $f^{(1)}: X \times \{1\} \times Y \to Y$ are the evident projections. A similar procedure produces compatible maps to $X \star Y$ from the other vertices of the pushout square defining $X \circ Y$. Note that the comparison map induces a bijection on vertices.

46.3. **Example.** We have

$$X \circ \Delta^0 \approx (X \times \Delta^1)/(X \times \{1\}), \quad \Delta^0 \circ Y \approx (\Delta^1 \times Y)/(\{0\} \times Y),$$

simplicial sets obtained by collapsing subcomplexes to a single point. These come with evident maps $X \circ \Delta^0 \to X^\triangleright$ and $\Delta^0 \circ Y \to Y^\triangleleft$.

Like the true join, $X \circ \emptyset \approx X \approx \emptyset \circ X$, and the functors $X \circ -: s\Set \to s\Set_{/X}$ and $- \circ Y: s\Set \to s\Set_{Y/}$ commute with colimits.

Warning: when $X$ and $Y$ are non-empty, $X \times \Delta^1 \times Y \to X \circ Y$ is surjective, but this is not the case when either $X$ or $Y$ are empty.

Unlike the true join, the alternate join is not monoidal: $(X \circ Y) \circ Z \neq X \circ (Y \circ Z)$ in general. Also, the alternate join of two quasicategories is not usually a quasicategory.

The alternate join is a categorically invariant construction.

46.4. **Proposition.** The alternate join $\circ$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \circ Y \to X \circ Y'$ and $Y \circ Z \to Y' \circ Z$.

**Proof.** The $\circ$ product is constructed using a “good” pushout, i.e., a pushout along a cofibration. The result follows because both products and good pushouts preserve categorical equivalences (43.11).

46.5. **Alternate slice.** Given $p: S \to X$ and $q: T \to X$, we define the **alternate slices** $X^{p/}$ and $X^{/q}$ via the bijective correspondences

$$\begin{cases} S \circ \emptyset \downarrow^p \to X \cong \{X \to X^{p/}\}, & \emptyset \circ T \downarrow^q \to X \cong \{K \to X^{/q}\}. \end{cases}$$

just as we defined ordinary slices using joins. These constructions give right adjoints to the alternate join functors:

$$S \circ (-): s\Set \rightleftarrows s\Set_{S/}:(p \mapsto X^{p/}), \quad (-) \circ T: s\Set \rightleftarrows s\Set_{T/}:(q \mapsto X^{/q}).$$

Alternate slices are “functorial” in exactly the sense that ordinary slices are (24.13): a sequence of maps $T \xrightarrow{f} S \xrightarrow{p} X \xrightarrow{f} Y$ induces $X^{p/} \to Y^{fpj/}$ and $X^{p/} \to Y^{qf/}$. 


46.6. Exercise. Show that there are pullback squares of the form

\[
\begin{array}{ccc}
X^{p/} & \longrightarrow & \text{Map}(S \times \Delta^1, X) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Map}(S \times \{0\}, X) \times \text{Map}(S \times \{1\}, X)
\end{array}
\quad \begin{array}{ccc}
X^{/p} & \longrightarrow & \text{Map}(\Delta^1 \times S, X) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Map}(\{0\} \times S, X) \times \text{Map}(\{1\} \times S, X)
\end{array}
\]

where \(\bar{p}: X \to \text{Map}(S, X)\) is adjoint to \(X \times S \xrightarrow{\text{proj}} S \xrightarrow{p} X\), and \(c: X \to \text{Map}(S, X)\) is adjoint to \(X \times S \xrightarrow{\text{proj}} X \xrightarrow{\text{id}} X\).

Using the adjunction relation between joins and slices, and alternate joins and slices, the natural comparison map

\[X \diamond Y \to X \cdot Y\]

induces natural comparison maps on alternate slices. That is, given \(p: S \to X\) and \(q: T \to Y\) we have natural comparison maps

\[X^{p/} \to X^{p/}\] and \[Y^{/q} \to Y^{/q}\].

46.7. Joins, slices, and function complexes. Recall the function complex \(\text{Map}(X, Y) \in s\text{Set}\), defined for any pair of simplicial sets \(X, Y\). Recall also (20.13) the relative function complex under \(S\), which for objects \(p: S \to X\) and \(q: S \to Y\) in \(s\text{Set}_{S/}\) is a simplicial set

\[\text{Map}_{S/}(X, Y) := \text{Map}(X, Y) \times \text{Map}(S, Y)\{q\}\]

with bijective correspondences

\[
\left\{ K \to \text{Map}_{S/}(X, Y) \right\} \iff \left\{ K \times S \xrightarrow{\text{proj}} S \xrightarrow{q} Y \right\}
\]

natural in the simplicial set \(K\). The set of vertices of \(\text{Map}_{S/}(X, Y)\) is precisely the set \(\text{Hom}_{s\text{Set}_{S/}}(X, Y)\) of morphisms in the category \(s\text{Set}_{S/}\).

Given \(X\) and \(p: S \to Y\), we have join/slice adjunctions

\[
\text{Hom}_{s\text{Set}}(X, Y_{p/}) \approx \text{Hom}_{s\text{Set}_{S/}}(S \ast X, Y), \quad \text{Hom}_{s\text{Set}}(X, Y_{/p}) \approx \text{Hom}_{s\text{Set}_{S/}}(X \ast S, Y).
\]

We now construct maps

\[
\text{Map}(X, Y_{p/}) \to \text{Map}_{S/}(S \ast X, Y), \quad \text{Map}(X, Y_{/p}) \to \text{Map}_{S/}(X \ast S, Y).
\]

which are natural in both \(X\) and \(p\), and which on vertices are exactly the join/slice adjunctions. We will call these the \textbf{enriched adjunction maps} for join/slice; they are not isomorphisms in general.

I write this out in the case of slice-over, by constructing a transformation

\[
\left\{ K \to \text{Map}(X, Y_{p/}) \right\} \implies \left\{ K \to \text{Map}_{S/}(S \ast X, Y) \right\}
\]

natural in the simplicial set \(K\). Applying the product/function complex adjunction, and the join/slice adjunction, this amounts to defining natural maps

\[
\left\{ S \ast \varnothing \xrightarrow{p} S \right\} \implies \left\{ K \times (S \ast \varnothing) \xrightarrow{\text{proj}} S \right\}
\]

Thus it suffices to produce natural maps

\[
K \times (S \ast X) \to S \ast (K \times X)
\]
which in the case that $X = \emptyset$ reduce to the projection map $K \times S \to S$. We take this to be the map corresponding by (23.11) to the triple $(g, f_0, f_1)$ so that $g$ is the composite

$$K \times (S \star X) \to K \times (\Delta^0 \star \Delta^0) \to \Delta^0 \star \Delta^0 = \Delta^1,$$

and

$$f^{(0)} = \text{proj}: K \times (S \star \emptyset) \to S, \quad f^{(1)} = \text{id}: K \times (\emptyset \star X) \to K \times X.$$

It is now straightforward to derive explicit formulas for the desired transformation (by specializing to $K = \Delta^n$), and to show that is is natural.

46.8. Exercise. Construct a natural “distributivity” map $K \times (X \star Y) \to (K \times X) \star (K \times Y)$.

46.9. Alternate joins, alternate slices, and function complexes. We can carry out the same procedure for alternate joins and slices, to obtain maps

$$\text{Map}(X, Y \uparrow p) \to \text{Map}_S(S \star X, Y), \quad \text{Map}(X, Y \uparrow p) \to \text{Map}_S(X \circ S, Y)$$

which are natural in both $X$ and $p$, and which on vertices are exactly the alternate join/slice adjunctions. We will call these the enriched adjunction maps for alternate join/slice.

Tracing through the same steps as in the previous section, we see that (in the first case) we need natural maps

$$K \times (S \circ X) \to S \circ (K \times X)$$

which when $X = \emptyset$ reduce to the projection map $K \times S \to S$. In this case it is entirely straightforward to construct such a map, since both objects are naturally quotients of the product $K \times S \times \Delta^1 \times X \approx S \times \Delta^1 \times K \times X$. In fact, examination of the constructions shows that the evident diagram

$$
\begin{array}{ccc}
K \times S & \xrightarrow{\text{proj}} & S \\
\downarrow & & \downarrow \\
K \times (S \circ X) & \longrightarrow & S \circ (K \times X)
\end{array}
$$

is a pushout square. (Exercise: prove this.) Given this consideration, we see that we have actually defined natural isomorphisms

$$\text{Map}(X, Y \uparrow p) \cong \text{Map}_S(S \star X, Y), \quad \text{Map}(X, Y \uparrow p) \cong \text{Map}_S(X \circ S, Y).$$

Furthermore, these natural isomorphisms are compatible with the transformations for join/slice.

46.10. Proposition. The evident diagrams

$$
\begin{array}{ccc}
\text{Map}(X, Y \uparrow p) & \longrightarrow & \text{Map}_S(S \star X, Y) \\
\downarrow & & \downarrow \\
\text{Map}(X, Y \uparrow p) & \cong & \text{Map}_S(S \circ X, Y)
\end{array} \quad \begin{array}{ccc}
\text{Map}(X, Y \uparrow p) & \longrightarrow & \text{Map}_S(X \circ S, Y) \\
\downarrow & & \downarrow \\
\text{Map}(X, Y \uparrow p) & \cong & \text{Map}_S(X \circ S, Y)
\end{array}
$$

commute.

Proof. This amounts to showing that the evident diagram

$$
\begin{array}{ccc}
K \times (S \circ X) & \longrightarrow & S \circ (K \times X) \\
\downarrow & & \downarrow \\
K \times (S \star X) & \longrightarrow & S \star (K \times X)
\end{array}
$$

commutes, which we leave to the reader. □
Below we will show that if \( Y = C \) is a quasicategory, then all of the maps in these diagrams are categorical equivalences. As a consequence, we will obtain categorical equivalences \( C_{p/} \to C^{p/} \) and \( C/p \to C/p' \).

47. Equivalence of the two join and slice constructions

47.1. The enriched adjunction map for joins/slices preserves isomorphism classes of objects. We now consider the natural maps

\[
\text{Map}(X, C_{p/}) \to \text{Map}_{S/}(S \star X, C), \quad \text{Map}(X, C/p) \to \text{Map}_{S/}(X \star S, C)
\]

in the case when \( p: S \to C \) is a map to a quasicategory \( C \). In this case both sources and targets of the natural maps in question are themselves quasicategories, and both induce bijections on sets of objects. Eventually we will show that these functors are categorical equivalences. Right now we will just prove that these functors induce bijections on isomorphism classes of objects.

47.2. Proposition. For \( X \) a simplicial set and \( p: S \to C \) a map to a quasicategory, the enriched adjunction map for join/slice induces bijections

\[
\pi_0(\text{Map}(X, C_{p/})^{\text{core}}) \cong \pi_0(\text{Map}_{S/}(S \star X, C)^{\text{core}}), \quad \pi_0(\text{Map}(X, C/_p)^{\text{core}}) \cong \pi_0(\text{Map}_{S/}(X \star S, C)^{\text{core}}),
\]

Proof. We give the proof in the slice-over case. Since the enriched adjunction map gives a bijection on objects, it suffices to prove injectivity on sets of isomorphism classes.

Let \( f_0, f_1: X \to C_{p/} \) be objects of \( \text{Map}(X, C_{p/}) \), which correspond to objects \( \tilde{f}_0, \tilde{f}_1: S \star X \to C \) of \( \text{Map}_{S/}(S \star X, C) \), with \( \tilde{f}_j|S = p \). If \( \tilde{f}_0 \) and \( \tilde{f}_1 \) are isomorphic objects, then there exists a map \( N\text{Iso} \to \text{Map}_{S/}(S \star X, C) \) representing such an isomorphism (32.15). The data of such a map amounts to a an arrow \( \tilde{f} \) fitting in the commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{p} & C \\
\downarrow & & \downarrow \\
S \star X & \xrightarrow{(\tilde{f}_0, \tilde{f}_1)} & C \times C
\end{array}
\]

where \( C \to \text{Map}(N\text{Iso}, C) \to C \times C \) are induced by restriction along \( * \leftarrow N\text{Iso} \leftarrow \{0,1\} \). Write \( D = \text{Map}(N\text{Iso}, C) \) and \( \overline{p}: S \to D \) for the map along the top of the rectangle. Applying the join/slice adjunction, we see that we have a diagram

\[
\begin{array}{ccc}
 & & D_{p/} \\
\downarrow & & \downarrow \\
X & \xrightarrow{(f_0, f_1)} & C_{p/} \times C/p
\end{array}
\]

That is, we have produced an object \( f \) in \( \text{Map}(X, D_{p/}) \) which under the two evident projections \( \pi_0, \pi_1: \text{Map}(X, D_{p/}) \to \text{Map}(X, C) \) is sent to \( f_0 \) and \( f_1 \) respectively.

We have that both projections \( D \to C \) are trivial fibrations, whence so are both projections \( D_{p/} \to C_{p/} \) (this needs a proof), and hence both projections \( \pi_0 \) and \( \pi_1 \). Considering the induced commutative diagram

\[
\begin{array}{ccc}
\text{Map}(X, C_{p/}) & \xrightarrow{\pi_0} & \text{Map}(X, D_{p/}) \\
\downarrow \text{id} & & \downarrow \pi_1 \\
\text{Map}(X, C_{p/}) & \xrightarrow{\pi_1} & \text{Map}(X, C_{p/})
\end{array}
\]

\[
\begin{array}{ccc}
\text{Map}(X, C_{p/}) & \xrightarrow{\pi_0} & \text{Map}(X, D_{p/}) \\
\downarrow \text{id} & & \downarrow \pi_1 \\
\text{Map}(X, C_{p/}) & \xrightarrow{\pi_1} & \text{Map}(X, C_{p/})
\end{array}
\]
we see that every arrow in this diagram is a categorical equivalence, and therefore both \( \pi_0 \) and \( \pi_1 \) induce the same bijection on isomorphism classes on objects. Therefore \( f_0 \) and \( f_1 \) are isomorphic objects, as desired. \( \square \)

47.3. **Equivalence of join and alternate join.** The key result of this section is the following.

47.4. **Proposition.** The canonical comparison map \( X \diamond Y \to X \star Y \) is a categorical equivalence for all simplicial sets \( X \) and \( Y \).

What we have proved implies the categorical invariance of the usual join.

47.5. **Corollary.** The join \( \star \) preserves categorical equivalences in either variable. That is, if \( Y \to Y' \) is a categorical equivalence, then so are \( X \star Y \to X \star Y' \) and \( Y \star Z \to Y' \star Z \).

Proof. Immediate using (47.4), the invariance of the alternate join under categorical equivalence (46.4), and the 2-out-of-3 property of categorical equivalences (22.9). \( \square \)

The proof is based on the following general strategy.

47.6. **Proposition.** Let \( \alpha : F \to F' \) be a natural transformation between functors \( sSet \to \mathcal{M} \), where \( \mathcal{M} \) is some model category. If

1. \( F \) and \( F' \) preserve colimits,
2. \( F \) and \( F' \) take monomorphisms to cofibrations,
3. \( F \) and \( F' \) take inner anodyne maps to weak equivalences in \( \mathcal{M} \), and
4. \( \alpha(\Delta^1) : F(\Delta^1) \to F'(\Delta^1) \) is a weak equivalence in \( \mathcal{M} \),

then \( \alpha(X) : F(X) \to F'(X) \) is a weak equivalence in \( \mathcal{M} \) for all simplicial sets \( X \).

Proof. [Lur09, 4.2.1.2] Consider the class of simplicial sets \( \mathcal{C} := \{ X \mid \alpha(X) \text{ is a weak equivalence} \} \). We use skeletal induction (44.14) to show that \( \mathcal{C} \) contains all simplicial sets.

It is clear that \( \mathcal{C} \) is closed under isomorphic objects. Because \( F \) and \( F' \) preserve colimits (1) and cofibrations (2), they take good colimit diagrams in \( sSet \) to good colimit diagrams in \( \mathcal{M} \). Since good colimits are weak equivalence invariant (43.9), (43.11), (43.13), we see that \( \mathcal{C} \) is closed under forming good colimits. It remains to show that \( \Delta^n \in \mathcal{C} \) for all \( n \).

We have \( \Delta^1 \in \mathcal{C} \) by (4). Since \( \Delta^0 \) is a retract of \( \Delta^1 \), we get that \( \Delta^0 \in \mathcal{C} \) since weak equivalences in \( \mathcal{M} \) are closed under retracts (41.3).

The spines \( I^n \) can be built from \( \Delta^0 \) and \( \Delta^1 \) by a sequence of good pushouts (glue on one 1-simplex at a time), so the \( I^n \in \mathcal{C} \). The inclusions \( I^n \subset \Delta^n \) are inner anodyne (12.12), so by (3) and the 2-out-of-3 property of weak equivalences in \( \mathcal{M} \) it follows that \( \Delta^n \in \mathcal{C} \). \( \square \)

We will apply this idea to functors \( sSet \to sSet_{X/} \), where the slice category \( sSet_{X/} \) inherits its model structure from the Joyal model structure on \( sSet \) (41.4).

**Proof of (47.4).** The functors \( X \diamond (\_) \), \( X \star (\_) \), \( (\_) \circ X \), \( (\_) \star X : sSet \to sSet_{X/} \) satisfy the first three properties required of the functors in the previous proposition (47.6). That is, they (1) preserve colimits, (2) take monomorphisms to monomorphisms, and (3) take inner anodyne maps to categorical equivalences. Condition (3) for \( \circ \) follows from (46.4), while condition (3) for \( \star \) this follows from (26.12) since \( \text{InnHorn} \subseteq \text{LHorn} \cap \text{RHorn} \).

Thus, to show \( X \diamond Y \to X \star Y \) is a categorical equivalence for a fixed \( X \) and arbitrary \( Y \), it suffices by the previous proposition to show that \( X \diamond \Delta^1 \to X \star \Delta^1 \) is a categorical equivalence.

The same argument lets us reduce to the case when \( X = \Delta^1 \), i.e., to showing that a single map \( \overline{f} : \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1 \) is a categorical equivalence.

We will show \( \overline{f} \) is a categorical equivalence by producing a map \( \overline{g} : \Delta^1 \star \Delta^1 \to \Delta^1 \diamond \Delta^1 \) such that \( \overline{f} \overline{g} = \text{id}_{\Delta^1 \star \Delta^1} \) and \( \overline{g} \overline{f} \) is preisomorphic to the identity map of \( \Delta^1 \diamond \Delta^1 \), via (20.7).
Since $\Delta^1 \diamond \Delta^1$ is a quotient of a cube, we start with maps involving the cube. I will write vertices in $(\Delta^1)^\times$ as sequences $(a_1 a_2 a_3)$ where $a_i \in \{0, 1\}$. Let

$$f: (\Delta^1)^{\times 3} \to \Delta^3 = \Delta^1 \ast \Delta^1$$

be the map which on vertices sends

$$(a_1 a_2 a_3) \mapsto \sup\{ k \mid a_k = 1 \}.$$ 

On passage to quotients this gives the comparison map $\overline{f}: \Delta^1 \diamond \Delta^1 \to \Delta^1 \ast \Delta^1$ of the proposition. Let $g: \Delta^3 \to (\Delta^1)^{\times 3}$ be the map classifying the simplex $((000), (100), (110), (111))$, and let $\overline{g}: \Delta^3 \to \Delta^1 \diamond \Delta^1$ be the composite with the quotient map. We have $fg = \text{id}_{\Delta^3} = \overline{f}g$.

Let $h \in \text{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$ and $a, b \in \text{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$ be as indicated in the following picture.

These pass to simplices $\overline{h}, \pi, \overline{b}$ in $\text{Map}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$. The edges $\pi$ and $\overline{b}$ are preisomorphisms, as one sees that for each vertex $v \in (\Delta^1 \diamond \Delta^1)$, the induced maps $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1)$ are preisomorphisms, as desired. Thus $\overline{f}g$ and $\overline{f}f$ are preisomorphic to identity maps, and hence $\overline{f}$ is a categorical equivalence as desired.

47.7. Equivalence of slice and alternate slice.

47.8. Proposition. For any quasicategory $C$ and map $p: S \to C$, the comparison maps $C_{p/} \to C^{p/}$ and $C/p \to C/p$ are categorical equivalences.

Proof. We do the first case. We use the following fact: if $f: A \to B$ is a functor between quasicategories, then $f$ is a categorical equivalence if and only if the induced maps $\pi_0(\text{Fun}(X, A)_{\text{core}}) \to (\text{Fun}(X, B)_{\text{core}})$ are bijections for all simplicial sets $X$. I probably did this before somewhere.

Recall (46.10) that we have a commutative diagram

$$\begin{align*}
\text{Map}(X, C_{p/}) &\longrightarrow \text{Map}_{S/}(S \ast X, C) \\
\downarrow & \\
\text{Map}(X, C^{p/}) &\longrightarrow \text{Map}_{S/}(S \circ X, C)
\end{align*}$$

in which the bottom map is an isomorphism. By (47.2) the top map is a bijection on isomorphism classes of objects By (47.4) $S \circ X \to S \ast X$ is a categorical equivalence, and thus the right-hand map is a categorical equivalence, and hence a bijection on isomorphism classes of objects. It follows that the left-hand map is a bijection on isomorphism classes of objects, and the proposition is proved.

47.9. Corollary. For any quasicategory $C$ and map $p: S \to C$, the enriched adjunction maps $\text{Fun}(X, C_{p/}) \to \text{Map}_{S/}(S \ast X, C)$ and $\text{Fun}(X, C/p) \to \text{Map}_{S/}(X \ast S, C)$ are categorical equivalences.
47.10. **Alternate pushout-join.** Just as we defined the “pushout-join” \( \bigcirc \), we can define the “alternate pushout-join” \( \bigdiamond \): given \( f: A \to B \) and \( g: K \to L \), we obtain

\[
f \bigdiamond g: (B \circ K) \cup_{A \circ K} (A \circ L) \to B \circ L.
\]

47.11. **Proposition.** We have that \( \text{RHorn} \bigcirc \bigdiamond \text{Cell} \cup \text{Cell} \bigcirc \text{LHorn} \subseteq \text{Cell} \cap \text{CatEq} \).

**Proof.** We’ll show that \( \text{RHorn} \bigcirc \bigdiamond \text{Cell} \subseteq \text{Cell} \cap \text{CatEq} \). It is straightforward to show that the \( \bigcirc \bigdiamond \)-product of two monomorphisms is a monomorphism. Thus, it suffices to show that for \( f: A \to B \) right anodyne and any inclusion \( g: K \to L \), the map \( f \bigdiamond g \) is a categorical equivalence. We know that \( \text{RHorn} \subseteq \text{InnHorn} \subseteq \text{CatEq} \) (26.12), so \( f \bigcirc g \) is a categorical equivalence. Furthermore, in

\[
\begin{array}{ccc}
(B \circ K) \cup_{A \circ K} (A \circ L) & \to & B \circ L \\
\downarrow & & \downarrow \\
(B \star K) \cup_{A \star K} (A \star L) & \to & B \star L
\end{array}
\]

the vertical maps are categorical equivalences; this uses the result proved above (47.4), as well as the fact that since \( f \) and \( g \) are monomorphisms, the domains of \( f \bigcirc g \) and \( f \bigcirc g \) are constructed from good pushouts.

**Question:** is \( \text{LHorn} \bigcirc \bigdiamond \text{Cell} \subseteq \text{InnHorn} \)?

47.12. **Proposition.** Given \( K \xrightarrow{j} L \xrightarrow{p} C \), if \( C \) is a quasicategory and \( j \) is a monomorphism, then \( C^{|p|} \to C^{|j|} \) is a left fibration, and \( C^{|p|} \to C^{|j|} \) is a right fibration.

**Proof.** Follows from \( \text{LHorn} \bigcirc \bigdiamond \text{Cell} \subseteq \text{Cell} \cap \text{CatEq} \) and \( \text{Cell} \bigcirc \text{RHorn} \subseteq \text{Cell} \cap \text{CatEq} \).

47.13. **Equivalence of various mapping spaces.** Finally we can prove our original goal.

47.14. **Proposition.** For any quasicategory \( C \) and object \( x \in C_0 \), the natural comparison maps \( \text{map}_C^R(x,y) \to \text{map}_C(x,y) \leftarrow \text{map}_C^L(x,y) \) are weak equivalences.

**Proof.** In

\[
\begin{array}{ccc}
C_{x/} & \xrightarrow{f} & C^{|p|} \\
\downarrow p & & \downarrow q \\
C & \xrightarrow{q} & C^{|j|}
\end{array}
\]

the map \( f \) is a categorical equivalence (47.8) and \( p \) and \( q \) are left fibrations by (??) and (??) respectively, and hence are categorical fibrations (??). It follows that the induced maps on fibers \( \text{map}_C^R(x,c) \to \text{map}_C(x,c) \) are categorical equivalences and hence weak equivalences, since the pullbacks describing the pullbacks are good pullbacks (with respect to the Joyal model structure).

47.15. **Slices as fibers.** Rewrite this in terms of the enriched adjunction maps.

The alternate slice \( C^{|j|} \) has another convenient characterization: it is the fiber over \( f \) of a map between functor categories.

47.16. **Proposition.** For a map \( f: S \to X \) of simplicial sets, the alternate slice \( X^{|f|} \) is isomorphic to the fiber of the restriction map

\[
\text{Map}(S \circ \Delta^0, X) \to \text{Map}(S, X).
\]

over \( f \).
Proof. Let $F$ be the fiber of the restriction map. There is an evident correspondence

\[
\begin{array}{ccc}
K & \xrightarrow{\sim} & F \\
\pi & \downarrow & \downarrow f \\
S \times K & \xrightarrow{\sim} & S \\
\end{array}
\]

The claim follows by showing that the evident quotient map

\[
S \times K \xrightarrow{\sim} (S \circ \Delta^0) \times K \cup_{S \times K} S
\]

compatible with the standard inclusions of $S$. \qed

We can also consider the fiber of the inclusion $S \subset S \star \Delta^0$ into the standard cone. This gives yet another version of the slice.

47.17. Corollary. Let $C$ be a quasicategory, and let $F(f) :=$ the fiber of $\text{Fun}(S^o, C) \to \text{Fun}(S, C)$ over $f$. Then there is a chain of categorical equivalences

\[
F(f) \to C^{f/} \leftarrow C_{f/}.
\]

Furthermore, $F(f)$ and $C_{f/}$ have the same set of 0-simplices, and both arrows above coincide on 0-simplices.

Proof. The second equivalence is just (47.8). For the first equivalence, note that

\[
\begin{array}{ccc}
\text{Fun}(S^o, C) & \xrightarrow{\sim} & \text{Fun}(S \circ \Delta^0, C) \\
\downarrow & & \downarrow \\
\text{Fun}(S, C) & \xrightarrow{\sim} & \text{Fun}(S, C)
\end{array}
\]

the top horizontal map is a categorical equivalence using (47.4), while the vertical maps are both categorical fibrations. Therefore the induced map on fibers over $f$ is a categorical equivalence, since the pullback squares in question are good.

The 0-simplices of $F(f)$ and $C_{f/}$ are exactly the set $\{S^o \to C\}$. Both inclusions $F(f)_0 \to (C^{f/})_0 \leftarrow (C_{f/})_0$ are induced by restriction along the standard comparison map $S \circ \Delta^0 \to S \star \Delta^0$. \qed

Part 6. Cartesian fibrations

Part 7. Old stuff

Note. From this point forward, these notes are not an organize narrative, but rather a collection of bits and pieces that might be worked into something useful at some point.

48. QUASICATEGORIES AS A QUASICATEGORY

48.1. Functors to sets and categories. In ordinary category theory, the category Set of sets plays a distinguished role. Notably, for every category\footnote{locally small category: the hom-sets of $C$ must actually be objects of Set} $C$ and every object $x \in \text{ob} \ C$, we have the representable and corepresentable functors

\[
\rho_x = \text{Hom}_C(-, x) : C^{\text{op}} \to \text{Set}, \quad \rho^x = \text{Hom}_C(x, -) : C \to \text{Set}.
\]

These are found everywhere, and are the subject of the Yoneda lemma: natural transformations $\rho_x \to F$ are in natural bijections with elements of $F(x)$.

In many situations we have examples of functors $C \to \text{Cat}$ to the category of categories.
48.2. Example. Let $C$ be the category of associative rings. We have a functor
\[ \text{Mod}: C^{\text{op}} \to \text{Cat}, \]
which sends $R$ to $\text{Mod}_R$, the category of left $R$-modules, and sends a homomorphism $f: R \to S$ to the restriction functor $f^*: \text{Mod}_S \to \text{Mod}_R$, so $f^*M$ has underlying abelian group $M$ and $R$-module structure given by $r \cdot m := f(r)m$.

Sometimes we have something that looks like a functor $C \to \text{Cat}$, but isn’t quite: these are pseudofunctors.

48.3. Example. Let $C$ be associative rings as in the last example. We have an operation
\[ \text{Mod}' : C \to \text{Cat}, \]
which sends $R$ to $\text{Mod}_R$, but sends a homomorphism $f: R \to S$ to the “extension of scalars functor”, given on objects by
\[ f_*M := S \otimes_R M. \]
Given $R \xrightarrow{f} S \xrightarrow{g} T$, we only have a natural isomorphism $T \otimes_S (S \otimes_R M) \approx T \otimes_R M$, so $g_*f_* \neq (gf)_*$, and thus $\text{Mod}'$ is not really a functor. If we incorporate the data of suitable natural isomorphisms $g_*f_* \approx (gf)_*$ and $\text{id}_* \approx \text{id}$ into the definition, we get the notion of a pseudofunctor. Give reference here.

48.4. Functors to $\infty$-groupoids and $\infty$-categories? Given a quasicategory $C$ and an object $x$, we would like to consider the functor represented by $x$. Roughly, this associates to an object $c$ in $C$ the object $\text{map}_C(c,x)$, which is a Kan complex. Thus, we might imagine that we get a functor
\[ \text{map}_C(-,x) : C^{\text{op}} \to \text{Kan}. \]
Of course, $\text{map}_C(-,x)$ is not a functor: given a morphism $f: c \to c'$ in $C$, we only know how to produce a zig-zag $\text{map}_C(c',x) \xrightarrow{\sim} \bullet \to \text{map}_C(c,x)$.

48.5. Three models for an $\infty$-category. We have touched on several different models for an $\infty$-category:

(1) quasicategories,
(2) categories enriched over Kan complexes.

Although $\text{QC}at$ and $\text{Kan}$ are not “naturally” quasicategories, they can be regarded naturally as of type (2).

Thus, we have a simplicially enriched category $\text{QC}at_{s,\text{core}}$, whose underlying objects are quasicategories, and whose mapping spaces are
\[ \text{Fun}(C,D)^{\text{core}}. \]

We can extract an actual quasicategory from this by using some kind of machine. For instance, Lurie defines
\[ \text{Cat}_\infty := \mathcal{N}(\text{QC}at_{s,\text{core}}) \]
where $\mathcal{N}$ denotes the “simplicial nerve” construction.

If we are interested in functors from $C^{\text{op}}$ to $\infty$-categories, we can model this by the simplicially enriched category
\[ (s\text{Set}_{/C}^{\text{Cart}})_{s,\text{core}}. \]
Here, $s\text{Set}_{/C}$ is the slice category, which has a simplicial enrichment with function objects
\[ \text{Map}_{/C}(p,p') \subseteq \text{Map}(E,E') \]
for $p: E \to C$ and $p': E' \to C$. We can consider a full subcategory consisting of $p: E \to C$ which are “Cartesian fibrations” (to be defined below). Cartesian fibrations are a special kind of inner
fibration, so $\text{Map}_{/C}(p, p')$ is a quasicategory in this case. We obtain a category enriched over Kan complexes, with function objects

$$\text{Map}_{/C}(p, p')^{\text{core}}.$$ 

This is the candidate for $\text{Fun}(C^{op}, \text{Cat}_\infty)$.

49. **Coherent nerve**

49.1. **The coherent nerve.** The coherent nerve $\mathcal{N}$ is a construction which turns a simplicially enriched category into a simplicial set, and in particular turns a Kan-enriched category into a quasicategory. It was invented by Cordier [Cor82]. The coherent nerve is constructed as right adjoint of a “realization/singular” pair

$$\mathfrak{C}: s\text{Set} \rightleftarrows s\text{Cat} : \mathcal{N}.$$ 

Given a finite totally ordered set $S$, define

$$\mathcal{P}(S) := \{ A \subseteq S \mid \{\text{min}, \text{max}\} \subseteq A \subseteq S \}.$$ 

This is a poset, ordered by set containment; here min, max denote the least and greatest elements of $S$ (possibly the same). If $S$ is empty, so is $\mathcal{P}(S)$.

Let $\mathfrak{C}(\Delta^n)$ denote the simplicially enriched category defined as follows.

- The objects are elements of $[n] = \{0, \ldots, n\}$.
- For objects $x, y \in [n]$, the function complex is

$$\text{Map}_{\mathfrak{C}(\Delta^n)}(x, y) := N\mathcal{P}([x, y]), \quad [x, y] := \{ t \in [n] \mid x \leq t \leq y \},$$

which is set to be empty if $x > y$.
- Composition is induced by the union operation on subsets:

$$(T, S) \mapsto T \cup S: \mathcal{P}([y, z]) \times \mathcal{P}([x, y]) \to \mathcal{P}([x, z]).$$

Every $f: [m] \to [n]$ in $\Delta$ gives rise to an enriched functor $\mathfrak{C}(f): \mathfrak{C}(\Delta^m) \to \mathfrak{C}(\Delta^n)$, which on objects operates as $f$ does on elements of the ordered sets, and is induced on morphisms by

$$S \mapsto f(S): \mathcal{P}([x, y]) \to \mathcal{P}([fx, fy]).$$

We obtain (after identifying $\Delta$ with its image in $s\text{Set}$) a functor $\mathfrak{C}: \Delta \to s\text{Cat}$.

Given a simplicially enriched category $C$, its **coherent nerve** (or **simplicial nerve**) is the simplicial set $\mathcal{N}C$ defined by

$$(\mathcal{N}C)_n = \text{Hom}_{s\text{Cat}}(\mathfrak{C}(\Delta^n), C).$$

49.2. **Quasicategories from simplicial nerves.**

49.3. **Proposition.** If $\mathcal{C}$ is a category enriched over Kan complexes, then $\mathcal{N}(\mathcal{C})$ is a quasicategory.

Proof. □

50. **Correspondences**

A **correspondence** is defined to be an inner fibration $p: M \to \Delta^1$. A map of correspondences is a morphism in the slice category $s\text{Set}/\Delta^1$. 
50.1. Correspondences of ordinary categories. If \( M \) is an ordinary category, then any functor \( p: M \to \Delta^1 \) is an inner fibration. Given such a functor, we can identify the following data:
- categories \( C := p^{-1}(\{0\}) \) and \( D := p^{-1}(\{1\}) \), the preimages of the vertices, and
- for each pair of objects \( c \in \text{ob} \, C, \, d \in \text{ob} \, D \), a set
  \[ \mathcal{M}(c, d) := \text{Hom}_M(c, d), \]
which
- fit together to give a functor
  \[ \mathcal{M}: C^{\text{op}} \times D \to \text{Set}. \]

Conversely, given the data of categories \( C \) and \( D \), and a functor \( \mathcal{M}: C^{\text{op}} \times D \to \text{Set} \), we can construct a category \( \mathcal{M} \) with functor \( p: \mathcal{M} \to \Delta^1 \) in the evident way, with
\[
ob \mathcal{M} := \text{ob} \, C \amalg \text{ob} \, D, \quad \text{mor} \mathcal{M} := \text{mor} \, C \amalg \left( \coprod_{c,d} \mathcal{M}(c,d) \right) \amalg \text{mor} \, D.
\]

Under the above, maps \( f: \mathcal{M} \to \mathcal{M}' \) between correspondences which are categories are sent to data consisting of: functors \( u: C \to C' \) and \( v: D \to D' \), and natural transformations \( \mathcal{M} \to \mathcal{M}' \circ (u \times v) \) of functors \( C^{\text{op}} \times D \to \text{Set} \).

50.2. Example. If \( C \) and \( D \) are categories, then the functor \( C \star D \to \Delta^0 \star \Delta^0 \approx \Delta^1 \) is an example of a correspondence. The corresponding functor \( \mathcal{M}: C^{\text{op}} \times D \to \text{Set} \) is the one with \( \mathcal{M}(c, d) = \{\ast\} \) for all objects.

50.3. Example. Let \( F: C \to D \) be a functor between categories. Then we get a functor \( \mathcal{M}: C^{\text{op}} \times D \to \text{Set} \) defined by
\[ \mathcal{M}(c, d) := \text{Hom}_D(F(c), D), \]
and thus an associated correspondence \( p: \mathcal{M} \to \Delta^1 \).

Similarly, let \( G: D \to C \) be a functor between categories. Then we get a functor \( \mathcal{M}': C^{\text{op}} \times D \to \text{Set} \) defined by
\[ \mathcal{M}'(c, d) := \text{Hom}_C(c, G(d)), \]
and thus an associated correspondence \( p': \mathcal{M}' \to \Delta^1 \).

50.4. Example. Suppose \( F: C \rightleftarrows D: G \) is an adjoint pair of functors. If we form \( \mathcal{M} \) and \( \mathcal{M}' \) as in the previous example, we see that the adjunction gives a natural isomorphism \( \mathcal{M} \approx \mathcal{M}' \) of functors \( C^{\text{op}} \times D \to \text{Set} \). The associated correspondences \( M \to \Delta^1 \) and \( M' \to \Delta^1 \) are isomorphic.

51. Cartesian and cocartesian morphisms

In the following, we fix an inner fibration \( p: M \to S \). We will often assume that \( S \) (and thus \( M \)) is a quasicategory.

Consider an edge \( f: x \to y \) in \( M \). We say that the edge represented by \( f: \Delta^1 \to M \) is \( p \)-cartesian if a lift exists in every diagram of the form

\[
\begin{array}{ccc}
\Delta^{\{n-1,n\}} & \xrightarrow{\Delta^n} & \Lambda^n_
 \\downarrow & \searrow \downarrow \quad \text{p} \quad \downarrow \\
& \Delta^n & \rightarrow S
\end{array}
\]

for all \( n \geq 2 \).

There is a dual notion of a \( p \)-cocartesian edge, where \( \Lambda^n_\n \) is replaced by \( \Lambda^n_0 \), and we use the leading edge of the simplex instead of the trailing edge.

We have already seen examples of this property.
• Let \( p \colon C \to * \) where \( C \) is a quasicategory. By the Joyal extension theorem (28.2), we have that an edge in \( C \) is \( p \)-cartesian if and only if it is \( p \)-cocartesian if and only if it is an isomorphism.

• Let \( p \colon M \to S \) be an inner fibration between quasicategories, and suppose \( f \in M_1 \) is an edge such that \( p(f) \) is an isomorphism in \( S \). By the Joyal lifting theorem (28.13), \( f \) is \( p \)-cartesian if and only if it is \( p \)-cocartesian if and only if \( f \) is an isomorphism in \( M \).

• If \( p \colon M \to S \) is a right fibration, then every edge in \( M \) is \( p \)-cartesian. Likewise, if \( p \) is a left fibration, then every edge in \( M \) is \( p \)-cocartesian.

Thus, Joyal’s theorem completely describes cartesian/cocartesian edges over an isomorphism in a quasicategory.

We have an equivalent formulation: \( f \) is \( p \)-cartesian if and only if
\[
M/f \to M/y \times_{S/p(y)} S/pf
\]

is a trivial fibration.

51.1. Cartesian edges and correspondence. Let \( p \colon M \to \Delta^1 \) be a correspondence, with \( M \) an ordinary category. We write
\[
C := p^{-1}([0]), \quad D := p^{-1}([1]), \quad \mathcal{M} : C^{\text{op}} \times D \to \text{Set}
\]

for the associated functor.

Suppose \( f \colon c \to d \) is an edge such that \( p(f) = \langle 01 \rangle \).

51.2. Lemma. The edge \( f \) is \( p \)-cartesian if and only if, for each \( u \colon x \to d \) with \( p(u) = \langle 01 \rangle \), there exists a unique \( v \colon x \to c \) such that \( fv = u \).

In particular, if \( f \) is \( p \)-cartesian, then composition
\[
f_* : \text{Hom}_M(x, c) \to \text{Hom}_M(x, d)
\]
is a bijection for all \( x \in \text{ob} C \). Equivalently, the map
\[
\text{Hom}_C(x, c) \to \mathcal{M}(x, d), \quad v \mapsto fv
\]
is a bijection, so \( \mathcal{M}(-, d) : C^{\text{op}} \to \text{Set} \) is represented by \( c \).

51.3. Box criterion for cartesian edges.

51.4. Proposition. [Lur09, 2.4.1.8] Let \( p \colon M \to S \) be an inner fibration, and \( f \in M_1 \) an edge. Then \( f \) is \( p \)-cartesian if and only if a lift exists in every diagram of the form
\[
\Delta^1 \times \{n\} \ar[r]^-f \ar[d] & (\{1\} \times \Delta^n) \cup_{\{1\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) \ar[r]^-p \ar[d] & M \\
\Delta^1 \times \Delta^n \ar[d] \ar[r] & S \\
\Delta^n \ar[r]^-b & \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)
\]

for all \( n \geq 1 \).

Proof. The if part is just like the proof of the box version of Joyal lifting. \( \square \)

We reformulate this criterion. Consider the box power map
\[
q := p^{\square(\{1\} \subset \Delta^1)} : \text{Map}(\Delta^1, M) \to \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M).
\]

Then the above proposition says that \( f \) is \( p \)-cartesian iff a lift exists in every diagram
\[
\partial \Delta^n \ar[r]^-a \ar[d] & \text{Map}(\Delta^1, M) \ar[d]^-q \\
\Delta^n \ar[r]_-b & \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)
\]
such that \( n \geq 1 \) and \( a(n) = f \in \text{Map}(\Delta^1, M)_0 \).

51.5. **Uniqueness of lifts to Cartesian edges.** Let \( U \subseteq \text{Map}(\Delta^1, M) \) be the full subsimplicial set spanned by the vertices which represent \( p \)-cartesian edges. Likewise, let \( V \subseteq \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M) \) denote the essential image of \( U \) under \( q \), i.e., the full subsimplicial set spanned by the vertices \( q(U_0) \). Obviously, the map \( q \) restricts to a map \( q': U \to V \).

Note in particular that \( V_0 \) is the subset of \( \{(g, y) \in S_1 \times M_0 \mid g_1 = p(y)\} \) such that there exists a Cartesian edge \( f \in M_1 \) with \( f_1 = y \) and \( p(f) = g \), and the preimage of \( (g, y) \) under \( q': U \to V \) is the set of all choices of lifts. The following in particular asserts a kind of uniqueness for choices of lifts.

51.6. **Proposition.** The map \( q': U \to V \) is a trivial fibration.

*Proof.* Consider

\[
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{a} & U \\
\downarrow & \downarrow \text{q} & \downarrow q \\
\Delta^n & \xrightarrow{q(U)} & \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M)
\end{array}
\]

If \( n \geq 1 \), then a lift \( s: \Delta^n \to \text{Map}(\Delta^1, M) \) exists by the previous proposition, since \( a(n) \in U_0 \subseteq \text{Map}(\Delta^1, M)_0 \) represents a \( p \)-cartesian edge. Because \( (\partial \Delta^n)_0 = (\Delta^n)_0 \) when \( n \geq 1 \), we see that \( s \) maps into the full subcomplex \( U \).

If \( n = 0 \), this amounts to \( U_0 \to V_0 \) being surjective, which holds by definition. \( \square \)

51.7. **Cartesian fibration.** A **cartesian fibration** is a map \( p: M \to S \) which is an inner fibration, and is such that for all \( (g, y) \in S_1 \times M_0 \) with \( g_1 = p(y) \), there exists a \( p \)-cartesian edge \( f \) with \( p(f) = g \) and \( f_1 = y \).

51.8. **Example.** Every left or right fibration is a cartesian fibration, since all edges are cartesian.

By the above, we see that an inner fibration \( p: M \to S \) is a cartesian fibration if and only if \( V = \text{Map}(\Delta^1, S) \times_{\text{Map}(\{1\}, S)} \text{Map}(\{1\}, M) \).

51.9. **Cartesian correspondences.** Given a map \( p: M \to S \), for any simplex \( a \in S_k \) write

\[ M_a := \text{Map}_S(\Delta^k, M) = \text{Map}_S(a, p). \]

Note that if \( a = bf \) for some simplicial operator \( f: [k] \to [l] \), we obtain an induced restriction map

\[ f^* : M_b \to M_a. \]

Given a correspondence \( p: M \to \Delta^1 \), we obtain

\[ C = M_{(0)} \xleftarrow{(0)^*} M_{(01)} \xrightarrow{(1)^*} M_{(1)} = D. \]

Note that these are all quasicategories. The objects of \( M_{(01)} \) are precisely the edges in \( M \) lying over \( (01) \).

51.10. **Proposition.** Let \( p: M \to S \) be a cartesian fibration, and let \( M^\text{cart}_{(01)} \subseteq M_{(01)} \) be the full subcategory spanned by elements corresponding to cartesian edges. Then \( M^\text{cart}_{(01)} \to M_{(1)} \) is a trivial fibration.
Proof. Every square in

\[
\begin{array}{ccc}
M_{(01)}^{\text{cart}} & \longrightarrow & U \\
\downarrow & & \downarrow i \\
M_{(01)} & \longrightarrow & \text{Map}(\Delta^1, M) \\
\downarrow & & \downarrow q \\
M_{(1)} & \longrightarrow & \text{Map}(\Delta^1, \Delta^1) \times_{\text{Map}\{1\}, \Delta^1} \text{Map}\{1\}, M) \\
\downarrow & & \downarrow \text{Map}\{1\}, M) \\
\{\text{id}_{\Delta^1}\} & \longrightarrow & \text{Map}(\Delta^1, \Delta^1) \times_{\text{Map}\{1\}, \Delta^1} \text{Map}\{1\}, \Delta^1) \\
\end{array}
\]

is a pullback. The result follows because \(q_i = q'\) is a trivial fibration. \(\square\)

More generally, given an inner fibration \(p: M \rightarrow S\) and a simplex \(a \in S_k\), the objects of the quasicategory \(M_a\) correspond to \(k\)-simplices \(b \in M_k\) such that \(p(b) = a\). Let \(M_{a}^{\text{cart}} \subseteq M_a\) denote the full subcategory spanned by objects corresponding to \(b \in M_k\) such that all edges of \(b\) are \(p\)-cartesian.

51.11. Proposition. Let \(p: M \rightarrow S\) be an inner fibration, and \(f \in M_1\) an edge. Consider

\[
\begin{array}{ccc}
\Delta^k \times \{n\} & \longrightarrow & (\Lambda_j^k \times \Delta^n) \cup_{\Lambda_j^k \times \partial \Delta^n} (\Delta^k \times \partial \Delta^n) \\
\downarrow & & \downarrow p \\
\Delta^1 \times \Delta^n & \longrightarrow & S
\end{array}
\]

where \(\Lambda_j^k \subset \Delta^k\) is a right horn inclusion, and \(f\) represents a \(p\)-cartesian edge. Then a lift exists whenever \(n, k \geq 1\), and also when \(k \geq 2, n = 0\).

Proof. This should also be like the box version of Joyal lifting. Note that if \(k = 0\), we recover the definition of \(p\)-cartesian edge. \(\square\)

This admits a reformulation: if \(f\) is \(p\)-cartesian, then if \(0 < j \leq k\) and \(a(n) = f\) is \(p\)-cartesian there is a lift in

\[
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \text{Map}(\Delta^k, M) \\
\downarrow a & & \downarrow q \\
\Delta^n & \longrightarrow & \text{Map}(\Delta^k, S) \times_{\text{Map}(\Lambda_j^k, S)} \text{Map}(\Lambda_j^k, M)
\end{array}
\]

when \(k \geq 1\) and \(n \geq 1\), or for all \(n \geq 0\) if \(k \geq 2\).

51.12. Cartesian fibrations and right fibrations.

51.13. Proposition. [Lur09, 2.4.2.4] A map \(p: M \rightarrow S\) is a right fibration iff it is a cartesian fibration whose fibers are Kan complexes.

Proof. We have already seen that a right fibration is a cartesian fibration, and has Kan complexes as fibers.

Now suppose \(p\) is cartesian fibration with Kan complex fibers. Let \(f: x \rightarrow y\) be an edge in \(M\). Since \(p\) is cartesian, there exists a \(p\)-cartesian edge \(f': x' \rightarrow y\) over \(p(f)\). Since \(p\) is cartesian fibration and \(f'\) a cartesian edge, there exists \(a \in M_2\) with \(a_{02} = f\) and \(a_{12} = f'\) and \(p(a) = (p(f))_{001}\). Thus \(g := a_{01}\) is an edge in the fiber over \((p(f))_0\), so is an isomorphism in that fiber. \(\square\)
51.14. **Mapping space criterion for cartesian edges.**

51.15. **Proposition.** [Lur09, 2.4.4.3] Let $p: C \to D$ be an inner fibration between quasicategories, and $f: x \to y$ a morphism in $C$. The following are equivalent.

1. $f$ is $p$-cartesian.
2. For every $c \in C_0$, the diagram

$$
\begin{array}{ccc}
\text{map}_C(c, x) & \xrightarrow{f_*} & \text{map}_C(c, y) \\
\downarrow & & \downarrow \\
\text{map}_D(p(c), p(x)) & \xrightarrow{p(f)_*} & \text{map}_D(p(c), p(y))
\end{array}
$$

is a homotopy pullback.

52. **Limits and colimits as functors**

Suppose $J$ and $C$ are categories. We say that $C$ has all $J$-colimits if every functor $F: J \to C$ has a colimit in $J$. It is a standard observation that if $F$ is such a functor, then we can assemble a functor

$$
colim_J: \text{Fun}(J, C) \to C.
$$

In fact, we can regard this functor as a composite of functors

$$
\text{Fun}(J, C) \xrightarrow{s} \text{Fun}(J^{op}, C) \xrightarrow{\text{eval. at } v} C,
$$

where $s$ is some section of the restriction functor $\text{Fun}(J^{op}, C) \to \text{Fun}(J, C)$ which takes values in colimit cones.

Even when $C$ does not have all $J$-colimits, we can assert the following. Consider the diagram

$$
\begin{array}{ccc}
\text{Fun}^{\text{colim cone}}(J^{op}, C) & \xrightarrow{p} & \text{Fun}(J^{op}, C) \\
\downarrow & & \downarrow \\
\text{Fun}^{\exists \text{colim}}(J, C) & \xrightarrow{\exists} & \text{Fun}(J, C)
\end{array}
$$

in which the objects on the left are the evident full subcategories of the corresponding objects on the right, i.e., the ones consisting of colimit cones, and of functors which admit colimits. Then $p$ is an equivalence of categories, and in fact is a trivial fibration. Therefore there is a contractible groupoid of sections of $p$, and any section $s$ gives rise to a colimit functor

$$
\text{Fun}^{\exists \text{colim}}(J, C) \xrightarrow{s} \text{Fun}^{\text{colim cone}}(J^{op}, C) \xrightarrow{\text{eval. at } v} C.
$$

We want to prove the analogous statement for quasicategories. Thus, given a quasicategory $C$ and a simplicial set $S$, let $\text{Fun}^{\text{colim cone}}(S^{op}, C) \subseteq \text{Fun}(S^{op}, C)$ denote the full subcategory spanned by $S^{op} \to C$ which are colimit cones, and let $\text{Fun}^{\exists \text{colim}}(S, C) \subseteq \text{Fun}(S, C)$ denote the full subcategory spanned by $S \to C$ for which a colimit exists.

52.1. **Proposition.** The induced projection $q: \text{Fun}^{\text{colim cone}}(S^{op}, C) \to \text{Fun}^{\exists \text{colim}}(S, C)$ is a trivial fibration.

We refer to this as the functoriality of colimits. We will prove it below.

The strategy is to show (1) that $q$ is an isofibration, and (2) that $q$ is fully faithful and essentially surjective. Then (39.1) applies to show that $q$ is a categorical equivalence, and so a trivial fibration by (40.1).

Parts of this are already clear. For instance, $q$ is certainly an inner fibration, since $p: \text{Fun}(S^{op}, C) \to \text{Fun}(S, C)$ is one, and $q$ is the restriction of $p$ to full subcategories. Likewise, $q$ is manifestly essentially surjective.
52.2. Conical maps. In what follows, \( C \) will be a quasicategory and \( S \) a simplicial set, and we write
\[
V = V(S) := \text{Fun}(S^\triangleright, C), \quad U = U(S) := \text{Fun}(S, C).
\]
Let \( p: V \to U \) be the evident restriction map.

Let’s say that a morphism \( \alpha: \hat{f} \to \hat{g} \) in \( V \) is conical if its evaluation \( \alpha(v): \hat{f}(v) \to \hat{g}(v) \) at the cone point of \( S^\triangleright \) is an isomorphism in \( C \).

What follows are two propositions involving conical maps. We will prove them soon. The first says that any morphism in \( U \) can be lifted to a conical morphism in \( V \) with prescribed target.

52.3. Proposition. Fix a quasicategory \( C \) and a simplicial set \( S \). Suppose given
- a functor \( \hat{g}: S^\triangleright \to C \), and
- a natural transformation \( \alpha: f \Rightarrow g \) of functors \( S \to C \) such that \( g = \hat{g}|S \).

Then there exists a conical morphism \( \alpha: \hat{f} \to \hat{g} \) in \( V \) such that \( \alpha|S = \alpha \).

The second says that morphisms in \( V \) can be “transported” along conical maps.

52.4. Proposition. Fix a quasicategory \( C \), simplicial set \( S \), and a map \( \alpha: \hat{f} \to \hat{g} \) in \( V \), and let \( \alpha: f \to g \) denote \( \hat{\alpha}|S \). For any object \( \hat{h} \) of \( V \), consider the square
\[
\begin{array}{ccc}
\map_V(\hat{h}, \hat{f}) & \xrightarrow{\alpha\circ} & \map_V(\hat{h}, \hat{g}) \\
\downarrow & & \downarrow \\
\map_U(h, f) & \xrightarrow{\alpha\circ} & \map_U(h, g)
\end{array}
\]
where \( h = \hat{h}|S \), and the horizontal maps are induced by postcomposition with \( \hat{\alpha} \) and \( \alpha \) respectively. If \( \hat{\alpha} \) is conical, then the above square is a homotopy pullback square.

We will explain and prove these two propositions soon. For the time being, you should convince yourself that if \( C \) is the nerve of an ordinary category, then both propositions are entirely straightforward to prove.

52.5. Proof of functoriality of colimits, using properties of conical maps. Recall that \( \hat{f}: S^\triangleright \to C \) extending \( f: S \to C \) is a colimit cone if and only if it corresponds to an initial object of \( C_f/ \). Using the categorical equivalences
\[
F(f) \to C_f/ \leftarrow C_f/
\]
where \( F(f) \subseteq V \) is the fiber of \( p: V \to U \) over \( f \), we see that it is equivalent to say that \( \hat{f} \) is initial in \( F(f) \).

The following gives a criterion for being a colimit cone in terms of the whole functor category \( V = \text{Fun}(S^\triangleright, C) \), rather than just in terms of the fiber over some \( f \).

52.6. Proposition. A functor \( \hat{f}: S^\triangleright \to C \) is a colimit cone if and only if
\[
p': \map_V(\hat{f}, \hat{g}) \to \map_U(f, g)
\]
is a weak equivalence for every \( \hat{g}: S^\triangleright \to C \), \( g = \hat{g}|S = p(\hat{g}) \).
Proof. Since $p: V \to U$ is a categorical fibration, the induced maps $p'$ on mapping spaces are Kan fibrations. Thus, $p'$ is a weak equivalence if and only if its fibers are contractible.

($\Leftarrow$) Suppose every $p'$ is a weak equivalence. Then in particular $p'$ is a weak equivalence for any $\hat{g}: S^\circ \to C$ such that $\hat{g}|S = f$. In this case, the fiber of $p'$ over $1_f \in \map_U(f, f)$ is precisely the mapping space $\map_{\map(f)}(\hat{f}, \hat{g})$ in the fiber quasicategory $F(f) \subseteq \Fun(S^\circ, C)$, and this fiber is contractible. Therefore, $\hat{f}$ is an initial object of $F(f)$, and therefore $\hat{f}$ is initial in $C_{f/}$ by the above discussion. We have shown that $\hat{f}$ is a colimit cone.

(\Rightarrow) Suppose $\hat{f}$ is a colimit cone. Therefore for $\hat{f}'$ such that $\hat{f}'|S = f$ the fiber of $\map_V(\hat{f}, \hat{f}') \to \map_U(f, f)$ over $1_f$ is contractible. We need to show that the fiber of $p': \map_V(\hat{f}, \hat{g}) \to \map_U(f, g)$ over a general $\alpha \in \map_U(f, g)$ is contractible.

Given such an $\alpha$, choose a conical map $\alpha: \hat{f}' \to \hat{g}$ with $\alpha|S = \alpha(52.3)$, and consider the resulting square

$$\begin{array}{ccc}
\map_V(\hat{f}, \hat{f}') & \xrightarrow{\alpha} & \map_V(\hat{f}, \hat{g}) \\
\map_U(f, f) & \xrightarrow{\alpha} & \map_U(f, g) \\
\downarrow{p'} & & \downarrow{p''} \\
1_f & \xrightarrow{\alpha} & \alpha
\end{array}$$

Since $\alpha$ is conical, the square is a homotopy pullback square (52.4). Therefore, the fiber of $p'$ over $\alpha$ is weakly equivalent to the fiber of $p'$ over $1_f$, which is contractible since $\hat{f}$ is a colimit cone. \hfill \Box

Proof of (5.21). First we show that $Q: \Fun_{\colim}{\con}(S^\circ, C) \to \Fun_{\colim}^3(S, C)$ is an isofibration; we have already observed that it is an inner fibration. Given an isomorphism $\alpha: f \to g$ between objects in $\Fun_{\colim}(S, C) \subseteq U$ and a choice of colimit cone $\hat{g}$ over $g$, chose a conical lift $\hat{\alpha}: \hat{f} \to \hat{g}$. The arrow $\alpha: S^\circ \times \Delta^1 \to C$ restricts to an isomorphism at each vertex of $S^\circ$, and so is a natural isomorphism by the objectwise criterion for natural isomorphisms. Thus $\hat{f}$ is also a colimit cone by (52.6), so $\alpha$ is an isomorphism in $\Fun_{\colim}{\con}(S^\circ, C)$.

We have already observed that $q$ is essentially surjective (in fact, it is surjective on 0-simplices). That $q$ is fully faithful is immediate from (52.6). \hfill \Box

52.7. Proof of properties of conical maps.

Proof of (52.3). Recall the situation: we are given a natural transformation $\alpha: f \Rightarrow g$ of functors $S \to C$, and a lift $\hat{g}: S^\circ \to C$ of the target to the cone, and we want to find a conical lift of $\alpha$:

$$\begin{array}{ccc}
\{1\} & \xrightarrow{\hat{g}} & \Fun(S^\circ, C) \\
\downarrow & & \downarrow \hat{\alpha} \\
\Delta^1 & \xrightarrow{\alpha} & \Fun(S, C)
\end{array}$$

We make use of a natural map

$$\kappa: S^\circ \times K \to (S \times K)^\circ.$$ 

Note that this map sends $\{v\} \times K$ to the cone point $\{v\}$. Consider the composite

$$\lambda: (S \times \Delta^1) \cup_{S \times \{1\}} (S^\circ \times \{1\}) \to S^\circ \times \Delta^1 \xrightarrow{\kappa} (S \times \Delta^1)^\circ$$

where the first map is the box-product $(S \subseteq S^\circ) \Box (\{1\} \subseteq \Delta^1)$. By inspection, we see that the composite map can be identified with the box-join

$$(S \times \{1\}) \subseteq S \times \Delta^1 \boxdot (\emptyset \subseteq \Delta^0).$$
Since RHorn\(\Box\)Cell \(\subseteq\) RHorn (45.4) we have that \((S \times \{1\} \subseteq S \times \Delta^1)\) is right anodyne. Likewise, since RHorn\(\Box\)Cell \(\subseteq\) InnHorn (26.12), we conclude that \(\lambda\) is inner anodyne. Therefore, an extension \(\pi\) exists in

\[
\begin{array}{c}
(S \times \Delta^1) \cup_{S \times \{1\}} (S^\circ \times \{1\}) \xrightarrow{(\alpha, \overline{\beta})} C \\
\downarrow \pi \\
(S \times \Delta^1)^\circ \\
\end{array}
\]

We set \(\tilde{\alpha} := \overline{\alpha} \circ \kappa\). It is clear that \(\tilde{\alpha}\) is conical: \(\hat{\alpha}(v)\) is the identity map of \(\overline{\alpha}(v)\).

For the proof of (52.4), let’s first note that, as stated, it actually doesn’t make sense! This proposition asserts that for conical \(\hat{\alpha}\), the diagram

\[
\begin{array}{ccc}
\text{map}_V(\hat{h}, \hat{f}) & \xrightarrow{\tilde{\alpha}} & \text{map}_V(\hat{h}, \hat{g}) \\
\downarrow & & \downarrow \\
\text{map}_U(h, f) & \xrightarrow{\alpha} & \text{map}_U(h, g)
\end{array}
\]

is a homotopy pullback. However, the horizontal maps (“postcomposition” with \(\alpha\) and \(\hat{\alpha}\)) are only defined as a homotopy class of maps in \(h\Kan\). For instance, “\(\alpha \circ \hat{\alpha}\)” is the homotopy class defined by the zig-zag around the left and top of the diagram

\[
\begin{array}{ccc}
\text{map}_U(h, f, g)_\alpha & \rightarrow & \text{map}_U(h, f, g) \xrightarrow{\text{comp}} \text{map}_U(h, g) \\
\sim & \downarrow & \sim \\
\text{map}_U(h, f) \times \{\alpha\} & \rightarrow & \text{map}_U(h, f) \times \text{map}(f, g)
\end{array}
\]

where the left-hand square is a pullback. The correct statement of (52.4) is that in

\[
\begin{array}{ccc}
\text{map}_V(\hat{h}, \hat{f}) & \xleftarrow{\sim} & \text{map}_V(\hat{h}, \hat{f}, \hat{g}) \tilde{\alpha} \rightarrow \text{map}_V(\hat{h}, \hat{g}) \\
\downarrow & & \downarrow \\
\text{map}_U(h, f) & \xleftarrow{\sim} & \text{map}_U(h, f, g) \alpha \rightarrow \text{map}_U(h, g)
\end{array}
\]

the right-hand square is a homotopy pullback.

We can refine this a little further. Fix a map \(e: \Delta^{1,2} \rightarrow C\). For a simplicial set \(S\), let \(K \subseteq S^\circ \times \Delta^2\) be a subcomplex containing the edge \(\{v\} \times \Delta^{1,2}\), and define \(\text{Map}(K, C)_e\) by the pullback square

\[
\begin{array}{c}
\text{Map}(K, C)_e \\
\downarrow \\
\{e\} \\
\end{array} \xrightarrow{\text{Map}(\{v\} \times \Delta^{1,2}, C)} \text{Map}(\{v\} \times \Delta^{1,2}, C)
\]

To prove our proposition, it suffices to show that for every isomorphism \(e\) in \(C\), the map

\[
\text{Map}(S^\circ \times \Delta^2, C)_e \rightarrow \text{Map}((S^\circ \times \Lambda^2_2) \cup_{S \times \Lambda^2_2} (S \times \Delta^2), C)_e
\]
is a trivial fibration. Equivalently, we must produce a lift in each diagram of the form

\[
\begin{array}{ccc}
\{v\} \times \Delta^{\{1,2\}} & \rightarrow & S^\circ \times \Delta^n \\
\downarrow & & \downarrow \downarrow \\
\Map(\Delta^2, C) & \rightarrow & \Map(\Delta^{\{1,2\}}, C)
\end{array}
\]

We reduce to producing a lift in

\[
\begin{array}{ccc}
\{n\} \times \Delta^{\{1,2\}} & \rightarrow & (\Delta^n \times \Lambda^2_2) \cup_{\partial \Delta^n \times \Lambda^2_2} (\partial \Delta^n \times \Delta^2) \\
\downarrow & & \downarrow \downarrow \\
\Delta^n \times \Delta^2 & \rightarrow & C
\end{array}
\]

where \(e\) is an isomorphism in \(C\). This is precisely the box-version of Joyal extension.

53. More stuff

Recall that the join constructions \(K \star -\) and \(- \star K\) are colimit preserving functors \(s\text{Set} \rightarrow s\text{Set}_K/\) to the category of simplicial sets under \(K\). In particular, viewed as functors \(s\text{Set} \rightarrow s\text{Set}\) to plain simplicial sets, they preserve pushouts, and transfinite compositions.

53.1. Proposition. If \(A\) is a class of maps in \(s\text{Set}\), then \(K \star A \subseteq K\star \{\} \leq \) and \(\overline{A} \star K \subseteq \overline{A} \star \overline{K}\).

Proof. Check that \(K \star -: s\text{Set} \rightarrow s\text{Set}\) preserves isomorphisms, transfinite composition, pushouts, and retracts. \(\square\)

53.2. Remark. Given \(f: X \rightarrow Y\) and \(K\), we have a factorization of \(K \star f\) as

\[
K \star X \rightarrow (K \star X) \amalg_{\varnothing \times X} (\varnothing \star Y) \rightarrow K \star Y.
\]

53.3. Proposition. We have \(\Delta^0 \star \text{Cell} \subseteq \text{LHorn}\) and \(\text{Cell} \star \Delta^0 \subseteq \text{RHorn}\).

53.4. Proposition. Let \(C\) be a quasicategory and \(x\) an object of \(C\). Then \(x\) is an initial object iff \(\{x\} \rightarrow C\) is left anodyne, and \(x\) is a terminal object iff \(\{x\} \rightarrow C\) is right anodyne.

Proof. \((\Rightarrow)\) Let \(x\) be terminal, and consider \(j: \{x\} \rightarrow C\). Since \(j^\circ\) is right anodyne, it suffices to show that \(j\) is a retract of \(j^\circ\). To do this, we construct a map \(r\) fitting into

\[
\begin{array}{ccc}
\{x\} & \xrightarrow{j} & \{x\} \\
\downarrow & & \downarrow \\
C & \xrightarrow{j^\circ} & C
\end{array}
\]

This amounts to solving the lifting problem

\[
\begin{array}{ccc}
C \cup \{x\}^\circ & \xrightarrow{\text{id}, 1_x} & C \\
\downarrow & & \downarrow \\
C^\circ & \xrightarrow{r} & C
\end{array}
\]

\[
\begin{array}{ccc}
\{x\} & \xrightarrow{1_x} & C_x^j \\
\downarrow & & \downarrow \\
C & \xrightarrow{r} & C
\end{array}
\]

\[
\begin{array}{ccc}
\{x\} & \xrightarrow{1_x} & C_x^j \\
\downarrow & & \downarrow \\
C & \xrightarrow{r} & C
\end{array}
\]
Since \( x \) is terminal, \( C_{\mathcal{X}} \to C \) is a trivial fibration \((??)\), so a lift exists.

\[(\iff) \text{Suppose } j: \{x\} \to C \text{ is right anodyne. Since } C_{\mathcal{X}} \to C \text{ is a right fibration, a lift exists in}
\]

\[
\begin{array}{ccc}
\{x\} & \xrightarrow{1_x} & C_{\mathcal{X}} \\
\downarrow & & \downarrow \\
C & \xrightarrow{a} & C
\end{array}
\]

which is equivalent to \( x \) being terminal. \( \Box \)

53.5. Corollary. Let \( p: D \to C \) be a right fibration between quasicategories, and let \( x \) be an object of \( C \). Then the induced map

\[
\text{Map}(C_{\mathcal{X}}, D) \to \text{Map}(\{1_x\}, D) \times_{\text{Map}(\{1_x\}, C)} \text{Map}(C_{\mathcal{X}}, C)
\]

is a trivial fibration. In particular, the map

\[
\text{Map}_C(C_{\mathcal{X}}, D) \to \text{Map}_C(\{1_x\}, D)
\]

induced by restriction over the projection map \((C_{\mathcal{X}} \to C) \in \text{Map}(C_{\mathcal{X}}, C)\) is a trivial fibration between Kan complexes.

54. Introduction: the cover/functor correspondence

Consider the following classes of maps between simplicial sets, called respectively covers, left covers and right covers.

\[
\text{Cover} := \text{(Horn} \sqcup \text{Horn}^\vee)\subseteq, \\
\text{LCover} := \text{(LHorn} \sqcup \text{LHorn}^\vee)\subseteq, \\
\text{RCover} := \text{(RHorn} \sqcup \text{RHorn}^\vee)\subseteq.
\]

Thus, Cover is the subclass of Kan fibrations, which admit unique lifting for every lifting problem with respect to horns, and therefore with respect to all anodyne maps. Similar statements hold for LCover and RCover.

We obtain for each simplicial set \( X \) full subcategories \( s\text{Set}^{\text{Cover}}/X \), \( s\text{Set}^{\text{LCover}}/X \), \( s\text{Set}^{\text{RCover}}/X \), of the slice category \( s\text{Set}/X \). We will see that each of these categories is equivalent to a certain category of functors from \( hX \) to sets.

54.1. Simplicial covering maps. Covers \( p: E \to X \) are precisely the simplicial analogues of covering maps of spaces. To see this, will first note that covers are “locally trivial”.

54.2. Lemma. Cover = \((S \sqcup S^\vee)\subseteq\), where \( S \) is the set of all maps \( \Delta^m \to \Delta^n \) between standard simplices. Thus, \( p: E \to X \) is a cover if and only if for each simplex \( a \in X_n \) and each simplicial operator \( \delta: [m] \to [n] \), the function \( x \mapsto x\delta: p^{-1}(a) \to p^{-1}(a\delta) \) is a bijection of sets.

Proof. Given a class \( B \) of maps, let \( \square B \) denote the class of maps \( f \) which have unique lifting with respect to maps in \( B \); i.e., such that \( \{f, f^\vee\} \sqsubseteq B \). An easy argument shows that \( \square B \) is weakly saturated, and also has the property that if \( f, g \circ f \in \square B \), then \( g \in \square B \).

Because any inclusion \( \Delta^0 \to \Delta^n \) of a vertex is anodyne and thus in \( \square \text{Cover} \), applying the observation of the previous paragraph to a composite \( \Delta^0 \to \Delta^m \xrightarrow{\delta} \Delta^n \) shows that every such \( \delta \in \square \text{Cover} \). Hence \( \text{Cover} \subseteq (S \sqcup S^\vee)\subseteq \).

Conversely, let \( B := (S \sqcup S^\vee)\subseteq \), so that \( S \subseteq \square B \). Since \( \square B \) is weakly saturated, we can show that \( \text{Horn} \subseteq \square B \), by building horn inclusions out of (injective) maps between simplices. Thus \( (S \sqcup S^\vee)\subseteq \subseteq \text{Cover} \). \( \Box \)
54.3. Corollary. A map $p \colon E \to X$ is a cover if and only if for each map $f \colon \Delta^n \to X$ there is a pullback square of the form

$$
\begin{array}{ccc}
\coprod \Delta^n & \longrightarrow & E \\
\downarrow \text{proj} & & \downarrow p \\
\Delta^n & \underset{f}{\longrightarrow} & X
\end{array}
$$

Proof. Left as an exercise, using the lemma. \qed

Given a cover $p \colon E \to X$, we define a functor $F \colon hX \to \text{Set}^\text{core}$ as follows. For each object $x \in X_0$, we let $F(x) := p^{-1}(x)$. For each edge $f \in X_1$, we define a bijection $F(f) : F(f_0) \to F(f_1)$ as the composite

$$p^{-1}(f_0) \xleftarrow{(0)} p^{-1}(f) \xrightarrow{(1)} p^{-1}(f_1),$$

where both maps are bijections by the lemma. For each 2-simplex $a \in X_2$, we have a commutative diagram of bijections which shows that $F(a_{12})F(a_{01}) = F(a_{02})$. By the universal property of fundamental category, this extends uniquely to a functor $F : hX \to \text{Set}^\text{core}$.

Conversely, given a functor $F : hX \to \text{Set}^\text{core}$, we define a cover $p : E \to X$ as follows. Set

$$E_n := \coprod_{x \in X_n} F(x_0),$$

and for each simplicial operator $\delta : [m] \to [n]$, define $\delta^* : E_n \to E_m$ by

$$\delta^*(x, t) := (x\delta, F(x_{\delta(0)})(t)).$$

54.4. Proposition. The above constructions give an adjoint equivalence of categories

$$\text{sSet}^\text{Cover}_/X \approx \text{Fun}(hX, \text{Set}^\text{core}).$$

Proof. Left as an exercise, including showing that $\text{Un}(F)$ is in fact a cover. \qed

Let $\Pi_1 X := (hX)_{\text{Gpd}}$, the groupoid obtained by formally inverting all morphisms in $hX$. The above passes to an equivalence of categories $\text{sSet}^\text{Cover}_/X \approx \text{Fun}(\Pi_1 X, \text{Set})$. This is the simplicial set analogue of the classification of covering spaces.

54.5. Left and right covers correspond to functors to sets. I’ll state the analogues of the above results for left and right covers. Proofs are left for the reader.

54.6. Lemma. \text{LCover} = (\text{S}I\text{I}S^\vee)^{\square}$ and \text{RCover} = (\text{T}I\text{I}T^\vee)^{\square}$, where $S = \{ \delta : \Delta^m \to \Delta^n \mid \delta(0) = 0 \}$ and $T = \{ \delta : \Delta^m \to \Delta^n \mid \delta(m) = n \}$.

Given a left cover $p : E \to X$, we define a functor $F : hX \to \text{Set}$ much as we did above, so that $F(x) := p^{-1}(x)$, while $F(f)$ is the map obtained as the composite

$$p^{-1}(f_0) \xleftarrow{(0)} p^{-1}(f) \xrightarrow{(1)} p^{-1}(f_1),$$

where the first map is a bijection by the lemma. We write $\text{St}(p) := F$ and call it the \textbf{straightening} of $p$.

Conversely, given a functor $F : hX \to \text{Set}$, we define a left cover $p : E \to X$ by

$$E_n := \coprod_{x \in X_n} F(x_0), \quad \delta^*(x, t) := (x\delta, F(x_{\delta(0)})(t)).$$
The constructions for right covers are similar, except that the corresponding functors are contravariant. Thus, given a right cover \( p: E \to X \) we obtain a straightening \( \text{St}(p) = F: hX^{\text{op}} \to \text{Set} \) given by \( F(x) = p^{-1}(x) \) and \( F(f) \) the composite
\[
p^{-1}(f_0) \xleftarrow{(0)} p^{-1}(f) \xrightarrow{(1)} p^{-1}(f_1),
\]
while for a functor \( F: hX^{\text{op}} \to \text{Set} \), we obtain a right cover \( p: E \to X \) by
\[
E_n := \prod_{x \in X_n} F(x_n), \quad \delta^*(x,t) := (x\delta, F(x_{\delta(m)},m)(t)).
\]

54.7. Proposition. The above constructions give adjoint equivalences of categories
\[
\text{St}: s\text{Set}^{\text{LCover}}_X \rightleftarrows \text{Fun}(hX, \text{Set}): \text{Un}, \quad \text{St}: s\text{Set}^{\text{RCover}}_X \rightleftarrows \text{Fun}(hX^{\text{op}}, \text{Set}): \text{Un}.
\]

Note also that left covers of \( X \) correspond to right covers of \( X^{\text{op}} \).

54.8. Straightening and unstraightening over a category. Suppose \( X = hX = C \) is itself a category and \( F: C \to \text{Set} \) a functor, and consider its unstraightening \( p: E \to C \). It is straightforward to show that \( E \) is the nerve of a category, namely the category of elements of \( F \), which has
- objects: pairs \((c, x), c \in C_0 \) and \( x \in F(c) \), and
- morphisms \((c, x) \to (c', x')\): morphisms \( \alpha: c \to c' \) in \( C \) such that \( F(\alpha)(x) = x' \).

The left cover map \( p \) is then just the evident forgetful functor. An analogous statement holds for contravariant functors.

Given any object \( x \) in \( C \), we have the slice projections \( C_x/ \to C \) and \( C/x \to C \).

54.9. Proposition. The projections \( C_x/ \to C \) and \( C/x \to C \) are left and right covers respectively. Their straightenings are representable functors: \( \text{St}(C_x/) \to C = \text{Hom}_C(x, -) \) and \( \text{St}(C/x \to C) = \text{Hom}_C(-, x) \).

Proof. Exercise. \( \square \)

54.10. Fiber product corresponds to tensor product. Let \( C \) be a category. Given functors \( F: C^{\text{op}} \to \text{Set} \) and \( G: C \to \text{Set} \), the tensor product is the set defined by
\[
F \otimes_C G := \text{colim} \left[ \prod_{c_0 \to c_1 \in C_1} F(c_1) \times G(c_0) \Rightarrow \prod_{c \in C} F(c) \times G(c) \right].
\]

54.11. Exercise. Given \( x \in C_0 \), let \( \rho_x := \text{Hom}_C(-, x): C^{\text{op}} \to \text{Set} \) and \( \rho^x := \text{Hom}_C(x, -): C \to \text{Set} \) denote the representable functors. Show that \( \rho_x \otimes_C G \approx G(x) \) and \( F \otimes_C \rho^x \approx F(x) \).

If \( A \to C \) and \( B \to C \) are the unstraightenings of \( F \) and \( G \) respectively, then
\[
F \otimes_C G \approx \pi_0(A \times_C B).
\]

54.12. The universal left cover. What happens if we unstraighten the identity functor of \( \text{Set} \)? We get the category of elements of \( \text{Set} \), which is precisely \( \text{Set}_x := \text{Set}_*/ \), the category of based sets.

55. Straightening and unstraightening

Let \( \mathcal{D}_{\partial^n}: \mathcal{E}(\Delta^n)^{\text{op}} \to s\text{Set} \) be the simplicially enriched functor defined as follows.
- For each object \( x \in \{0, \ldots, n\} \), set
  \[
  \mathcal{D}_{\partial^n}(x) := \mathcal{NP}_t(x),
  \]
  the nerve of the poset
  \[
  \mathcal{P}_t(x) := \{ S \mid \{x\} \subseteq S \subseteq [x, n]\}
  \]
of subsets of the interval \([x, n]\) = \(\{x, \ldots, n\}\) which contain the left endpoint.
The structure of enriched functor is induced by the union operation on subsets:
\[(T, S) \mapsto T \cup S : \mathcal{P}(x, y) \times \mathcal{P}_t(y) \to \mathcal{P}_t(x).\]

For each map \(\delta : \Delta^m \to \Delta^n\), we define a natural transformation
\[D_\delta : D_{\Delta^m} \to D_{\Delta^n} \circ \mathcal{C}(\delta)^{op}\]
of simplicially enriched functors \(\mathcal{C}(\Delta^m)^{op} \to sSet\), which at each object \(x\) of \(\mathcal{C}(\Delta^m)^{op}\) is a map \(D_{\Delta^m}(x) \to D_{\Delta^n}(\delta x)\) induced by the map of posets
\[S \mapsto \delta(S) : \mathcal{P}_t(x) \to \mathcal{P}_t(\delta x).\]

55.1. Remark. The functor \(D_{\Delta^n} : \mathcal{C}(\Delta^n)^{op} \to sSet\) is isomorphic to the representable functor \(\text{Map}_{\mathcal{C}(\Delta^n)^{op}}((-), v)\), where \(v\) represents the cone point of \((\Delta^n)^{op}\). Likewise, the natural transformation \(D_\delta : D_{\Delta^m} \to D_{\Delta^n} \circ \mathcal{C}(\delta)^{op}\) coincides with the transformation
\[\text{Map}_{\mathcal{C}(\Delta^m)^{op}}((-), v) \xrightarrow{\delta^\to} \text{Map}_{\mathcal{C}(\Delta^n)^{op}}((\delta(-), v)\]
induced by \(\delta^\to : (\Delta^m)^{op} \to (\Delta^n)^{op}\).

Fix a simplicial set \(S\), and consider a simplicially enriched functor \(F : \mathcal{C}(S)^{op} \to sSet\). We define a morphism
\[\text{Un}_S(F) : X \to S\]
of simplicial sets, called the unstraightening of \(F\) over \(S\), as follows.

- An \(n\)-simplex of \(\text{Un}_S F\) is a pair
  \[f : \Delta^n \to S, \quad t : D_{\Delta^n} \to F \circ \mathcal{C}(f),\]
  where \(f\) is a map of simplicial sets, and \(t\) is a map of simplicially enriched functors \(\mathcal{C}(\Delta^n)^{op} \to sSet\).
- To a map \(\delta : \Delta^m \to \Delta^n\) we have an induced map \((\text{Un}_S F)_n \to (\text{Un}_S F)_m\), which sends an \(n\)-simplex \((f, t)\) to the pair
  \[\Delta^m \xrightarrow{\delta} \Delta^n \xrightarrow{f} S, \quad D_{\Delta^m} \xrightarrow{D_\delta} D_{\Delta^n} \circ \mathcal{C}(\delta)^{op} \xrightarrow{t \circ \mathcal{C}(\delta)^{op}} F \circ \mathcal{C}(f) \circ \mathcal{C}(\delta)^{op} .\]

56. Pullback of right anodyne map along left fibration

The arguments here are based on the appendix to [Mos15].

56.1. Lemma. Consider a pullback square
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow p \\
\Lambda^n_k & \rightarrow & \Delta^n
\end{array}
\]

If \(p\) is a left fibration and \(k \geq 1\), then \(A \to B\) is right anodyne.

56.2. Contraction maps. Let \(\Delta_R\) denote the category whose

- objects are totally ordered sets \([n]_R := [n] \sqcup \{R\} = \{0 < 1 < \cdots < n < R\}\) for \(n \geq -1\), and
- morphisms are order preserving functions which take the “right basepoint” \(R\) to \(R\).

There is an evident “inclusion” functor \(\Delta \to \Delta_R\) sending \([n] \mapsto [n]_R\).

A right contraction of a simplicial set \(X\) is a choice of extension of \(X : \Delta^{op} \to \text{Set}\) to a functor \(X : (\Delta_R)^{op} \to \text{Set}\). There is an evident dually defined category \(\Delta_L\) and corresponding notion of left contraction.

A right contraction of a simplicial set \(X\) is completely determined by the inclusion \(X_{n-1} \to X_0\) together with its contraction operators, which are the maps \(X_{n-1} \to X_n\) for \(n \geq 0\) induced by the surjective map \(Q : [n]_R \to [n-1]_R\) which sends \(n \mapsto R\).
Let $\Delta_{R,\text{surj}}^{\text{nc}} \subset \Delta_{R}^{\text{nc}}$ denote the subcategory consisting of all objects, and all surjective maps. Given a right-contracted simplicial set $X: (\Delta_{R})^{\text{op}} \to \text{Set}$, say that $a \in X_{n}$ is **contracted** if there exists a non-identity $\sigma: [n]_{R} \to [k]_{R}$ in $\Delta_{R,\text{surj}}^{\text{nc}}$ and $b \in X_{k}$ such that $a = b\sigma$. That is, a contracted simplex is one which is either degenerate or in the image of the contraction operators $Q$. Say $a \in X_{n}$ is **non-contracted** if $a = b\sigma$ for $\sigma \in \Delta_{R,\text{surj}}^{\text{nc}}$ we have $\sigma = \text{id}$.

We have an analogue of the Eilenberg-Zilber lemma (15.11) for right contracted simplicial sets, which is proved in the same way.

56.3. **Proposition.** For $a$ in $X$, there exists a unique pair $(b, \sigma)$ consisting of a map $\sigma \in \Delta_{R,\text{surj}}^{\text{nc}}$ and a non-contracted simplex $b \in X$ such that $a = b\sigma$.

56.4. **Corollary.** For any right contracted simplicial set, the evident maps

$$\prod_{k \geq 0} \prod_{b \in X_{k}^{\text{nc}}} \text{Hom}_{\Delta_{R,\text{surj}}^{\text{nc}}}([n]_{R}, [k]_{R}) \to X([n]_{R})$$

defined by $(b, \sigma) \mapsto b\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective morphisms $[n]_{R} \to [n']_{R}$.

There is an analogue of the skeletal filtration: let $F_{n}X \subseteq X$ denote the smallest subobject containing all simplices of dimensions $\leq n$. Then the simplices of $F_{n}X$ consist of those which are degeneracies or contractions of noncontracted simplices of dimension $\leq n$.

Comparing this with the skeletal filtration, we discover the following.

56.5. **Lemma.** For all $n \geq 0$, the contraction operators $Q: X_{n-1} \to X_{n}$ restrict to bijections $X_{n-1}^{\text{nc}} \to (X_{n}^{\text{nc}} \setminus X_{n-1}^{\text{nc}})$.

56.6. **Proposition.** Let $X$ be a simplicial set equipped with a right contraction. Consider the evident inclusion $f: S \to X$, where $S$ is the discrete simplicial set with underlying set $X_{-1}$. Then $f$ is right anodyne; in fact, $f \in \{ \Lambda^{n}_{R} \subset \Delta^{n} \mid n \geq 1 \}$.

We note that for $x \in X_{n-1}$,

- $d_{i}Q(x) = Qd_{i}(x)$ if $i \in \{0, \ldots, n-1\}$, i.e., the contraction operators commute with most face maps;
- we have $d_{n}Q(x) = x$;
- $s_{i}Q(x) = Qs_{i}(x)$ if $i \in \{0, \ldots, n-1\}$, i.e., the contraction operators commute with most degeneracy maps;
- we have $s_{n}Q(x) = QQ(x)$.

In fact, a right contraction for a simplicial set $X$ is equivalent to choosing data $(X_{-1},d_{0}: X_{0} \to X_{-1}, \{Q: X_{n-1} \to X_{n}\})$, where the $Q$ satisfy the above identities.

There is an evident complementary notion of **left contraction**.

If $X$ admits a right contraction, then it is “right contractible” to a discrete simplicial set.

56.7. **Proposition.** Let $X$ be a simplicial set equipped with a right contraction. Consider the evident inclusion $f: S \to X$, where $S$ is the discrete simplicial set with underlying set $X_{-1}$. Then $f$ is right anodyne; in fact, $f \in \{ \Lambda^{n}_{R} \subset \Delta^{n} \mid n \geq 1 \}$.

**Proof.** Consider the collection $X_{n}^{\text{nd}}$ of non-degenerate simplices of $X$ not contained in $S$. Partition $X_{n}^{\text{nd}}$ into disjoint subsets $X_{n}^{\text{I}} \sqcup X_{n}^{\text{II}}$, where $X_{n}^{\text{I}} = (X_{n}^{\text{nd}} \setminus S^{\text{nd}}) \cap Q(X)$, the set of nondegenerate simplices which are in the image of the contraction operators.

The claim is that the contraction operators define bijections $\phi: X_{n-1}^{\text{II}} \to X_{n}^{\text{I}}$ for $n \geq 1$. To see this, we note the following.

- If $Qx = s_{i}y \in X_{n}$ with $i \in \{0, \ldots, n-2\}$, then $x = d_{n}Qx = d_{n}s_{i}y = s_{i}d_{n}y$. Thus, $x$ is degenerate.
• If \( Qx = s_{n-1}y \in X_n \), then \( x = d_nQx = d ns_{n-1}y = y \) and \( Qd_{n-1}x = d_{n-1}Qx = d_{n-1}s_{n-1}y = y \). Thus, \( x \) is in the image of \( Q \).

• Taken together, the last two statements imply that \( Q \) applied to an element of \( X^{II} \) is non-degenerate. Thus, \( \phi \) is a well-defined map.

• If \( x = s_iy \in X_{n-1} \) with \( i \in \{0, \ldots, n-2\} \), then \( Qs_iy = s_iQy \), i.e., \( Qx \) is degenerate whenever \( x \) is degenerate. Thus, \( \phi \) is surjective.

• We have \( d_nQx = x \) for \( x \in X_{n-1} \). Therefore, \( \phi \) is injective.

Now we can filter \( X \) by subcomplexes \( E_n \), where \( E_{-1} = S \), while \( E_n \) is the smallest subcomplex containing \( Sk_{n-1}X \) and \( X^I_n \). For each \( Qx \in X^I_n \) we have \( d_iQx = Qd_ix \in E_{n-1} \) when \( i \in \{0, \ldots, n-1\} \), while \( d_nQx = x \notin E_{n-1} \). Thus each inclusion \( E_{n-1} \subseteq E_n \) is obtained by attaching the collection \( X^I_n \) of \( n \)-simplices along \( \Lambda^n_\alpha \subseteq \Delta^n \).

56.8. Proposition. Let \( X \) be a simplicial set and \( x \in X_0 \). Then the inclusion \( \{1_x \} \rightarrow X_x/1 \) is right anodyne, and the inclusion \( \{1_x \} \rightarrow X_x/ \) is left anodyne.

Proof. The functor \( \Delta \rightarrow sSet_\star \) to pointed simplicial sets defined by \([n] \rightarrow (\Delta^n)^\circ \) manifestly extends to a functor \( \Delta^R \rightarrow sSet_\star \). From this we obtain a canonical contraction on \( X_x/1 \).

56.9. Proposition. Suppose \( X \) is a contractible Kan complex. Then for any choice of vertex \( * \in X_0 \), the inclusion \( \{*\} \rightarrow X \) admits a right contraction.

Proof. We define a right contraction by inductively constructing contraction operators. Set \( X_{-1} := \{\star\} \), and let \( Q : X_{-1} \rightarrow X_0 \) denote the evident inclusion.

Suppose we have already defined contraction operators \( Q : X_{k-1} \rightarrow X_k \) for \( k < n \), which satisfy all the identities which make sense. That is, we have a presheaf of sets on the subcategory of \( \Delta^R \) generated by \( \Delta \) and by the contraction operators \([k]_R \rightarrow [k-1]_R \) for \( k < n \). We will construct \( Q : X_{n-1} \rightarrow X_n \) so that the following hold.

(1) If \( x = s_iy \), then \( Qx = s_iQy \).

(2) If \( x = Qy \), then \( Qx = s_{n-1}Qy \).

(3) We have that \( d_nQx = x \) and \( d_iQx = Qd_ix \) for \( i \in \{0, \ldots, n-1\} \).

Doing so precisely gives an extension of the given presheaf to the subcategory of \( \Delta^R \) generated by \( \Delta \) and contraction operators \([k]_R \rightarrow [k-1]_R \) for \( k \leq n \).

The idea is to use the above as a prescription for \( Q : X_{n-1} \rightarrow X_n \): that is,

(1) if \( x = s_iy \), set \( Qx := s_iQy \);

(2) if \( x = Qy \), set \( Qx := s_{n-1}Qy \);

(3) otherwise, choose any \( Qx \) such that \( d_nQx = x \) and \( d_iQx = Qd_ix \) for \( i \in \{0, \ldots, n-1\} \).

We need to check that this makes sense: that cases (1) and (2) agree when they overlap, and that a choice in (3) is always possible. We consider the various cases, for \( x \in X_{n-1} \).

• Suppose \( x = s_iy = s_jz \) for some \( i, j \in \{0, \ldots, n-2\} \). Thus, both \( y \) and \( z \) are common degeneracies of some non-degenerate simplex, i.e., \( y = \sigma u \) and \( z = \tau u \) for some \( u \in X_d \) with \( d \leq n - 3 \), while \( \sigma, \tau : [n-2] \rightarrow [d] \) are surjective operators with \( \sigma s_i = \tau s_j \). Then \( yQs_i = u\sigma Qs_i = u\sigma Qs_i = u\sigma \tau s_j = u\tau Qs_j = zQs_j \).

• Suppose \( x = s_iy = Qz \) for some \( i \in \{0, \ldots, n-2\} \). Then there exists \( Qz \) such that \( d_nQz = x \) and \( d_iQz = Qd_iy \) for \( i \in \{0, \ldots, n-1\} \).

That a choice in (3) exists follows from the fact that the collection of \( Qd_0x, \ldots, Qd_{n-1}x, x \) fit together to give a map \( \partial \Delta^n \rightarrow X \).

Fix a vertex \( k \) of \( \Delta^n \). We define the excess of a simplex \( \delta : \Delta^m \rightarrow \Delta^n \) (relative to \( k \)) as follows.

• If some vertex of \( \Delta^n \) other than \( k \) is not in the image of \( \delta \), the excess of \( \delta \) is \(-1 \).

• Otherwise, the excess of \( \delta \) is

\[ |\{ j \in (\Delta^m)_{0} \mid \delta(j) \neq k \} | - n. \]
Note that the simplices of excess \(-1\) form the subcomplex \(\Lambda_k^n\).

Given a map \(p: B \to \Delta^n\), define the excess of a simplex \(\Delta^m \to B\) (relative to \(k\) and \(p\)) to be the excess of \(px: \Delta^m \to \Delta^n\). Observe that the collection of simplices \(B\) with excess \(\leq d\) form a subcomplex \(E_d \subseteq B\), and that \(E_{-1}\) is the pullback of \(B\) over \(\Lambda_k^n\).

56.10. Lemma. If \(k \in \{1, \ldots, n\}\) and \(p\) is a left fibration, then each map \(E_{d-1} \to E_d\) is right anodyne.

Here is a formalism for "simplicial operators"\(^{32}\). Given a finite totally ordered set \(I\), we can define \(\Delta^I := N(I)\). We have that \(\Delta^I\) is isomorphic to a unique standard simplex \(\Delta^n\) (or to the empty simplicial set if \(I\) is empty) by a unique isomorphism. The \(I\)-simplices \(X_I\) of a simplicial set \(X\) are the set of maps \(\Delta^I \to X\); we have a canonical identification \(X_I \approx X_n\) where \(|I| = n + 1\).

We define the following maps between totally ordered sets.

- \(d^a: I \setminus \{a\} \to I\) for \(a \in I\) is the evident inclusion.
- \(s^b_{a,c}: I \to I \cup \{b,c\}\) for consecutive elements \(b < c\) in \(I\) is a projection map; the target is the totally ordered set obtained by identifying \(b\) and \(c\) and relabelling the new element as "\(a\)" (which is assumed not to be an element of \(I \setminus \{b, c\}\), though we are allowed to consider \(s_{a,b}^a\) or \(s_{a,b}^b\)).
- \(s_0^a: I \to I \cup \{a\}\) is an isomorphism, where the target is obtained by removing \(b\) and replacing it with \(a\).

The resulting operators on a simplicial set are \(d_a: X_I \to X_I \setminus \{a\}\) (face operators), \(s^b_{a,c}: X_{I \cup \{b,c\}} \{a\} \to X_I\) (degeneracy operators), and \(s_0^a: X_{I \cup \{a\}} \{a\} \to X_I\). Then we have the following simplicial identities:

- \(d_a d_b = d_b d_a\),
- \(s^b_{a,c} e f = e f s^b_{a,c}\) if \(a, b, c, d, e, f\) are pairwise distinct,
- \(s^c_{b,c} s^b_{a,c} = s^b_{a,c} s^c_{b,c}\),
- \(d_a s^b_{a,c} = s^b_{a,c} d_a\) if \(a, b, c, d\) pairwise distinct,
- \(d_b s^b_{a,c} = s^b_{a,c} d_b\).

In other words, these identities say that simplicial operators generally commute, except when we compose \(f s^b_{a,c}\) where \(f\) is an operator involving \(b\) or \(c\).

Fix a map \(p: B \to \Delta^n\) and a vertex \(k\) of \(\Delta^n\). For each simplex \(u: \Delta^d \to B\) such that \(pu\) factors through the face \(\Delta_{[n]\setminus\{k\}} \subseteq \Delta^n\) opposite \(k\), we obtain a coaugmented simplicial set \(B(u): (\Delta^+)^{op} \to \text{Set}\). Here \(B(u)_j\) is the the set of fillers in

\[
\Delta^d \approx \Delta^{d_L} \star \emptyset \star \Delta^{d_R} \xrightarrow{u} B \xrightarrow{p} \Delta^{d_{\ell}} \star \Delta^j \star \Delta^{d_R} \twoheadrightarrow \Delta^{k-1} \star \{k\} \star \Delta^{n-k-1} \approx \Delta^n
\]

Note that any simplicial operator \(\delta: \Delta^{d'} \to \Delta^d\) induces a map \(B(\delta): B(u) \to B(u\delta)\) of coaugmented simplicial sets.

56.11. Lemma. Suppose there exist, for each simplex \(u: \Delta^d \to B\) such that the image of \(pu\) is equal to \(\Delta_{[n]\setminus\{k\}}\), a right contraction \(Q\) of \(B(u)\), and that these contraction maps are compatible with degeneracy operators, in the sense that for surjective \(\sigma: \Delta^{d'} \to \Delta^d\) we have \(Q \circ B(\sigma) = B(\sigma) \circ Q\). Then \(A \to B\) is right anodyne.

Proof. Observe that the set of simplices of \(B\) in the complement of \(A\) is precisely the disjoint union of the \(B(u)\), where \(u\) ranges over simplices such that \(pu\) surjects to the face opposite \(k\).

---

\(^{32}\)Adapted from the appendix to [Mos15].
Let $S_n := B_n^\text{nd} \setminus A_n^\text{nd}$, the set of non-degenerate $n$-simplices in the complement of $A$. Let $S'_n$ be the set of such which are in the image of the degeneracy operators, and let $S''_n$ be its complement in $S_n$. We claim that the $Q$s induce a bijection

$$\phi: S''_{n-1} \rightarrow S'_n.$$ 

To see this, we observe the following for simplices in the complement of $A$.

- If $Q(x)$ is a degenerate simplex, then $x$ is also degenerate. Therefore $Q$ induces a function $\phi: S''_{n-1} \rightarrow S'_n$.
- If $x$ is a degenerate simplex, then so is $Q(x)$. Therefore $\phi$ is surjective.
- We have that $d_z(Q(x)) = x$, where $z = \sup V_x$. Therefore $\phi$ is injective.

Furthermore, we have for $x \in S''_{n-1}$ that

- For $a \in V_Q(x)$ but $a \neq \sup V_Q(x)$, we have that $d_a(Q(x)) = Q(d_a(x))$. Therefore these faces $d_a(Q(x))$ are either degenerate or are in the image of $Q$.
- For $a \in [n] \setminus V_Q(x)$, we have that $d_a(Q(x))$ has strictly smaller excess than $x$ and $Q(x)$.
- $z = \sup V_Q(x)$ is not equal to 0 (since $k > 0$).

We can now construct $B$ from $A$ by attaching right horns. Let $E_{r,n} \subseteq E_r$ denote the smallest subcomplex of $E_r$ containing $S'_d$ for $d \leq n$; let $E_{r,-1} = E_{r-1}$. Then each inclusion $E_{r,n-1} \rightarrow E_{r,n}$ is obtained by attaching right horns of dimension $n$, according to the elements of $S'_n$ which have excess $r$.

56.12. Lemma. Suppose $B \rightarrow \Delta^n$ is a left fibration. Then contraction maps as in the previous lemma exist.

Proof. For non-degenerate simplices in the complement of $A$, we define $Q$ inductively, as follows.

1. If $x = s^b_c y$, set $Q_z x := s^b_c Q_z y$.
2. If $x = Q_w y$, set $Q_z x := s^{w,z} Q_w y$.
3. Otherwise, choose $Q_z x$ to be any $n$-simplex such that $d_z Q_z x = x$ and $d_a Q_z x = Q_z d_a x$ for $a \in V_x$.

57. Cartesian fibrations

Let $p: C \rightarrow D$ a functor between ordinary categories. A morphism $f: x' \rightarrow x$ in $C$ is called $p$-Cartesian if for every object $c$ of $C$ the evident commutative square

$$
\begin{array}{ccc}
\text{Hom}_C(c,x') & \overset{f_0}{\longrightarrow} & \text{Hom}_C(c,x) \\
p \downarrow & & \downarrow p \\
\text{Hom}_D(p(c), p(x')) & \overset{p(f)_0}{\longrightarrow} & \text{Hom}_D(p(c), p(x))
\end{array}
$$

is a pullback square of sets.

Given an object $x \in \text{ob} C$ and a morphism $g: y' \rightarrow p(x)$ in $D$, a Cartesian lift of $g$ at $x$ is a $p$-Cartesian morphism $f: x' \rightarrow x$ such that $p(f) = g$.

We say that $p: C \rightarrow D$ is a Cartesian fibration of categories if every pair $(x \in \text{ob} C, g: y' \rightarrow p(x) \in D)$ admits a Cartesian lift.

Here are some observations, whose verification we leave to the reader. Fix a functor $p: C \rightarrow D$.

- Every isomorphism in $C$ is $p$-Cartesian.
- Every Cartesian lift of an isomorphism in $D$ is itself an isomorphism.
- If $f: x' \rightarrow x$ is $p$-Cartesian, then for any $g: x'' \rightarrow x'$ in $C$, we have that $g$ is $p$-Cartesian if and only if $gf$ is $p$-Cartesian.
Any two Cartesian lifts of $g$ at $x$ are “canonically isomorphic”.
Explicitly, fix $g: y' \to y$ in $D$ and an object $x$ in $C$ such that $y = p(x)$. If $f_1: x'_1 \to x$ and $f_2: x'_2 \to x$ are any two Cartesian lifts of $g$, then there exists a unique map $u: x'_1 \to x'_2$ such that $p(u) = 1_{y'}$ and $f_2u = f_1$; the map $u$ is necessarily an isomorphism.

The map $p$ is a right fibration if and only if it is a Cartesian fibration and every morphism in $C$ is $p$-Cartesian.

Now suppose that $p: C \to D$ is a Cartesian fibration. For an object $y$ of $D$, we write $C_y := p^{-1}(y)$ for the fiber of $C$ over $y$.

- The map $p$ is an isofibration.
- For each morphism $g: y' \to y$ in $D$ and object $x$ in $C$ with $p(x) = y$, fix a choice of Cartesian lift $\tilde{g}_x$ of $g$ at $x$. Using this data, we obtain functors
  $$g^! : C_y \to \mathcal{C}_y$$
  so that for morphism $\alpha: x_1 \to x_2$ in $C_y$, the map $g^!(\alpha)$ in $\mathcal{C}_y$ is the unique one fitting into
  $$\begin{array}{ccc}
  x'_1 & \xrightarrow{\tilde{g}_x} & x_1 \\
  \downarrow^{g(\alpha)} & & \downarrow^{\alpha} \\
  x'_2 & \xrightarrow{\tilde{g}_x} & x_2
  \end{array}$$

  The functor $g^!$ depends on the choices of Cartesian lifts of $g$. Any two set of choices of lifts give rise to isomorphic functors.
- For each pair of morphisms $y'' \xrightarrow{h} y' \xrightarrow{g} y$, we obtain a natural isomorphism of functors
  $$\gamma : h^! \circ g^! \cong (hg)^! : C_y \to \mathcal{C}_{y''}.$$  

  This natural transformation is given by the unique maps $\gamma_x$ in $\mathcal{C}_{y''}$ fitting into
  $$\begin{array}{ccc}
  x'' & \xrightarrow{\tilde{h}_x} & x' \\
  \downarrow^{\gamma_x} & & \downarrow^{\tilde{g}_x} \\
  x''' & \xrightarrow{(gh)_x} & x
  \end{array}$$

  Similarly, there is a natural isomorphism $\text{id} \cong (1_y)_1 : C_y \to C_y$. The data of the functors $g^!$ together with these natural isomorphisms define a pseudofunctor $D^{\text{op}} \to \mathbf{Cat}$, which on objects sends $y \mapsto C_y$.
- We can produce an actual functor $F : D^{\text{op}} \to \mathbf{Cat}$ with $F(y)$ equivalent to $C_y$ as follows. Given functors $p' : C' \to D$ and $p : C \to D$, let $\text{Fun}_D(C', C)$ denote the category of fiberwise functors and natural transformations; i.e., the fiber of $p_0 : \text{Fun}(C', C) \to \text{Fun}(C', D)$ over $q$. Let $\text{Fun}_D^+(C', C) \subseteq \text{Fun}_D(C', C)$ denote the full subcategory of functors $f : C' \to C$ which take $p'$-Cartesian morphisms to $p$-Cartesian morphisms.

  We obtain a functor $F : D^{\text{op}} \to \mathbf{Cat}$, given on objects by
  $$F(y) := \text{Fun}_D^+(D/y, C).$$

  One can show that restriction to $\{1_y\} \subseteq D/y$ defines an equivalence of categories $F(y) \to C_y$.
- Given $D$, there is a 2-category $\mathcal{F}_D$, whose objects are Cartesian fibrations $p : C \to D$; for any two objects $p : C \to D$ and $p' : C' \to D$ we take $\text{Fun}_D^+(C', C)$ as the category of morphisms from $p'$ to $p$. One can show that $\mathcal{F}_D$ is 2-equivalent to the 2-category $\text{Fun}(D^{\text{op}}, \mathbf{Cat})$. 

Part 8. Appendices

58. Appendix: Generalized horns

A generalized horn\(^{33}\) is a subcomplex \(\Lambda^n_S \subset \Delta^n\) of the standard \(n\)-simplex, where \(S \subseteq [n]\) and
\[
(\Lambda^n_S)_k := \{ f : [k] \to [n] \mid S \not\subseteq f([k]) \}.
\]
In other words, a generalized horn is a union of some codimension 1 faces of the \(n\)-simplex:
\[
\Lambda^n_S = \bigcup_{s \in S} \Delta^{[n] \setminus s}.
\]
In particular,
\[
\Lambda^n_{[n]} = \partial \Delta^n, \quad \Lambda^n_{[n] \setminus j} = \Lambda^n_j, \quad \Lambda^n_{\{j\}} = \Delta^{[n] \setminus j}, \quad \Lambda^n_{\emptyset} = \emptyset.
\]
In general \(S \subseteq T\) implies \(\Lambda^n_S \subseteq \Lambda^n_T\).

58.1. Proposition (Joyal [Joy08a, Prop. 2.12]). Let \(S \subseteq [n]\) be a proper subset.
(1) \((\Lambda^n_S \subset \Delta^n) \in \text{Horn} \text{ if } S \neq \emptyset\).
(2) \((\Lambda^n_S \subset \Delta^n) \in \overline{\text{Horn}} \text{ if } n \in S\).
(3) \((\Lambda^n_S \subset \Delta^n) \in \text{RHorn} \text{ if } 0 \in S\).
(4) \((\Lambda^n_S \subset \Delta^n) \in \text{InnHorn} \text{ if } S \text{ is not an "interval"; i.e., if there exist } a < b < c \text{ with } a, c \in S \text{ and } b \notin S\).

Proof. We start with an observation. Consider \(S \subseteq [n]\) and \(t \in [n] \setminus S\). Observe the diagram
\[
\begin{array}{ccc}
\Delta^{[n] \setminus t} \cap \Lambda^n_S & \rightarrow & \Delta^{[n] \setminus t} \\
\downarrow & & \downarrow \\
\Lambda^n_S & \rightarrow & \Lambda^n_{S \cup t} \rightarrow \Delta^n
\end{array}
\]
in which the square is a pushout, and the top arrow is isomorphic to the generalized horn \(\Lambda^n_{S \setminus t} \subset \Delta^{[n] \setminus t}\). Thus, \((\Lambda^n_S \subset \Delta^n)\) is contained in the weak saturation of any set containing the two inclusions
\[
\Lambda^n_{S \setminus t} \subset \Delta^{[n] \setminus t} \quad \text{and} \quad \Lambda^n_{S \cup t} \subset \Delta^n.
\]
Each of the statements of the proposition is proved by an evident induction on the size of \([n] \setminus S\), using the above observation. I’ll do case (4), as the other cases are similar. If \(S \subseteq [n]\) is not an interval, there exists some \(s < u < s'\) with \(s, s' \in S\) and \(u \notin S\). If \([n] \setminus S = \{u\}\) then we already have an inner horn. If not, then choose \(t \in [n] \setminus (S \cup \{u\})\), in which case \(S \cup t\) is not an interval in \([n]\), and \(S\) is not an interval in \([n] \setminus t\). Therefore both \(\Lambda^n_{S \setminus t} \subset \Delta^{[n] \setminus t}\) and \(\Lambda^n_{S \cup t} \subset \Delta^n\) are inner anodyne by the inductive hypothesis. The proofs of the other cases are similar. \(\square\)

58.2. Proposition (Joyal [Joy08a, Prop. 2.13]). For all \(n \geq 2\), we have that \((I^n \subset \Delta^n) \in \overline{\text{InnHorn}}\).

Proof. We can factor the inclusion spine inclusion as \(h_n = g_n f_n:\)
\[
I^n \xrightarrow{f_n} \Delta^{\{1, \ldots, n\}} \cup I^n \xrightarrow{g_n} \Delta^n.
\]
We show by induction on \(n\) that \(f_n, g_n, h_n \in \overline{\text{InnHorn}}\), noting that the case \(n = 2\) is immediate.

---

\(^{33}\)This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our \(\Lambda^n_S\) is Joyal’s \(\Lambda^{[n] \setminus S}\). I find my notation easier to follow, but note that it does conflict with the standard notation for horns. Maybe I should use something like \(\Lambda^{n,S}\)?
To show that \( f_n \in \text{InnHorn} \), consider the pushout square
\[
\begin{array}{ccc}
I^{\{1,\ldots,n\}} & \to & \Delta^{\{1,\ldots,n\}} \\
\downarrow & & \downarrow \\
I^n & \to & \Delta^{\{1,\ldots,n\}} \cup I^n
\end{array}
\]
in which the top arrow is isomorphic to \( h_{n-1} \), which is in \( \text{InnHorn} \) by induction.

To show that \( g_n \in \text{InnHorn} \), consider the diagram
\[
\begin{array}{ccc}
\Delta^{\{1,\ldots,n-1\}} \cup I^{\{0,\ldots,n-1\}} & \to & \Delta^{\{0,\ldots,n-1\}} \\
\downarrow & & \downarrow \\
\Delta^{\{1,\ldots,n\}} \cup I^n & \to & \Delta^{\{1,\ldots,n\}} \cup \Delta^{\{0,\ldots,n-1\}} \to \Delta^n
\end{array}
\]
in which the square is a pushout, the top horizontal arrow is isomorphic to \( g_{n-1} \), an element of \( \text{InnHorn} \) by induction, and the bottom right horizontal arrow is equal to \( \Lambda_n^{\{0,n\}} \subset \Delta^n \), which is in \( \text{InnHorn} \) by (58.1)(4).

59. Appendix: Box product lemmas

Here is where I'll prove various statements mentioned in the text.

- \( \text{LHorn} \Box \text{Cell} \subseteq \text{LHorn} \) (45.4), proved in (59.1) below.
- \( \text{RHorn} \Box \text{Cell} \subseteq \text{RHorn} \) (45.4), proved in (59.1) below.
- \( \text{Horn} \Box \text{Cell} \subseteq \text{Horn} \), is a consequence of the above, since \( \text{Horn} = \text{LHorn} \cup \text{RHorn} \) and \( \text{LHorn} \cup \text{RHorn} \subseteq \text{Horn} \).
- \( \text{InnHorn} \Box \text{Cell} \subseteq \text{InnHorn} \) (16.9), proved in (59.3) below.

59.1. Left and right horns. We prove the case of \( \text{LHorn} \Box \text{Cell} \subseteq \text{LHorn} \) here. Given this \( \text{RHorn} \Box \text{Cell} \subseteq \text{RHorn} \) follows since \( \text{op}: \text{sSet} \to \text{sSet} \) carries \( \text{LHorn} \) to \( \text{RHorn} \) and preserves \( \text{Cell} \).

Joyal [Joy08a, 2.25]\(^{34}\) observes that \( (\Lambda^n_k \subset \Delta^n) \) is a retract of \( (\Lambda^n_k \subset \Delta^n) \Box (\{0\} \subset \Delta^1) \) when \( 0 \leq k < n \). The retraction is
\[
\Delta^n \xrightarrow{s} \Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n
\]
defined by \( s(x) = (x, 1) \) and
\[
r(x, 0) = \begin{cases} 
  x & \text{if } x \leq k, \\
  k & \text{if } x \geq k, 
\end{cases} \quad r(x, 1) = x.
\]
Note that \( r(\Delta^n[k] \times \Delta^1) = \Delta^n[k] \times \Delta^1 \) if \( j \neq k \), and \( r(\Delta^n \times \{0\}) = \Delta^{\{0,\ldots,k\}} \subseteq \Delta^n[k] \times (k+1) \), so this gives the desired retraction.

The existence of the retraction reduces showing \( \text{LHorn} \Box \text{Cell} \subseteq \text{LHorn} \) to proving
\[
(\{0\} \subset \Delta^1) \Box \text{Cell} \subseteq \text{LHorn},
\]
since \( (\Lambda^n_k \subset \Delta^n) \in \text{Cell} \) and thus \( (\Lambda^n_k \subset \Delta^n) \Box \text{Cell} \subseteq \text{Cell} \).

59.2. Lemma. We have that \( (\{0\} \subset \Delta^1) \Box \text{Cell} \subseteq \text{LHorn} \).

\(^{34}\)Lurie [Lur09, 2.1.2.6] states this incorrectly.
Proof. . . Let \( K = \{(0) \times \Delta^n \} \cup (\Delta^1 \times \partial \Delta^n) \), so that \( \{(0) \subset \Delta^1 \} \square (\partial \Delta^n \subset \Delta^n) \) is the inclusion \( K \to \Delta^1 \times \Delta^n \). We will show that we can build \( \Delta^1 \times \Delta^n \) from \( K \) by an explicit sequence of steps, where in each case we attach an \((n+1)\)-sequence along a left horn.

For each \( 0 \leq a \leq n \) let \( \tau_a \) be the \((n+1)\)-simplex of \( \Delta^1 \times \Delta^n \) defined by
\[
\tau_a = \langle (0,0), \ldots, (0,a), (1,a), \ldots, (1,n) \rangle.
\]
We obtain an ascending filtration of \( \Delta^1 \times \Delta^n \) by starting with \( K \) and attaching simplices in the following order:
\[
\tau_n, \tau_{n-1}, \ldots, \tau_1, \tau_0.
\]
The \( \tau \)s range through all non-degenerate \((n+1)\)-subsimplices of \( \Delta^1 \times \Delta^n \), so \( K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n \).

(Here I am using the same notation for elements \( \tau \Lambda \).)

We have (17.5) that
\[
1 \subseteq \tau \Lambda \n \subseteq \tau \Lambda + 1 \subseteq \partial \Delta^n \subset \Delta^n.
\]

Thus both assertions follow from the following.

- These explicitly exhibit \( \tau \Lambda \n \) of \( \tau \Lambda + 1 \)-subcomplex of \( \Delta \) following order:
  \[
  \tau \Lambda \n \subseteq \partial \Delta^n \subset \Delta^n.
  \]

The claim is that each attachment is along a specified horn inclusion. More precisely, for each \( 0 \leq a \leq n \)
\[
\Lambda_a \n \subseteq \partial \Delta^n \subset \Delta^n,
\]
\( \tau \) contained in any \( \Lambda_a \n \) horn inclusion. Note that if when \( a > 0 \) this is an inner horn, while when \( a = 0 \)
\( \tau \) is the inclusion \( \Lambda_0 \n+1 \) in \( \Delta^n \); in either case, it is a left horn. Given the claim, it follows that
\[
\{(0) \subset \Delta^1 \} \square (\partial \Delta^n \subset \Delta^n) \in \text{InnHorn} \text{ as desired.}
\]

The proof of the claim amounts to the following list of elementary observations about \( \tau_a \):

- Every codimension-one face is contained in \( \Delta^1 \times \partial \Delta^n \) except: the face opposite vertex \((0,a)\),
  and the face opposite vertex \((1,a)\).
- The face opposite vertex \((1,a)\) is contained in \( \{0\} \times \Delta^n \) if \( a = n \), or is a face of \( \tau_{a+1} \) if \( a < n \).
- The face opposite vertex \((0,a)\) is not contained in \( \Delta^1 \times \partial \Delta^n \), nor in \( \{0\} \times \Delta^n \). Nor is it contained in any \( \tau \) with \( i > a \) (because the vertex \((1,a)\) is in this face but not in \( \tau \) with \( i > a \)).

Taken together these show that \( \tau_a \cap (K \cup \bigcup_{k>a} \tau_k) \) is the \( a \)th horn in the \((n+1)\)-simplex \( \tau_a \).

\[\square\]

59.3. Inner horns. Here is an argument for the key case for inner horns.

Consider \( \Delta^n \to \Delta^2 \times \Delta^n \to \Delta^n \), the unique maps which are given on vertices by
\[
s(y) = \begin{cases} 
(0,y) & \text{if } y < j, \\
(1,y) & \text{if } y = j, \\
(2,y) & \text{if } y > j,
\end{cases} \quad r(x,y) = \begin{cases} 
y & \text{if } x = 0 \text{ and } y < j, \\
y & \text{if } x = 2 \text{ and } y > j, \\
j & \text{otherwise.}
\end{cases}
\]

These explicitly exhibit \( \Lambda_j^n \subset \Delta^n \) as a retract of \( \Lambda_j^2 \subset \Delta^2 \) \( \square \) \( \Lambda_j^n \subset \Delta^n \), so
\[
\text{InnHorn} \subseteq \{ \Lambda_j^2 \subset \Delta^2 \} \square \text{Cell}.
\]

We have (17.5) that \( \text{Cell} \square \text{Cell} \subseteq \text{Cell} \), so the above implies that \( \text{InnHorn} \square \text{Cell} \subseteq \{ \Lambda_j^2 \subset \Delta^2 \} \square \text{Cell} \).
Thus the assertions “\text{InnHorn} \square \text{Cell} \subseteq \text{InnHorn}” and “\{ \Lambda_j^2 \subset \Delta^2 \} \square \text{Cell} \subseteq \text{InnHorn}” are equivalent.
Thus both assertions follow from the following.

59.4. Lemma. For all \( n \geq 0 \) we have that \( \Lambda_j^2 \subset \Delta^2 \square (\partial \Delta^n \subset \Delta^n) \in \text{InnHorn} \).

Proof. [Lur09, 2.3.2.1].

For each \( 0 \leq a \leq b < n \), let \( \sigma_{ab} \) be the \((n+1)\)-simplex of \( \Delta^2 \times \Delta^n \) defined by
\[
\sigma_{ab} = \langle (0,0), \ldots, (0,a), (1,a), \ldots, (1,b), (2,b+1), \ldots, (2,n) \rangle.
\]

For each \( 0 \leq a \leq b \leq n \), let \( \tau_{ab} \) be the \((n+2)\)-simplex of \( \Delta^2 \times \Delta^n \) defined by
\[
\tau_{ab} = \langle (0,0), \ldots, (0,a), (1,a), \ldots, (1,b), (2,b), \ldots, (2,n) \rangle.
\]

The set \( \{ \tau_{ab} \} \) consists of all the non-degenerate \((n+2)\)-simplices. Note that \( \sigma_{ab} \) is a face of \( \tau_{ab} \) and
of \( \tau_{a,b+1} \), but not a face of any other \( \tau \).
We attach simplices to $K := (\Lambda_2^1 \times \Delta^n) \cup (\Delta^2 \times \partial \Delta^n)$ in the following order:

$$\sigma_{00}, \sigma_{01}, \sigma_{11}, \sigma_{02}, \sigma_{12}, \cdots \sigma_{0,n-1}, \cdots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00}, \tau_{01}, \tau_{11}, \tau_{02}, \tau_{12}, \cdots, \tau_{0,n}, \cdots, \tau_{n,n}.$$  

The $\tau$s range through all the non-degenerate $(n+2)$-simplices of $\Delta^2 \times \Delta^n$, so that $K \cup \sigma_{a,b} \cup \tau_{a,b} = \Delta^2 \times \Delta^n$.

The claim is that each attachment is along an inner horn inclusion. More precisely, each $\sigma_{a,b}$ gets attached along the horn at the vertex $(1,a)$ in $\sigma_{ab}$, i.e., via a $\Lambda_{a+1} \subset \Delta^{n+1}$ horn inclusion, which is always inner since $a \leq b < n$. Likewise, each $\tau_{a,b}$ gets attached along the horn at vertex $(1,a)$ in $\tau_{ab}$, i.e., via a $\Lambda_{a+2} \subset \Delta^{n+2}$ horn inclusion, which is always inner since $a \leq b \leq n$.

The proof of the claim amounts to the following lists of elementary observations.

For $\sigma_{a,b}$:

- Every codimension-one face is contained in $\Delta^2 \times \partial \Delta^n$, except the following: the face opposite vertex $(0,a)$, and the face opposite vertex $(1,a)$.
- The face opposite vertex $(0,a)$ is either contained in $\Lambda_0^2 \times \Delta^n$ if $a = 0$, or a face of $\sigma_{a-1,b}$ if $a > 0$.
- The face of $\sigma_{a,b}$ opposite vertex $(1,a)$ is not contained in $\Delta^2 \times \partial \Delta^n$, nor in $\Lambda_0^2 \times \Delta^n$, nor in any $\sigma_{i,b}$ with $i < a$ (because of the vertex $(0,a)$), nor in any $\sigma_{i,j}$ with $i \leq j < b$ (because of the vertex $(1,b)$ if $a < b$, or the vertex $(0,a)$ if $a = b$).

For $\tau_{a,b}$ when $a < b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial \Delta^n$ except the following: the face opposite vertex $(0,a)$, the face opposite vertex $(1,a)$, the face opposite vertex $(1,b)$, and the face opposite vertex $(2,b)$.
- The face opposite vertex $(2,b)$ is $\sigma_{a,b}$, while the face opposite vertex $(1,b)$ is $\sigma_{a,b-1}$.
- The face opposite vertex $(0,a)$ is either contained in $\Lambda_1^2 \times \Delta^n$ if $a = 0$, or a face of $\tau_{a-1,b}$ if $a > 0$.
- The face opposite vertex $(1,a)$ is not contained in $\Delta^2 \times \partial \Delta^n$, nor in $\Lambda_1^2 \times \Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(1,b)$ and $(2,b)$), nor in any $\tau_{i,b}$ with $i < b$ (because of the vertex $(0,a)$), nor in any $\tau_{i,j}$ with $i \leq j < b$ (because of the vertex $(1,b)$).

For $\tau_{a,b}$ when $a = b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial \Delta^n$ except the following: the face opposite vertex $(0,a)$, the face opposite vertex $(1,a) = (1,b)$, and the face opposite vertex $(2,b)$.
- The face opposite vertex $(2,b)$ is $\sigma_{a,b}$.
- The face opposite vertex $(0,a)$ is contained in $\Lambda_1^2 \times \Delta^n$ if $a = 0$, or a face of $\tau_{a-1,b}$ if $a > 0$.
- The face opposite vertex $(1,a) = (1,b)$ is not contained in $\Delta^2 \times \partial \Delta^n$, nor in $\Lambda_1^2 \times \Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(0,a)$ and $(2,b)$), nor in any $\tau_{i,b}$ with $i < b$ (because of the vertex $(0,a)$), nor in any $\tau_{i,j}$ with $i \leq j < b$ (because of the vertex $(0,a)$).

\[ \square \]

59.5. **A pushout-product version of Joyal lifting.** Rewrite using proof of (59.2).

We now give a proof of (29.12): we will prove the case of $(i,j) = (0,0)$, i.e., given $p: C \to D$ an inner fibration of quasicategories, $n \geq 1$, and

\[
\begin{array}{ccc}
\Delta^1 \times \{0\} & \xrightarrow{f} & \{(0) \times \Delta^n\} \cup \{0\} \times \partial \Delta^n \to \Delta^1 \times \Delta^n \\
\downarrow & & \downarrow \\
C & \xrightarrow{p} & D
\end{array}
\]
such that \( f \) represents an isomorphism in \( C \), we will construct a lift. (Note that if \( n = 0 \) such a lift does not generally exist.)

For each \( 0 \leq a \leq n \) let \( \tau_a \) be the \((n+1)\)-simplex of \( \Delta^1 \times \Delta^n \) defined by

\[
\tau_a = \langle (0,0), \ldots, (0,a), (1,a), \ldots, (1,n) \rangle.
\]

Let \( K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial \Delta^n) \). We obtain an ascending filtration of \( \Delta^1 \times \Delta^n \) by starting with \( K \) and attaching simplices in the following order:

\[
\tau_n, \tau_{n-1}, \ldots, \tau_1, \tau_0.
\]

The \( \tau \)s range through all non-degenerate \((n+1)\)-simplices of \( \Delta^1 \times \Delta^n \), so \( K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n \).

The claim is that each attachment is along a specified horn inclusion. More precisely, for \( a \leq n \) the simplex \( \tau_a \) is attached to \( K \cup \bigcup_{k > a} \tau_k \) along the horn at the vertex \((0,a)\) in \( \tau_a \), i.e., via a \( \Lambda_{a+1}^n \subset \Delta^n \) horn inclusion. Note that if when \( a > 0 \) this is an inner horn, while when \( a = 0 \) this is the inclusion \( \Lambda_0^{n+1} \subset \Delta^n \) and the edge \( \Delta^1 \times \{0\} \) is the leading edge of \( \tau_0 \).

Given the claim, we thus construct the desired lift by inductively choosing a lift defined on each \( \tau_a \) relative to the given lift on its \( \Lambda_a^{n+1} \)-horn. When \( a > 0 \) such a lift exists because \( p \) is an inner fibration, while when \( a = 0 \) a lift exists by Joyal lifting (28.13).

The proof of the claim amounts to the following list of elementary observations. For \( \tau_a \):

- Every codimension-one face is contained in \( \Delta^1 \times \partial \Delta^n \) except: the face opposite vertex \((0,a)\), and the face opposite vertex \((1,a)\).
- The face opposite vertex \((1,a)\) is contained in \( \{0\} \times \Delta^n \) if \( a = n \), or is a face of \( \tau_{a+1} \) if \( a < n \).
- The face opposite vertex \((0,a)\) is not contained in \( \Delta^1 \times \partial \Delta^n \), nor in \( \{0\} \times \Delta^n \), nor in any \( \tau_i \) with \( i > a \) (because of the vertex \((1,a)\)).

The proof of the other case \((i,j) = (1,n)\) proceeds analogously; in that case simplices are attached to \( K \) in the order: \( \tau_0, \tau_1, \ldots, \tau_n \).

### References


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