POWER OPERATIONS IN MORAVA E-THEORY: STRUCTURE AND CALCULATIONS (DRAFT)

CHARLES REZK

Abstract. We review what is known about power operations for height 2 Morava E-theory, and carry out some sample calculations.

Contents

1. Introduction

2. Dramatis personae

In this section, we give an minimal overview of definitions and results which lead to the promised calculations. We hope enough detail is given to convey the global structure of the ideas; we refer to other papers or later sections (if they've been written yet) for more information.

2.1. **Commutative ring spectra and Morava E-theory.** We use a convenient category of structured commutative ring spectra; the category of [EKMM97] is appropriate, although the particular choice of model will not play an important role in the statement of results.

Fix a formal group G_0 over a perfect field k of characteristic p, which is of finite height h, and let $E = E_{G_0/k}$ denote the associated **Morava E-theory** spectrum. By the theorem of Hopkins-Miller, E this has an essentially unique structure as a commutative S-algebra, and we fix such a structure [GH04].

Date: September 13, 2013.

Let $\widehat{\mathcal{R}}$ denote the homotopy theory of **$K(h)$-local commutative E-algebras**, where $K(h)$ is the Morava K-theory of height h at the prime p. We can take this homotopy theory to be the EKMM model category of commutative E-algebras equipped with a suitable model category structure, where we say a map $f\colon R \to R'$ is a weak equivalence if it induces an isomorphism in $K(h)$-homology. Let $\mathcal{M}$ denote the homotopy theory of **$K(h)$-local E-modules**.

We write $\pi_\star M$ for the the graded homotopy groups of an E-module or commutative E-algebra; it is naturally a graded $E_\star = \pi_\star E$-module. It turns out to be convenient for our purposes to package these homotopy groups as $\mathbb{Z}/2$-graded objects, rather than $\mathbb{Z}$-graded objects; we can get away with this because E is even periodic. (The reader should probably ignore the distinction between $\mathbb{Z}$ and $\mathbb{Z}/2$-grading for now. See §5.9 for discussion of the $\mathbb{Z}/2$-grading.) The category of $\mathbb{Z}/2$-graded $E_\star$-modules is denoted $\mathrm{Mod}_{E_\star}$.

Let $\widehat{\mathcal{R}}_E$ denote the homotopy theory of **augmented $K(h)$-local commutative E-algebras**. We can take this to be the slice model category over E of the EKMM category of commutative E-algebras, with $K(h)_*$-homology weak equivalences.

We are interested in computing the homotopy groups of derived mapping spaces $\widehat{\mathcal{R}}_E(R, F)$ between two $K(h)$-local augmented commutative E-algebras R and F. To do this, we need to understand the algebraic structure inherited by the the homotopy groups of an object of $\widehat{\mathcal{R}}_E$.

2.2. **Γ-modules and Γ-rings.** The homotopy groups $\pi_\star R$ of an object R of $\widehat{\mathcal{R}}$ carry quite a bit of structure.

(1) $\pi_\star R$ is a **strongly graded commutative $E_\star$-algebra**; "strongly graded commutative" means that elements in odd degree anticommute and square to zero.
(2) The $E_\star$-module $\pi_\star R$ carries the structure of a **$\mathbb{Z}/2$-graded Γ-module**.
(3) The category $\mathrm{Mod}^\star_\Gamma$ of $\mathbb{Z}/2$-graded Γ-modules is actually a tensor category (compatibly with the underlying tensor product on $E_\star$-modules), and multiplication $\pi_\star R \otimes_{E_\star} \pi_\star R \to \pi_\star R$ is a morphism in $\mathrm{Mod}^\star_\Gamma$. That is, $\pi_\star R$ is a **$\mathbb{Z}/2$-graded Γ-ring**.

See §§5.1–5.9 for more detail.

The Γ-module structure on $\pi_\star R$ encodes the action of "power operations" on the homotopy groups. The structure of Γ-modules is determined by the theory of deformations of subgroups of the formal group G_0/k. The category $\mathrm{Mod}^\star_\Gamma$ will be the main subject of much of this paper, notably §§3–4 and §7; the special cases of heights 1 and 2 are discussed in §8 and §9.

2.3. **$\mathbb{T}$-algebras.** We continue the discussion of structure on the homotopy groups of an object R of $\widehat{\mathcal{R}}$.

(4) The Γ-ring $\pi_\star R$ satisfies the **Frobenius congruence**. In short, for a Γ-ring B in there is a naturally defined $E_\star$-module map $\mathrm{can}_*\colon B \to B/pB$ (which relates to the "canonical subgroup" of a formal group in characteristic p). The Frobenius congruence for B asserts that for every element $x \in B$ in even degree, we have

$$\mathrm{can}_*(x) \equiv x^p \mod pB.$$

(See §4.2.)

The Frobenius congruence on $\pi_\star R$ is "witnessed" by a natural non-additive operation on $\pi_0 R$. The paradigm for this is the case of p-adic K-theory, whose homotopy groups are a "θ-ring" [Bou96]. That is, the homotopy groups of a $K(1)$-local commutative K-algebra R admit a natural ring map $\psi^p\colon \pi_0 R \to \pi_0$, and a natural operation $\theta^p\colon \pi_0 R \to \pi_0 R$, such that

$\psi^p(x) = x^p + p\,\theta^p(x)$. We can think of $\theta^p(x)$ as "witnessing" the congruence $\psi^p(x) \equiv x^p \bmod p$.

In the general setting, there is a monad $\mathbb{T}$ on $\mathrm{Mod}_{E_\star}$, so that $\mathbb{T}$-algebras are Γ-rings equipped with a "witness of the Frobenius congruence". In particular, the above points (1)–(4) are subsumed in the following.

(5) For R in $\widehat{\mathcal{R}}$, $\pi_\star R$ is naturally equipped with the structure of a $\mathbb{T}$-algebra.

We write $\mathcal{T}$ for the category of $\mathbb{T}$-algebras; thus, homotopy is a functor $\pi_\star\colon h\widehat{\mathcal{R}} \to \mathcal{T}$. It is not easy to give an entirely self-contained description of the category of $\mathbb{T}$-algebras. However, we have the following **congruence criterion**, which lets us understand $\mathbb{T}$-algebras which are p-torsion free.

2.4. **Theorem** ([Rez09, Theorem A]). *If B is a $\mathbb{Z}/2$-graded Γ-ring which is p-torsion free as an abelian group, then B admits the structure of a $\mathbb{T}$-algebra if and only if B satisfies the Frobenius congruence. If such a $\mathbb{T}$-algebra structure exists, it is unique.*

See §5.11 for more on $\mathbb{T}$-algebras, which were introduced in [Rez09].

2.5. **Analytic completion.** Recall that $\pi_0 E \approx \mathbb{W}k[\![u_1, \dots, u_{h-1}]\!]$. Let $\mathfrak{m} = (p, u_1, \dots, u_{h-1}) \subset \pi_0 E$ be the maximal ideal, and consider $\mathfrak{m}$-adic completion $M \mapsto M^\wedge_\mathfrak{m}$, which is a functor $\mathrm{Mod}_{E_\star} \to \mathrm{Mod}_{E_\star}$. This completion functor is not right exact, but admits a natural best approximation by a right exact functor $\mathcal{A}\colon \mathrm{Mod}_{E_\star} \to \mathrm{Mod}_{E_\star}$, which comes with natural comparison maps $M \to \mathcal{A}(M) \to M^\wedge_\mathfrak{m}$. The functor $\mathcal{A}$ is often called L_0, as it is in fact the 0th left-derived functor of analytic completion. The properties of this functor have been studied in [GM92] and [HS99, App. A].

There is an equivalent and more elementary description of the functor $\mathcal{A}$, which is sometimes useful to know. Namely, for an $E_\star$-module M, we have

$$\mathcal{A}(M) \approx M[\![x_0, \dots, x_{h-1}]\!]/(x_0 - p, x_1 - u_1, \dots, x_{h-1} - u_{h-1})M[\![x_0, \dots, x_{h-1}]\!].$$

Here $M[\![x_0, \dots, x_{h-1}]\!]$ represents the set of formal power series with coefficients in the module M; it is naturally a module over $E_\star[\![x_0, \dots, x_{h-1}]\!]$. The canonical coaugmentation $\eta\colon M \to \mathcal{A}(M)$ is the map induced by inclusion of constant power series.

For this reason, we like to refer to $\mathcal{A}\colon \mathrm{Mod}_{E_\star} \to \mathrm{Mod}_{E_\star}$ as the **analytic completion functor**, and we say that M is **analytically complete** (or just **analytic**) if the map $\eta\colon M \to \mathcal{A}(M)$ is an isomorphism. We note that any $\mathfrak{m}$-adically complete module is analytically complete, but not conversely [HS99, Thm. A.6]; however, the natural comparison map $\mathcal{A}(M) \to M^\wedge_\mathfrak{m}$ is often an isomorphism, in particular when M is flat [HS99, Thm. A.2 (b)]. An exposition of the properties of $\mathcal{A}$ from this power series point of view is given in [Rez13]; however, most of what we need can be found in [HS99, App. A] where what we call "analytic completion" is there called "L-completion".

An E-module spectrum M is $K(h)$-local if and only if $\pi_\star M$ is analytically complete (**??**). In particular, for any object R of $\widehat{\mathcal{R}}$, the object $\pi_\star R$ is analytically complete.

By a result of [BF13], the analytic completion functor lifts to the category $\mathcal{T}$. That is, there is a functor $\mathcal{A}_\mathbb{T}\colon \mathcal{T} \to \mathcal{T}$ which on underlying $E_\star$-modules coincides with $\mathcal{A}$; we usually just write $\mathcal{A}$ for the lifted functor. We'll say that an object of $\mathcal{T}$ is **analytically complete** if its underlying $E_\star$-algebra is.

Say that an $E_\star$-module M is **tame** if the higher left derived functors of analytic completion vanish on it, i.e., if $\mathbf{L}_j\mathcal{A}(M) = 0$ for $j \geq 1$. (Note that the $\mathbf{L}_j\mathcal{A}$ coincide with the higher derived functors of $\mathfrak{m}$-adic completion, denoted L_j in [HS99].) We say that a $\mathbb{T}$-algebra is **tame** if its underlying $E_\star$-module is tame. We note that projective modules are tame [HS99, Thm. A.2 (b)], and also that analytically complete modules are tame [HS99, Thm. A.6 (b)].

2.6. **Cohomology of augmented $\mathbb{T}$-algebras.** Let $\mathcal{T}_{E_\star}$ denote the slice category of $\mathcal{T}$ over $E_\star = \pi_\star E$. That is, an object B of $\mathcal{T}_{E_\star}$ is a $\mathbb{T}$-algebra equipped with a $\mathbb{T}$-algebra map $B \to E_\star$. It is clear that taking homotopy groups defines a functor $\pi_\star\colon \widehat{\mathcal{R}}_E \to \mathcal{T}_{E_\star}$. (In fact, the image of this functor is contained in $\widehat{\mathcal{T}}_{E_\star} \subset \mathcal{T}_{E_\star}$, the full subcategory of analytically complete objects.)

Let $\mathrm{ab}\,\mathcal{T}_{E_\star}$ denote the category of abelian group objects in $\mathcal{T}_{E_\star}$. There is a pair of adjoint functors

$$Q\colon \mathcal{T}_{E_\star} \rightleftarrows \mathrm{ab}\,\mathcal{T}_{E_\star} : \mathcal{J},$$

where the right adjoint $\mathcal{J}$ is a fully faithful functor, identifying $\mathrm{ab}\,\mathcal{T}_{E_\star}$ with the full subcategory of $\mathcal{T}_{E_\star}$ consisting of objects $\phi\colon B \to E_\star$ such that $\overline{B}^2 = 0$, where $\overline{B} = \operatorname{Ker}\phi$ is the augmentation ideal. We will typically represent objects in $\mathrm{ab}\,\mathcal{T}_{E_\star}$ by their augmentation ideals, and write $E_\star \rtimes M$ for $\mathcal{J}(M)$ above. The left adjoint Q is the indecomposable quotient functor $Q(B) = \overline{B}/\overline{B}^2$.

We can define a Quillen-type cohomology theory for objects of $\mathcal{T}_{E_\star}$, denoted

$$H^n_{\mathcal{T}_{E_\star}}(B, N),$$

where B is in $\mathcal{T}_{E_\star}$ and N is in $\mathrm{ab}\,\mathcal{T}_{E_\star}$. It may be defined as the set of (derived) homotopy classes of maps $B \to K(N, n)$ for the category $s\mathcal{T}_{E_\star}$ of simplicial objects in $\mathcal{T}_{E_\star}$, equipped with a suitable model category structure.

2.7. **Mapping space spectral sequence.** Now we can describe the main spectral sequence.

2.8. **Proposition.** *Let R and F be $K(h)$-local augmented commutative E-algebras. There is a conditionally convergent spectral sequence of the form*

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\widehat{\mathcal{R}}_E(R, F),$$

with

$$E_2^{s,t} = \begin{cases} \mathcal{T}_{E_\star}(\pi_\star R, \pi_\star F) & \text{if } (s,t) = (0,0), \\ H^s_{\mathcal{T}_{E_\star}}(\pi_\star R, \pi_\star \Omega^t \overline{F}) & \text{otherwise.} \end{cases}$$

2.9. **Composite functor spectral sequence.** To compute the cohomology of $\mathbb{T}$-algebras in our setting, we can use a composite functor type spectral sequence. Like $\mathcal{T}$, the category $\mathrm{ab}\,\mathcal{T}_{E_\star}$ of abelian group objects admits a lift of the analytic completion functor, which coincides with the usual one on underlying $E_\star$-modules. Thus, we may define $\widehat{Q} = \mathcal{A}Q\colon \mathrm{ab}\,\mathcal{T}_{E_\star} \to \mathrm{ab}\,\mathcal{T}_{E_\star}$, the analytic completion of indecomposables.

2.10. **Proposition.** *Let B be an object of $\mathcal{T}_{E_\star}$, and let N be an object of $\mathrm{ab}\,\mathcal{T}_{E_\star}$ whose underlying $E_\star$-module is analytically complete. Then there is a spectral sequence of the form*

$$E_2^{i,j} = \operatorname{Ext}^i_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(\mathbf{L}_j\widehat{Q}(B), N) \Longrightarrow H^{i+j}_{\mathcal{T}_{E_\star}}(B, N).$$

Here $\mathbf{L}_j\widehat{Q}$ are left derived functors of $\widehat{Q}$.

In certain situations, we can identify $L_j\widehat{Q} \circ \mathcal{A}$ with $\mathcal{A} \circ L_jQ$.

2.11. **Proposition.** *Let B be a tame object of $\mathcal{T}_{E_\star}$, such that the $\mathbf{L}_j Q(B)$ are also tame. Then*

$$\mathbf{L}_j \widehat{Q}(\mathcal{A}(B)) \approx \mathcal{A}(\mathbf{L}_j Q(B)).$$

2.12. **Computing with abelian group objects.** The category $\mathrm{ab}\,\mathcal{T}_{E_\star}$ of abelian group objects is in fact equivalent to the category $\mathrm{Mod}_\Gamma^\star$. However, this equivalence does not manifest itself in the most obvious way. There is a pair of functors

$$\mathrm{Mod}_\Gamma^\star \xrightarrow{\mathcal{S}} \mathrm{ab}\,\mathcal{T}_{E_\star} \xrightarrow{\mathcal{U}} \mathrm{Mod}_\Gamma^\star.$$

The functor $\mathcal{U}$ is the "obvious" one, which associates to an abelian group object N its underlying augmentation ideal, which is naturally a Γ-module. This functor $\mathcal{U}$ is *not* an equivalence of categories. However, the functor $\mathcal{S}$ *is* an equivalence of categories, and furthermore there is a natural isomorphism of $\mathbb{Z}/2$-graded Γ-modules

$$\mathcal{U}\mathcal{S}(M) \approx \omega^{1/2} \otimes M.$$

Here we write

$$\omega^{t/2} = \mathrm{Ker}[\pi_\star E^{S^t_+} \to \pi_\star E]$$

for the underlying $\mathbb{Z}/2$-graded Γ-module of the augmentation ideal of $E^{S^t_+}$, the E-cochains of the t-sphere, for $t \geq 0$.

Thus we have an isomorphism

$$\mathrm{Ext}^i_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(M,N) \approx \mathrm{Ext}^i_{\mathrm{Mod}_\Gamma^\star}(\mathcal{S}^{-1}(M), \mathcal{S}^{-1}(N)).$$

In practice, it often seems more convenient to write these sorts of things in terms of the underlying Γ-modules of M and N. To do this, we note that if M is a p-torsion free object of $\mathrm{ab}\,\mathcal{T}_{E_\star}$, then there exists an essentially unique object M' of $\mathrm{Mod}_\Gamma^\star$ such that $\omega^{1/2} \otimes M' \approx \mathcal{U}(M)$ as Γ-modules. We will typically write "$\omega^{-1/2} \otimes M$" for this object M' when it exists — an abuse of notation, since $\omega^{1/2}$ is not an invertible object in $\mathrm{Mod}_\Gamma^\star$. Thus, if M and N are p-torsion free objects, we have an isomorphism

$$\mathrm{Ext}^i_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(M,N) \approx \mathrm{Ext}^i_{\mathrm{Mod}_\Gamma^\star}(\omega^{-1/2} \otimes M, \omega^{-1/2} \otimes N).$$

Furthermore, if M and N happen to also be concentrated in even degrees, then we can "remove" another $\omega^{1/2}$ in the same way, and we get an isomorphism

$$\mathrm{Ext}^i_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(M,N) \approx \mathrm{Ext}^i_{\mathrm{Mod}_\Gamma^\star}(\omega^{-1} \otimes M, \omega^{-1} \otimes N).$$

2.13. **Example 1.** Let $R = E^{S^{2m-1}_+}$ where $2m-1$ is an odd positive integer, and let $F = E \rtimes \Omega^t E$, the "square-zero extension" of E by $\Omega^t E$, where $t \in \mathbb{Z}$. Then

$$\pi_0 \widehat{\mathcal{R}}_E(E^{S^{2m-1}_+}, E \rtimes \Omega^t E) \approx \pi_t \mathcal{F}(\mathrm{TAQ}^{S_{K(h)}}((S_{K(h)})^{S^{2m-1}_+}), E),$$

the E-cohomology of the $K(h)$-localized topological André-Quillen homology of the spectrum of $S_{K(h)}$-valued cochains on the sphere. Behrens-Rezk identify these groups with $\pi_t(E \wedge \Phi_h S^{2m-1})_{K(h)}$. The space $\widehat{\mathcal{R}}_E(E^{S^{2m-1}_+}, E \rtimes \Omega^t E)$ is an infinite loop space.

The underlying Γ-module of $\mathrm{Ker}(\pi_\star(E \rtimes \Omega^t E) \to \pi_\star E)$ is isomorphic to $\omega^{t/2} \otimes \mathrm{nul}$, where nul is a certain Γ-module whose underlying $E_\star$-module is isomorpic to $E_\star$, but which has "trivial" Γ-module structure.

We have that $B = \pi_\star E^{S^{2m-1}_+}$ is a free strongly graded commutative $E_\star$-algebra on one generator, whence $\mathbf{L}_j Q(B) \approx 0$ for $j \geq 1$, and $Q(B)$ itself has underlying Γ-module $\mathcal{U}Q(B) \approx$

$\omega^{(2m-1)/2}$. As the objects in question are analytically complete, they are tame, so we have a similar result for $\mathbf{L}_j\widehat{Q}(B)$. Thus the composite functor spectral sequence collapses to give

$$\begin{aligned} H^s(\pi_\star R, \pi_\star \Omega^t \overline{F}) &\approx \operatorname{Ext}^s_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(\omega^{(2m-1)/2}, \omega^{t/2}\otimes \mathrm{nul}) \\ &\approx \operatorname{Ext}^s_{\mathrm{Mod}^\star_\Gamma}(\omega^{m-1}, \omega^{(t-1)/2}\otimes \mathrm{nul}). \end{aligned}$$

For the cases of heights $h=1$ and 2, we will show (by explicit calculation) that these groups vanish except when $s=h$. Thus the resulting mapping space spectal sequence also collapses to give, for $h=1$

$$\begin{aligned} \pi_0\widehat{\mathcal{R}}_E(E^{S^{2m-1}_+}, E\rtimes \Omega^t E) &\approx \operatorname{Ext}^1_{\mathrm{Mod}^\star_\Gamma}(\omega^{m-1}, \omega^{t/2}\otimes \mathrm{nul}) \\ &\approx \begin{cases} E_0/p^{m-1}E_0 & \text{if } t \text{ even,} \\ 0 & \text{if } t \text{ odd.} \end{cases} \end{aligned}$$

For $h=2$, we get

$$\begin{aligned} \pi_0\widehat{\mathcal{R}}_E(E^{S^{2m-1}_+}, E\rtimes \Omega^t E) &\approx \operatorname{Ext}^2_{\mathrm{Mod}^\star_\Gamma}(\omega^{m-1}, \omega^{(t+1)/2}\otimes \mathrm{nul}) \\ &\approx \begin{cases} A_1/(s(A_0)+b'^{m-1}A_1) & \text{if } t \text{ odd,} \\ 0 & \text{if } t \text{ even.} \end{cases} \end{aligned}$$

The object $P_m = A_1/(s(A_0)+b'^{m-1}A_1)$ is a certain E_0-module, for which the reader will need to read ahead to understand. We note here that $P_1=0$, and that $P_2\approx (E_0/pE_0)^{\oplus(p-1)}$, while in general P_m is isomorphic to some complicated quotient of $(E_0/p^{m-1}E_0)^{\oplus(p-1)}$.

2.14. **Example 2.** Let $R=\Sigma^\infty_+\mathbb{Z}$, and let $F=E\times E$. Then

$$\widehat{\mathcal{R}}_E((E\wedge \Sigma^\infty_+\mathbb{Z})_{K(h)}, E\times E)\approx (\text{Comm } S\text{-alg})(\Sigma^\infty_+\mathbb{Z}, E).$$

This space is an H-space, using the evident coproduct on $\Sigma^\infty_+\mathbb{Z}$. In fact, it is an infinite loop space; it is equivalent to the function spectrum $\mathcal{F}(H\mathbb{Z}, \mathrm{gl}_1 E)$.

We have that $B=\pi_\star R$ is the analytic completion of a Laurent polynomial algebra: $B\approx \mathcal{A}B'$, where $B'=E_\star[t,t^{-1}]$. This algebra is smooth, so $\mathbf{L}_jQ(B')\approx 0$ for $j\geq 1$ and $Q(B')\approx \det$, where det is a certain Γ-module whose underlying $E_\star$-module is $E_\star$. We conclude that $\mathbf{L}_j\widehat{Q}(B)\approx 0$ for $j\geq 1$ and $\widehat{Q}(B)\approx \det$.

The underlying Γ-module of $\pi_\star\Omega^t\overline{F}$ is $\omega^{t/2}$.

The composite functor spectral sequence thus degenerates to give

$$\begin{aligned} H^s_{\mathcal{T}_{E_\star}}(\pi_\star R, \pi_\star\Omega^t\overline{F}) &\approx \operatorname{Ext}^s_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(\det, \omega^{t/2}) \\ &\approx \begin{cases} \operatorname{Ext}^s_{\mathrm{Mod}^\star_\Gamma}(\omega^{-1}\otimes\det, \omega^{t/2-1}) & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases} \end{aligned}$$

Assume now that G_0/k where $k=\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$. For the cases of heights $h=1$ and 2, we will show (by explicit calculation) that these Ext-groups vanish except when $s=h-1$ and $t=2h$, in which case

$$\operatorname{Ext}^{h-1}_{\mathrm{Mod}^\star_\Gamma}(\omega^{-1}\otimes\det, \omega^{h-1})\approx \mathbb{Z}_p$$

for $h=1,2$. Also, in either case (11.7) we have that

$$\mathcal{T}_{E_\star}(\pi_\star R, \pi_\star F) = \mathcal{T}(\pi_\star R, \pi_\star E)\approx \overline{\mathbb{F}}_p^\times.$$

Thus, for height 1, the resulting mapping space spectral sequence collapses to give

$$\pi_n(\mathrm{Comm}\ S\ \mathrm{alg})(\Sigma^\infty_+\mathbb{Z}, E) \approx \begin{cases} \overline{\mathbb{F}}_p^\times & \text{if } n=0,\\ \mathbb{Z}_p & \text{if } n=2,\\ 0 & \text{otherwise.}\end{cases}$$

For height 2, we get

$$\pi_n(\mathrm{Comm}\ S\ \mathrm{alg})(\Sigma^\infty_+\mathbb{Z}, E) \approx \begin{cases} \overline{\mathbb{F}}_p^\times & \text{if } n=0,\\ \mathbb{Z}_p & \text{if } n=3,\\ 0 & \text{otherwise.}\end{cases}$$

This proves a conjecture of Lurie in the case of height 2.

3. Deformations

The theory of power operations for Morava E-theory is controlled by the deformation theory of finite subgroups of its formal group G_{univ}, which is the universal deformation of a formal group G_0 over a perfect field k. Any finite subgroup $K \leq G$ of a deformation G determines a quotient homomorphism $f\colon G \to G/K$, and the quotient G/K is provided with the structure of a deformation of G_0 via factorization of a power of the Frobenius isogeny of G_0. In this section we review this deformation theory, which is a consequence of work of Strickland.

3.1. **The category of deformations.** We recall the "deformation of Frobenius" graded category scheme associated to a formal group, as described in [Rez09, §11]. In the following, "formal groups" are assumed to be one-dimensional and commutative.

Fix a prime p, an integer $h \geq 1$, a perfect field k of characteristic p, and a formal group G_0 over k of height h.

If R is a ring of characteristic p, we write $\phi\colon R \to R$ for $\phi(r) = r^p$. For each formal group G over a ring R of characteristic p, the **Frobenius isogeny** $\mathrm{Frob}\colon G \to \phi^*G$ is the homomorphism of formal groups induced by the relative Frobenius map on rings. We write $\mathrm{Frob}^r\colon G \to (\phi^r)^*G$ for the homomorphism inductively defined by $\mathrm{Frob}^r = \phi^*(\mathrm{Frob}^{r-1}) \circ \mathrm{Frob}$.

Given a complete local ring R, with maximal ideal $\mathfrak{m} \subseteq R$ such that $p \in \mathfrak{m}$, and quotient map $\pi\colon R \to R/\mathfrak{m}$, we define a category $\mathrm{Def}(R) = \mathrm{Def}_{G_0/k}(R)$ as follows.

- *Objects* (G, i, α) are deformations of G_0 to R. That is, G is a formal group over R, $i\colon k \to R/\mathfrak{m}$ is an inclusion of fields, and $\alpha\colon \pi^*G \to i^*G_0$ is an isomorphism of formal groups over $R/\mathfrak{m}$.
- *Morphisms* $f\colon (G, i, \alpha) \to (G', i', \alpha')$ are deformations of a power of Frobenius. That is, $f\colon G \to G'$ is a homomorphism of formal groups over R for which there exists an $r \geq 0$ such that (i) $i \circ \phi^r = i'$ as maps $k \to R/\mathfrak{m}$ (so that $(i\circ\phi^r)^*G_0 = (i')^*G_0$), and (ii) the square

(3.2)
$$\begin{array}{ccc} \pi^*G & \xrightarrow{f} & \pi^*G' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ i^*G_0 & \xrightarrow{\mathrm{Frob}^r} & (i\circ\phi^r)^*G_0 \end{array}$$

  of homomorphisms of formal groups over $R/\mathfrak{m}$ commutes.

Every morphism of $\mathrm{Def}(R)$ is a deformation of Frob^r for a unique $r \geq 0$, called the **height** of the morphism. Let $\mathrm{Def}(R)^0 \subset \mathrm{Def}(R)$ be the subcategory consisting of morphisms of height 0, i.e., of isomorphisms between deformations.

3.3. *Remark.* There is an evident functor $U\colon \mathrm{Def}_{G_0}(R) \to \mathrm{Isog}(R)$ from the category of deformations to the category of formal groups over R and isogenies, on objects sending $(G, i, \alpha) \mapsto G$. The functor U can be identified as the "Grothendieck construction" of a functor $\mathcal{D}_{G_0/k}\colon \mathrm{Isog}(R) \to \mathrm{Sets}$, as we now describe.

To each formal group G over R, let $\mathcal{D}_{G_0/k}(G)$ be the set of pairs (i, α) such that (G, i, α) is a deformation of G_0 to R. For an isogeny $f\colon G \to G'$ of rank p^r, let $\mathcal{D}_{G_0/k}(f)\colon \mathcal{D}_{G_0/k}(G) \to \mathcal{D}_{G_0/k}(G')$ be the function that sends $(i, \alpha) \mapsto (i \circ \phi^r, \alpha')$, where α' is the unique isomorphism of formal groups making (3.2) commute. We can think of $\mathcal{D}_{G_0/k}(G)$ as the set of "G_0-deformation structures" on the formal group G, and we have just described how to "push forward" a G_0-deformation structure along any isogeny. Now $\mathrm{Def}_{G_0}(R)$ is a category whose objects are pairs (G, d), where $d \in \mathcal{D}_{G_0/k}(G)$, and whose morphisms $(G, d) \to (G', d')$ are isogenies $f\colon G \to G'$ such that $\mathcal{D}_{G_0/k}(f)(d) = d'$.

3.4. **Representability of the deformation category.** By the deformation theory of Lubin-Tate, for any two objects of $\mathrm{Def}(R)^0$ there is at most one isomorphism between them. Thus, it makes sense to form the quotient category $\mathrm{Sub}(R) \overset{\mathrm{def}}{=} \mathrm{Def}(R)/\mathrm{Def}(R)^0$, by identifying isomorphic objects. The quotient functor $\mathrm{Def}(R) \to \mathrm{Sub}(R)$ is an equivalence of categories, and $\mathrm{Sub}(R)$ is a "gaunt" category, i.e., every morphism is an identity map. The notation comes from the fact that there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{morphisms in } \mathrm{Sub}(R) \\ \text{with source } (G, i, \alpha) \end{array} \right\} \longleftrightarrow \{\text{finite subgroups of } G\},$$

which associates a morphism $f\colon G \to G'$ with its kernel $\mathrm{Ker}\, f \leq G$.

Let $\mathrm{Sub}^r(R)$ denote the set of morphisms of height r in $\mathrm{Sub}(R)$, which correspond to subgroups of degree p^r.

3.5. **Proposition** (Lubin-Tate; Strickland)**.** *Let G_0/k be of height h over a perfect field k. For each $r \geq 0$, there exists a complete local ring A_r which carries a universal height r morphism $f^r_{\mathrm{univ}}\colon (G_s, i_s, \alpha_s) \to (G_t, i_t, \alpha_t) \in \mathrm{Sub}^r(A_r)$. That is, the operation $f^r \mapsto g^*(f^r)$ defines a bijective correspondence from the set of local homomorphisms $g\colon A_r \to R$ to the set* $\mathrm{Sub}^r(R)$ *of height r-morphisms in the category* $\mathrm{Sub}(R)$*. Furthermore, we have that:*

1. $A_0 \approx \mathbb{W}_p k[\![a_1, \dots, a_{h-1}]\!]$.
2. *Under the map $s\colon A_0 \to A_r$ which classifies the source of the universal height r map, A_r is finite and free as an A_0-module.*

Proof. For $r = 0$ this is the theory of Lubin and Tate [LT66]. For $r > 0$ this is a theorem of Strickland [Str97]. □

Thus, $\mathrm{Sub} = \coprod \mathrm{Sub}^r$ is a "affine graded-category scheme". In particular, there are ring maps

$$s = s_k, t = t_k\colon A_0 \to A_k, \qquad \mu = \mu_{\ell,k}\colon A_{k+\ell} \to A_k{}^s\!\otimes_{A_0}{}^t\!A_\ell$$

classifying source, target, and composition of morphisms. Note that s_0, t_0, $\mu_{\ell,0}$, and $\mu_{0,k}$ are all isomorphisms (since Sub^0 consists entirely of identity maps), and that $\mu \circ s = 1 \otimes s$,

$\mu \circ t = t \otimes 1$, and $(\mu \otimes 1) \circ \mu = (1 \otimes \mu) \circ \mu$ (because Sub is a category object). As a consequence, μ is map of A_0-bimodules, which we represent with the notation

$$\mu\colon {}^tA_{k+\ell}{}^s \to {}^tA_k{}^s \otimes_{A_0} {}^tA_\ell{}^s.$$

3.6. **Canonical subgroups.** For any deformation (G, i, α) of G_0 to a ring R of characteristic p, there is for each $r \geq 0$ a morphism in $\mathrm{Def}^r(R)$ of the form

$$\mathrm{Frob}^r\colon (G, i, \alpha) \to ((\phi^r)^* G, i \circ \phi^r, (\phi^r)^*(\alpha)).$$

That is, Frobenius is a deformation of Frobenius. The kernel of Frob^r is the **canonical subgroup** of G of rank p^r.

The universal example of this map is classified by a ring homomorphism

$$\mathrm{can}_r\colon A_r \to A_0/(p),$$

which satisfies $\mathrm{can}_r \circ s = \pi$ and $\mathrm{can}_r \circ t = \phi^r \circ \pi$, where $\pi\colon A_0 \to A_0/(p)$ denotes the evident projection. We note that the further projection

$$A_r \xrightarrow{\mathrm{can}_r} A_0/(p) \to A_0/\mathfrak{m} = k$$

is the map classifying $\mathrm{Frob}^r\colon G_0 \to (\phi^r)^* G_0$.

4. Γ-modules and p-isogeny modules

In this section, we give two equivalent descriptions of a category of "quasi-coherent sheaves over $\mathrm{Def}_{G_0/k}$", called Γ-modules and p-isogeny modules. This summarizes some of the material discussed in [Rez09, §11].

4.1. **The category of Γ-modules.** For a given height h formal group G_0/k, we define a category of $\Gamma = \Gamma_{G_0/k}$-modules. A **Γ-module** is an A-module M equipped with A_0-module homomorphisms

$$P_k = P_{k,M}\colon M \to {}^tA_k{}^s \otimes_{A_0} M$$

for $k \geq 0$, such that $P_0 = \mathrm{id}$, and for all $k, \ell \geq 0$ the square

$$\begin{array}{ccc} M & \xrightarrow{P_k} & A_k{}^s \otimes_{A_0} M \\ \downarrow{\scriptstyle P_{k+\ell}} & & \downarrow{\scriptstyle \mathrm{id} \otimes P_\ell} \\ A_{k+\ell}{}^s \otimes_{A_0} M & \xrightarrow[\mu \otimes \mathrm{id}]{} & A_k{}^s \otimes_{A_0} {}^tA_\ell{}^s \otimes_A M \end{array}$$

commutes. (In this definition, $A_k{}^s \otimes_{A_0} M$ is made into an A-module via the ring homomorphism $t\colon A_0 \to A_k$, hence our notation.) A morphism of Γ-modules is a map $M \to N$ of A_0-modules which commutes with the structure maps P_k.

Given two Γ-modules M and N, their tensor product is the Γ-module with underlying A_0-module $M \otimes_{A_0} N$, and with structure map

$$P_k\colon M \otimes_{A_0} N \to A_k{}^s \otimes_{A_0} (M \otimes_{A_0} N)$$

defined by

$$P_k(m \otimes n) \overset{\mathrm{def}}{=} \sum a'b' \otimes m'' \otimes n'', \quad \text{where } P_k(m) = \sum a' \otimes m' \text{ and } P_k(n) = \sum b' \otimes n''.$$

This tensor product makes Γ-modules into a symmetric monoidal category, with unit object $\mathbb{1}$ being A_0 equipped with structure maps

$$P_k = t_k \colon A_0 \to A_k{}^s\!\otimes_{A_0} A_0 \approx A_k.$$

We write $\mathrm{Mod}_{\Gamma_{G_0/k}}$ for the category of Γ-modules for G_0/k, or simply Mod_Γ if the formal group is clear.

4.2. **Γ-rings.** A **Γ-ring** is a commutative A_0-algebra B together with a Γ-module structure on B, so that multiplication $B \otimes_{A_0} B \to B$ is a morphism of Γ-modules. The initial Γ-ring is $\mathbb{1} = A_0$.

A **Γ-ideal** in a Γ-ring B is an ideal $I \subseteq B$ which is also a Γ-submodule of B; that is, for all $k \geq 0$ we have

$$P_k(I) \subseteq A_k{}^s\!\otimes_{A_0} I.$$

If $I \subseteq B$ is a Γ-ideal, then B/I inherits the structure of a Γ-ring. In particular, $pA_0 \subseteq A_0$ is a Γ-ideal, and thus $A_0/(p)$ is a Γ-ring (in a unique way). However, note that the maximal ideal $\mathfrak{m} \subset A_0$ is *not* a Γ-ideal if $h \geq 2$.

A Γ-ring B is said to satisfy the **Frobenius congruence** if the diagram

$$\begin{array}{ccccc} B & & \xrightarrow{\quad P_1 \quad} & & A_1{}^s\!\otimes_{A_0} B \\ \big\downarrow & & & & \big\downarrow {\scriptstyle \mathrm{can} \otimes \mathrm{id}} \\ B/pB & \xrightarrow[\phi]{} & B/pB & = & (A_0/p) \otimes_{A_0} B \end{array}$$

commutes.

4.3. **The ring Γ of operations.** The associative ring Γ is a graded ring $\Gamma \approx \bigoplus \Gamma[k]$, where $\Gamma[k] = \mathrm{Hom}_{A_0}({}^sA_k, A_0)$ is the A_0-linear dual of A_k (where A_k is viewed as an A_0-module using s_k), and the ring structure of Γ is induced by the maps $\mu_{k,\ell}$. The category comodules for (A_r, s, t, μ) described above is isomorphic to the category of modules for the ring Γ. Explicitly, the isomorphism of categories is obtained by associating $P_k \colon M \to {}^tA_k{}^s\!\otimes_{A_0} M$ with its adjoint $\Gamma[k] \otimes_{A_0} M \subset \Gamma \otimes_{A_0} M \to M$.

In fact, the structure of graded affine category scheme (A_r, s, t, μ) makes Γ into a *twisted commutative bialgebra*, as described in [Rez09, §5]. We remind the reader that although Γ contains $\Gamma[0] = A_0$ as its degree 0 part, the subring A_0 is not central in general.

In this paper, we will usually use the coalgebraic formulation of Γ-modules as described above, but will nonetheless call them "Γ-modules".

4.4. **A remark on "handedness" conventions.** In this paper, we are regarding coactions as happening on the left, i.e., via maps $M \to A_k \otimes_{A_0} M$. For this reason, it seems most convenient here to regard the adjoint action as also happening on the left, i.e., via maps $\Gamma[k] \otimes_{A_0} M \to M$. This is not necessarily the same convention used in other papers.

The only reason we need to talk about Γ at all is so that we can quote the results of [Rez11], which we will reinterpret in the language of coactions in §7. Our choice here is consistent with [Rez11], where Γ is also regarded as acting on the left.

In [Rez09], we used the same left coaction convention we have here. However, in that paper, we regarded Γ as acting on the right (so that properly speaking, what is here called Γ is the opposite of the Γ in that paper).

In [Rez12], we used conventions consistent with having coactions on the right, and actions on the left, although neither actually appear explicitly. In particular, this means that the description of $\Gamma \otimes \mathbb{Z}/p$ (there simply called Γ) given in §4 of that paper, is consistent with treatment of Γ here.

The author finds this business somewhat confusing, and apologizes for any resulting confusion in the reader.

4.5. **p-isogeny modules over deformations.** Here we briefly describe an equivalent formulation of the notion of a Γ-modules called p-isogeny modules , which will provide us with convenient language for certain constructions. (These were described in [Rez09, §11.13] as "quasi-coherent sheaves over Def".)

Let G_0/k be a height h formal group. A **p-isogeny module over deformations of** G_0/k is data $\underline{M} = \{\underline{M}_R, \underline{M}_g\}$ consisting of

- for each complete local ring R, a contravariant functor $\underline{M}_R \colon \mathrm{Def}(R)^{\mathrm{op}} \to \mathrm{Mod}_R$ from the category of deformations of G_0/k to R, to the category of R-modules, and
- for each local homomorphism $g \colon R \to R'$, a natural isomorphism
$$\underline{M}_g \colon R' \otimes_R \underline{M}_R \Longrightarrow \underline{M}_{R'} \circ g^* \colon \mathrm{Def}(R)^{\mathrm{op}} \to \mathrm{Mod}_{R'}.$$
where $g^* \colon \mathrm{Def}(R) \to \mathrm{Def}(R')$ is the evident functor induced by base change along g.

We require that for $R \xrightarrow{g} R' \xrightarrow{h} R''$ that both ways of constructing a natural isomorphism $R'' \otimes_R \underline{M}_R \to \underline{M}_{R''} \circ M_h \circ M_g$ coincide (up to the evident coherence isomorphism), and that M_{id_R} is the identity transformation. A morphism of p-isogeny modules $\underline{M} \to \underline{N}$ is a collection of natural maps $\underline{M}_R \to \underline{N}_R$ commuting with all the structure.

It is straightforward to check [Rez09, Prop. 11.16] that the category of p-isogeny modules is equivalent to the category of Γ-modules. Explicitly, given a Γ-module $(M, \{P_r\})$, the associated p-isogeny module $\underline{M}$ is given by

$$\underline{M}_R((G, i, \alpha)) = R^{\rho_G} \otimes_{A_0} M,$$

where $\rho_G \colon A_0 \to R$ classifies $(G/R, i, \alpha)$, and $f^* = \underline{M}_R(f) \colon \underline{M}_R((G_2, i_2, \alpha_2)) \to \underline{M}_R((G_1, i_1, \alpha_1)$ is the composite

$$R^{\rho_f} \otimes_{A_r} {A_r}^t \otimes_{A_0} M \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes P_r} R^{\rho_f} \otimes_{A_r} {A_r}^t \otimes_{A_0} {}^t{A_r}^s \otimes_{A_0} M \xrightarrow{\mathrm{id} \otimes \mathrm{mult} \otimes \mathrm{id}} R^{\rho_f} \otimes_{A_r} {A_r}^s \otimes_{A_0} M,$$

where $\rho_f \colon A_r \to R$ classifies $f \colon (G_1, i_1, \alpha_1) \to (G_2, i_2, \alpha_2)$. Conversely, a p-isogeny module determines a Γ-module, by evaluating at the universal deformation (defined over A_0), and at the universal height r isogenies (defined over A_r).

We will use the equivalence of Γ-modules and p-isogeny modules without comment in this paper.

4.6. **p-isogeny rings over deformations.** A **p-isogeny ring** is a commutative ring object in p-isogeny modules. The initial p-isogeny ring is $\underline{\mathcal{O}}$, defined by $\underline{\mathcal{O}}_R((G, i, \alpha)) = R$. A p-isogeny $\underline{B}$ ring satisfies the **Frobenius congruence** if and only if for every deformation (G, i, α) over a ring R of characteristic p, the map

$$R^\phi \otimes_R \underline{B}_R((G, i, \alpha)) \underset{\sim}{\xrightarrow{B_\phi}} \underline{B}_R((\phi^* G, i \circ \phi, \phi^* \alpha)) \xrightarrow{\underline{B}_R(\mathrm{Frob})} \underline{B}_R(G, i, \alpha)$$

is equal to the relative Frobenius map on the ring $\underline{B}_R(G, i, \alpha)$; i.e., $\underline{B}$ carries the relative Frobenius map of deformations to the relative Frobenius map of rings. This evidently coincides with the "Frobenius congruence" condition for Γ-rings.

4.7. **The Γ-ring $\mathcal{O}_{G_{\text{univ}}}$.** Given a deformation (G, i, α) of G_0/k to R, let $\mathcal{O}_G$ denote the ring of functions on G. It is isomorphic as an algebra to $R[\![x]\!]$. Tautologically, $(G, i, \alpha)/R \longmapsto \mathcal{O}_G$ is a p-isogeny ring. It corresponds to a Γ-ring $\mathcal{O}_{G_{\text{univ}}}$, whose underlying algebra is the ring of functions on the universal deformation of G_0/k.

4.8. **The Γ-module ω of invariant 1-forms.** Given a deformation (G, i, α) of G_0 to R, let ω_G denote the set of invariant 1-forms on the formal group G. Then ω_G is naturally an R module, free of rank 1, and it is compatible with base change, in the sense that if $\theta\colon R \to R'$ is a local homomorphism, then there is a canonical isomorphism

$$\omega_{\theta^* G} \approx {R'}^{\theta} \otimes_R \omega_G.$$

Furthermore, given a morphism $f\colon (G, i, \alpha) \to (G', i', \alpha')$ in $\mathrm{Def}(R)$, pullback of 1-forms defines a map $f^*\colon \omega_{G'} \to \omega_G$.

Thus, ω naturally carries the structure of a p-isogeny module, and thus we obtain a Γ-module ω, with underlying A_0-module $\omega_{G_{\text{univ}}}$, and with structure map $P_r\colon \omega \to {A_r}^s\otimes_{A_0} \omega$ the map

$$\omega = A_0 \otimes_{A_0} \omega \xrightarrow{t\otimes \mathrm{id}} {A_r}^t \otimes_{A_0} \omega \xrightarrow{f^*} {A_r}^s \otimes_{A_0} \omega,$$

where $f\colon s^* G_{\text{univ}} \to t^* G_{\text{univ}}$ is the universal deformation of Frobenius of height r. It is clear that as a Γ-module, $\omega \approx \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{G_{\text{univ}}} \to \mathbb{1})$.

4.9. **Frobenius-trivial Γ-modules and inverting ω.** Although ω is a rank one A_0-module, it is not invertible as a Γ-module; i.e., there is no Γ-module M such that $\omega \otimes M \approx \mathbb{1}$. However, there are circumstances in which it is possible to unambiguously define a module "$\omega^{-1} \otimes M$", namely when M is *p-torsion free and Frobenius-trivial.*

Choose a basis $u \in \omega$. Then $P_{k,\omega}(u) = b_k \otimes u$ for some $b_k \in A_k$.

4.10. **Lemma.** *The element $b_k \in A_k$ is a divisor of p^k, i.e., $p^k = b_k c$ for some $c \in A_k$.*

Proof. A map $f\colon (G_1, i_1, \alpha_1) \to (G_2, i_2, \alpha_2)$ in $\mathrm{Def}^k(R)$ has kernel killed by $[p^k]$, and so $[p^k]_{G_1} = g \circ f$ for some isogeny $g\colon G_2 \to G_1$. If u is a basis of ω_{G_1}, then $p^k u = f^*(g^*(u))$. Applied to the universal example of a deformation of Frob^k, this proves the claim. ☐

4.11. **Proposition.** *If M, N are Γ-modules, and if N has no p-torsion, then*

$$\omega \otimes -\colon \mathrm{Hom}_\Gamma(M, N) \to \mathrm{Hom}_\Gamma(\omega \otimes M, \omega \otimes N)$$

is an isomorphism.

Proof. This is a straightforward verification, using the fact that according to the hypothesis and (4.10), multiplication by b_k on ${A_k}^s \otimes_{A_0} (\omega \otimes N)$ is injective. ☐

Say that a Γ-module M is **Frobenius-trivial** if the composite of

$$M \xrightarrow{P_{1,M}} {A_1}^s \otimes_{A_0} M \xrightarrow{\mathrm{can}_1 \otimes \mathrm{id}_M} A_0/pA_0 \otimes_{A_0} M = M/pM$$

is 0. In terms of the associated p-isogeny module $\underline{M}$, Frobenius-triviality is equivalent to the following: for any deformation (G, i, α) of G_0/k to a ring R of characteristic p, the map $\underline{M}_R(\mathrm{Frob}_G)\colon \underline{M}_R(\phi^* G, i \circ \phi, \phi^* \alpha) \to \underline{M}_R(G, i, \alpha)$ is equal to 0.

As a consequence, if M is Frobenius trivial, we have that $\underline{M}_R(f \circ \mathrm{Frob}_{G_1}) = 0$ for any map $f\colon (\phi^* G_1, i_1 \circ \phi, \phi^* \alpha_1) \to (G_2, i_2, \alpha_2)$ in $\mathrm{Def}(R)$ where R is of characteristic p. Recall that a morphism $g\colon (G_1, i, \alpha_1) \to (G_2, i_2, \alpha_2)$ in $\mathrm{Def}(R)$ factors as $g = f \circ \mathrm{Frob}_{G_1}$ if and only if

$g^*\colon \omega_{G_2} \to \omega_{G_1}$ is equal to 0. Applied to the universal example this means that when M is Frobenius-trivial, the composite

$$M \xrightarrow{P_{k,M}} {A_k}^s \otimes_{A_0} M \to (A_k/(b_k))^s \otimes_{A_0} M$$

is 0 for all $k \geq 1$, where $b_k \in A_k$ are the elements associated to the basis $u \in \omega$ introduced above.

Given a Γ-module M, let $\mathcal{D}(M)$ denote the solution groupoid of the equation $\omega \otimes X \approx M$. That is, objects of $\mathcal{D}(M)$ are pairs $(N, f\colon \omega \otimes N \to M)$, where N is a Γ-module and f is an isomorphism of Γ-modules, and whose maps $(N, f) \to (N', f')$ are Γ-module isomorphisms $g\colon N \to N'$ such that $f' \circ (g \otimes \mathrm{id}) = f$.

4.12. **Proposition.** *If M is a Γ-module with no p-torsion which is Frobenius-trivial, then then $\mathcal{D}(M)$ is contractible.*

Proof. As noted above, Frobenius-triviality implies that $P_{k,M}(M) \subseteq b_k {A_k}^s \otimes_{A_0} M$, while by (4.10) the p-torsion free condition implies that multiplication by b_k is injective on ${A_k}^s \otimes_{A_0} M$. Thus we may define a Γ-module N with the same underlying A_0-module as M, so that $P_{k,N}(x) = b_k^{-1} P_{k,M}(x)$, with an evident isomorphism $\omega \otimes N \approx M$. Thus $\mathcal{D}(M)$ is non-empty; contractibility follows using (4.11) □

We can summarize the above results as follows.

4.13. **Proposition.** *The functor $\omega \otimes -\colon \mathrm{Mod}_\Gamma \to \mathrm{Mod}_\Gamma$ given by tensoring with ω restricts to an equivalence*

$$\omega \otimes -\colon (\mathrm{Mod}_\Gamma)_{\mathrm{tf}} \xrightarrow{\sim} (\mathrm{Mod}_\Gamma)_{\mathrm{tf,Ft}}$$

from the full subcategory of p-torsion free Γ-modules, to the full subcategory of p-torsion free and Frobenius-trivial Γ-modules.

4.14. **The null Γ-module.** Let nul denote the Γ-module with underlying A_0-module $\mathrm{nul} = A_0$, and with $P_{k,\mathrm{nul}} = 0$ for all $k \geq 1$. Thus, nul has "trivial Γ-action".

Given an A_0-module M, we abuse notation and write $\mathrm{nul} \otimes M$ for the Γ-module with underlying A_0-module M and trivial Γ-action. The induced functor

$$\mathrm{nul} \otimes -\colon \mathrm{Mod}_{A_0} \to \mathrm{Mod}_\Gamma$$

is fully faithful; in fact, $\mathrm{Hom}_\Gamma(\mathrm{nul}, \mathrm{nul}) \approx A_0$ is the endomorphism ring of nul as a Γ-module.

4.15. **The pth power map and the operation Ψ.** Every formal group has a pth power endomorphism $[p]\colon G \to G$. If G is a defomation of a height h formal group G_0, then $[p]$ is an isogeny of rank p^h. Here we point out a subtlety in the way that the pth power map becomes a deformation of Frob^h.

Because G_0/k has height h, $\mathrm{Ker}[p] = \mathrm{Ker}\,\mathrm{Frob}^h$, and thus there is a commutative diagram of homomorphisms

$$\begin{array}{ccc} G_0 & \xrightarrow{\mathrm{Frob}^h} & (\phi^h)^* G_0 \\ & {\scriptstyle [p]}\searrow \quad \nearrow{\scriptstyle \psi_0}\ \sim & \\ & G_0 & \end{array}$$

where ψ_0 is an isomorphism. In particular,

$$[p]\colon (G_0, \mathrm{id}, \mathrm{id}) \to (G_0, \phi^h, \psi_0)$$

describes a morphism in Def^h. More generally, for an arbitrary deformation (G, i, α) of G_0 to R, we get a morphism
$$[p]\colon (G, i, \alpha) \to (G, i \circ \phi^h, i^*\psi_0 \circ \alpha)$$
in $\mathrm{Def}(R)$. Note that this morphism is *not* generally an endomorphism of an object of $\mathrm{Def}(R)$.

There are ring homomorphisms
$$\Psi\colon A_0 \to A_0, \qquad [p]\colon A_h \to A_0,$$
which represent the operations
$$(G, i, \alpha) \mapsto (G, i \circ \phi^h, i^*\psi_0 \circ \alpha), \qquad (G, i, \alpha) \mapsto \big([p]\colon (G, i, \alpha) \to (G, i \circ \phi^h, i^*\psi_0 \circ \alpha)\big),$$
and which fit into a commutative diagram
$$\begin{array}{ccccc} A_0 & \xrightarrow{s} & A_h & \xleftarrow{t} & A_0 \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow {\scriptstyle [p]} & \swarrow {\scriptstyle \Psi} & \\ & & A_0 & & \end{array}$$
It is immediate from the above discussion that the ring homomorphism $\Psi\colon A_0 \to A_0$ is identical to the automorphism $(\phi^h, \widehat{\psi}_0)^*\colon A_0 \to A_0$ induced by the map $(\phi^h, \widehat{\psi}_0) \in \mathrm{FmlGp}_h(G_0/k, G_0/k)$, using the notation of §6.2, and where $\widehat{\psi}_0$ is the composite $G_0 \xrightarrow{\psi_0} (\phi^h)^*G_0 \to G_0$ covering $\phi^h\colon \operatorname{Spec}\overline{\mathbb{F}}_p \to \operatorname{Spec}\overline{\mathbb{F}}_p$.

For any Γ-module M, define $\Psi_M\colon M \to M$ to be the composite
$$M \xrightarrow{P_h} {A_h}^s\otimes_{A_0} M \xrightarrow{[p]\otimes \mathrm{id}} A_0 \otimes_{A_0} M = M.$$
This map Ψ_M is Ψ-linear, in the sense that $\Psi_M(\alpha m) = \Psi(\alpha)\Psi_M(m)$ for $\alpha \in A_0$ and $m \in M$. For the unit Γ-module $\mathbb{1} = A_0$, the map $\Psi_{\mathbb{1}}\colon \mathbb{1} \to \mathbb{1}$ coincides with the ring homomorphism $\Psi\colon A_0 \to A_0$ described above.

4.16. *Remark.* There is an important special case, in which (i) G_0 is defined over $\mathbb{F}_{p^h}$ (so that there is a canonical identification $(\phi^h)^*G_0 = G_0$), and (ii) $\mathrm{Frob}^h\colon G_0 \to G_0$ is central in the ring of endomorphisms of $(G_0)_{\overline{\mathbb{F}}_p}/\overline{\mathbb{F}}_p$. Given (i), condition (ii) is equivalent to the assertion that $\psi_0 = [\lambda]$ for some $\lambda \in \mathbb{Z}_p^\times$. In this case, for every deformation (G, i, α) the endomorphism $[\lambda p]\colon (G, i, \alpha) \to (G, i, \alpha)$ is a deformation of Frob^h.

In this special case, the ring homomorphism $\Psi\colon A_0 \to A_0$ is the identity map, and for a Γ-module M the map $\Psi_M\colon M \to M$ is a map of A_0-modules. On the module ω of invariant 1-forms, $\Psi_\omega\colon \omega \to \omega$ is given by $\Psi(u) = \lambda p u$.

5. Γ-modules and power operations

We briefly review the relation between power operations on Morava E-theory, and the theory of Γ-modules described above. The punchline is that the homotopy groups of $K(h)$-local commutative E-algebras are "analytically complete $\mathbb{T}$-algebras" for a certain monad $\mathbb{T}$. (§5.11). We also discuss abelian group objects in $\mathbb{T}$-algebras (§5.12).

5.1. $K(h)$**-local commutative** E**-algebras and** E**-modules.** Fix a height h formal group G_0/k over a perfect field k, and let $E = E_{G_0/k}$ be its associated Morava E-theory spectrum, which is canonically a commutative S-algebra. Recall that $\pi_* E$ is even periodic, and that $\pi_0 E = A_0$, the ring which classifies deformations of G_0/k.

Let $\mathcal{M} = \mathcal{M}_{G_0/k}$ denote the homotopy theory of E-module spectra, and let $\mathcal{R} = \mathcal{R}_{G_0/k}$ denote the homotopy theory of E-algebra spectra. Both these homotopy theories are realized by the model category structure described in EKMM, in which weak equivalences are maps which are weak equivalences on underlying spectra.

Let $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_{G_0/k}$ denote the homotopy theory of $K(h)$-local E-module spectra, and let $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_{G_0/k}$ denote the homotopy theory of $K(h)$-local E-algebra spectra. These are localizations of $\mathcal{M}$ and $\mathcal{R}$, with weak equivalences the maps which are $K(h)$-homology isomorphisms on underlying spectra.

5.2. **Homotopy of** E**-modules and** $\mathbb{Z}/2$**-graded** $E_\star$**-modules.** Recall [Rez09, §2] that we may define a category $\mathrm{Mod}_{E_\star}$ of $\mathbb{Z}/2$-graded E_0-modules, whose objects are pairs $M = \{M_0, M_{-1}\}$ of E_0-modules. We will call such objects $E_\star$**-modules**. This category becomes a symmetric monoidal category via an ω**-twisted tensor product**, defined by

$$M \otimes N \overset{\text{def}}{=} \{(M_0 \otimes_{E_0} N_0) \oplus (M_{-1} \otimes_{E_0} N_{-1} \otimes_{E_0} \omega),\ (M_0 \otimes_{E_0} N_{-1}) \oplus (M_{-1} \otimes_{E_0} N_0)\}, \tag{5.3}$$

where $\omega = \pi_2 E$. We write ω for the $E_\star$-module $\{\pi_2 E, 0\}$ and $\omega^{1/2}$ for the $E_\star$-module $\{0, \pi_0 E\}$, whence $\omega^{1/2} \otimes \omega^{1/2} \approx \omega$.

We define a functor

$$\pi_\star \colon h\mathcal{M} \to \mathrm{Mod}_{E_\star}, \qquad \pi_\star M = \{\pi_0 M, \pi_{-1} M\}$$

from E-modules to the category of $E_\star$-modules. It is straightforward to check that this functor is weakly monoidal, in the sense that there is an evident map

$$\pi_\star M \otimes \pi_\star N \to \pi_\star (M \wedge_E N)$$

satisfying suitable coherence properties. Observe that with these conventions we have $\pi_\star \Sigma^{-k} E \approx \omega^{k/2}$ for all $k \in \mathbb{Z}$. (Note that $\omega^{1/2}$ is invertible in $\mathrm{Mod}_{E_\star}$.)

We can recover the usual $\mathbb{Z}$-graded homotopy groups of a module from the $\mathbb{Z}/2$-graded ones, by

$$\pi_k M \approx \mathrm{Hom}_{\mathrm{Mod}_{E_\star}}(\omega^{-k/2}, \pi_\star M).$$

It is not hard to show that this describes a monoidal equivalence between $\mathrm{Mod}_{E_\star}$ and the more familiar category of $\mathbb{Z}$-graded E_*-modules.

5.4. $\pi_0 R$ **as a** Γ**-ring.** We now recapitulate the following statement, which is described in detail in [Rez09]: the homotopy groups of an object $R \in \widehat{\mathcal{R}}$ are naturally equipped with the structure of a "$\mathbb{Z}/2$-graded Γ-ring satisfying the Frobenius congruence".

Given $m \geq 1$, let ρ denote the m-dimensional real permutation representation of Σ_m, and let $\overline{\rho} \subset \rho$ denote the reduced representation (of codimension 1.) Recall that given a map of spectra $x \colon S^k \to R$, the commutative ring structure on R gives a "total mth power" map $\mathcal{P}_m(x) \colon S^k \wedge B\Sigma_m^{k\overline{\rho}} \approx B\Sigma_m^{k\rho} \to R$ for all $m \geq 0$. Applied to $m = p^r$, this construction produces abelian group homomorphisms

$$P_r \colon \pi_k R \to (E^0 B\Sigma_{p^r}^{k\overline{\rho}}/I) \otimes_{E_0} \pi_k R,$$

where I denotes the ideal generated by the image of transfer maps along the restriction to $\Sigma_i \times \Sigma_{p^r - i} \subset \Sigma_{p^r}$ for all $0 < i < p^r$.

Strickland's theorem [Str98] asserts a canonical isomorphism of rings

$$E^0 B\Sigma_{p^r}/I \approx A_r,$$

Using this, we obtain a Γ-module structure on $\pi_0 R$ by

$$P_r \colon \pi_0 R \to (E^0 B\Sigma_{p^r}/I) \otimes_{E_0} \pi_0 R = {A_r}^s \otimes_{A_0} \pi_0 R.$$

With this structure, $\pi_0 R$ is in fact a Γ-ring which satisfies the *Frobenius congruence* (§4.2).

5.5. *Remark.* In fact, the ring Γ of §4.3 is precisely the ring of *additive* operations on π_0 of a $K(h)$-local commutative E-algebra. That is, it is the endomorphisms of the functor $\widehat{\mathcal{R}} \to \mathrm{Ab}$ defined by $R \mapsto \pi_0 R$. This is the point of view taken in [Rez09, §6]; see also [Rez11, §3.8].

We note some significant examples.

- The natural Γ-ring structure on $\pi_0 E$ is precisely the initial Γ-ring $\mathbb{1} = A_0$.
- The natural Γ-ring structure on $\pi_0 E^{\mathbb{CP}^\infty_+} = E^0 \mathbb{CP}^\infty$ is precisely the Γ-ring $\mathcal{O}_{G_{\mathrm{univ}}}$ of functions on the universal deformation of G_0.
- The natural Γ-module structure on $\pi_2 E \approx \mathrm{Ker}[\pi_0 E^{\mathbb{CP}^1_+} \to \pi_0 E]$ is precisely the Γ-module ω of invariant 1-forms.

5.6. $\mathbb{Z}/2$**-graded Γ-modules and Γ-rings.** Recall Recall [Rez09, §2 and §7], that we may define a category $\mathrm{Mod}_\Gamma^\star$ of $\mathbb{Z}/2$**-graded Γ-modules**, whose objects are pairs $M^\star = \{M^0, M^1\}$ of Γ-modules, which becomes a symmetric monoidal category via an ω-twisted tensor product. The formula for this tensor product is exactly that of (5.3), though now ω represents the Γ-module of invariant 1-forms (which is naturally isomorphic to $\pi_2 E$.)

As a notational short-hand, we identify Mod_Γ with $\mathrm{Mod}_\Gamma^{\mathrm{even}} \subset \mathrm{Mod}_\Gamma^\star$, the full subcategory of $\mathbb{Z}/2$-graded Γ-modules concentrated in even degree. Thus, we write ω for the $\mathbb{Z}/2$-graded Γ-module $\{\omega, 0\} = \{\pi_2 E, 0\}$.

As in §5.2, we write $\omega^{1/2}$ for the $\mathbb{Z}/2$-graded Γ-module $\{0, \pi_0 E\}$. As before, we have $\omega^{1/2} \otimes \omega^{1/2} \approx \omega$. Furthermore, there are evident isomorphisms

$$\pi_\star E^{S^k} = \mathrm{Ker}[\pi_\star E^{S^k_+} \to \pi_\star E] \approx \omega^{k/2}$$

of Γ-modules for $k \geq 0$. Note that $\omega^{1/2}$ is *not invertible* as a $\mathbb{Z}/2$-graded Γ-module, though it is invertible as an $E_\star$-module.

Commutative monoid objects in $\mathrm{Mod}_\Gamma^\star$ form a category $\mathrm{Ring}_\Gamma^\star$ of $\mathbb{Z}/2$**-graded Γ-rings**.

5.7. $\frac{1}{2}$**-Frobenius-triviality and inverting** $\omega^{1/2}$**.** Say that a $\mathbb{Z}/2$-graded Γ-module $M = \{M_0, M_{-1}\}$ is $\frac{1}{2}$**-Frobenius-trivial** if the Γ-module M_0 is Frobenius-trivial in the sense of §4.9.

5.8. **Proposition.** *The functor* $\omega^{1/2} \otimes - \colon \mathrm{Mod}_\Gamma^\star \to \mathrm{Mod}_\Gamma^\star$ *given by tensoring with* $\omega^{1/2}$ *restricts to an equivalence*

$$\omega^{1/2} \otimes - \colon (\mathrm{Mod}_\Gamma^\star)_{\mathrm{tf}} \xrightarrow{\sim} (\mathrm{Mod}_\Gamma^\star)_{\mathrm{tf}, \frac{1}{2}\mathrm{Ft}}$$

from the full subcategory of p-torsion free Γ-modules, to the full subcategory of p-torsion free and $\frac{1}{2}$-Frobenius-trivial Γ-modules.

Proof. This is immediate from (4.13) and the definition of the ω-twisted tensor product. □

Clearly, we can apply the above proposition iteratively. Thus, for any $k \geq 0$ there are equivalences of full subcategories $\omega^{k/2} \otimes - \colon (\mathrm{Mod}_\Gamma^\star)_{\mathrm{tf}} \xrightarrow{\sim} (\mathrm{Mod}_\Gamma^\star)_{\mathrm{tf}, \frac{k}{2}\mathrm{Ft}}$, the definitions and verifications of which we leave as an exercise for the reader.

5.9. **$\pi_\star R$ as a $\mathbb{Z}/2$-graded Γ-ring.** It is a fact (see discussion in [Rez09, §7.5]) that the zero-section inclusion $B\Sigma_m^{-\bar\rho} \to B\Sigma_m^0$ induces an isomorphism

$$A_r = E^0 B\Sigma_{p^r}/I \xrightarrow{\sim} E^0 B\Sigma_{p^r}^{-\bar\rho}/I.$$

The induced map

$$P_r\colon \pi_{-1}R \to (E^0 B\Sigma_{p^r}^{-\bar\rho}/I) \otimes_{E_0} \pi_{-1}R = A_r{}^s\otimes_{A_0} \pi_{-1}R,$$

defines a Γ-module structure on $\pi_{-1}R$.

The power construction $\mathcal{P}_m$ is multiplicative, in the sense that the diagram

$$\begin{array}{ccc} B\Sigma_m^{(a+b)\rho} & \xrightarrow{\mathcal{P}_m(xy)} & R \\ {\scriptstyle\text{diag}}\downarrow & & \uparrow{\scriptstyle\text{mult}} \\ B\Sigma_m^{a\rho} \wedge B\Sigma_m^{b\rho} & \xrightarrow[\mathcal{P}_m(x)\wedge\mathcal{P}_m(y)]{} & R\wedge R \end{array}$$

commutes for $x \in \pi_a R$, $y \in \pi_b R$. Applied to a three-fold product xyu with $x, y \in \pi_{-1}R$ and $u \in \omega = \pi_2 E \subseteq \pi_2 R$, this multiplicativity implies that

$$\pi_{-1}R \otimes_{E_0} \pi_{-1}R \otimes_{E_0} \omega \xrightarrow{\text{mult}\otimes\text{id}} \pi_{-2}R \otimes_{E_0} \omega \approx \pi_0 R$$

is a map of Γ-modules. Thus, we obtain a functor

$$\pi_\star\colon h\widehat{\mathcal{R}} \to \mathrm{Ring}_\Gamma^\star, \qquad \pi_\star R = \{\pi_0 R, \pi_{-1}R\}$$

from $K(h)$-local commutative E-algebras to the category of $\mathbb{Z}/2$-graded Γ-rings. Furthermore, $\pi_\star R$ satisfies the *Frobenius congruence*, which just means that the even degree part $\pi_0 R$ satisfies the Frobenius congruence as noted above. (That is, "Frobenius congruence" does not impose a condition on odd degree.)

5.10. **Square-zero extension rings.** Let $\mathrm{nul} \in \mathrm{Mod}_\Gamma^\star$ denote the $\mathbb{Z}/2$-graded Γ-module $\{\mathrm{nul}, 0\}$, where $\mathrm{nul} \in \mathrm{Mod}_\Gamma$ is the null module of §4.14. The evident functor

$$\mathrm{nul} \otimes -\colon \mathrm{Mod}_{E_\star} \to \mathrm{Mod}_\Gamma^\star$$

is fully faithful.

Given an E-module M, we may form the **square-zero extension** $E \rtimes M$, which is an augmented commutative E-algebra with "trivial" multiplication on the augmentation fiber M. We have that

$$\mathrm{Ker}[\pi_\star(E \rtimes M) \to \pi_\star E] \approx \mathrm{nul} \otimes \pi_\star M$$

as $\mathbb{Z}/2$-graded Γ-modules.

5.11. **The monad $\mathbb{T}$.** The functor $\pi_\star\colon \widehat{\mathcal{R}} \to \mathrm{Ring}_\Gamma^\star$ described above lifts even further to a functor

$$\pi_\star\colon \widehat{\mathcal{R}} \to \mathcal{T},$$

where $\mathcal{T}$ is the category of $\mathbb{Z}/2$-graded $\mathbb{T}$-algebras, where $\mathbb{T}$ is a certain monad on $\mathbb{Z}/2$-graded $\pi_0 E$-modules, as mentioned in §2.3, and which is analyzed at inordinate length in [Rez09], and the reader is referred there for more information.

In brief, a $\mathbb{T}$-algebra is a $\mathbb{Z}/2$-graded Γ-ring equipped with an additional non-additive operation which "witnesses" the Frobenius congruence. Theorem A of [Rez09] asserts that a p-torsion free $\mathbb{Z}/2$-graded Γ-ring B admits the structure of a $\mathbb{T}$-algebra (necessarily uniquely) if and only if B satisfies the Frobenius congruence.

In fact, the above functor factors through a full subcategory

$$\pi_\star\colon \widehat{\mathcal{R}} \to \widehat{\mathcal{T}}$$

of *analytically complete* objects. We'll say more about this later.

5.12. **Abelian group objects.** Let $\mathcal{T}_{E_\star}$ denote the slice category, whose objects are objects of $\mathcal{T}$ equipped with an augmentation to $E_\star = \pi_\star E$. We write $\mathrm{ab}\,\mathcal{T}_{E_\star}$ for the category of abelian group objects in $\mathcal{T}_{E_\star}$. It is easy to see that an object $f\colon B \to E_\star$ of $\mathcal{T}_{E_\star}$ admits an abelian group structure if and only if the augmentation ideal $\overline{B} = \operatorname{Ker} f$ is such that $\overline{B}^2 = 0$, and that if such an abelian groups structure exists, it is unique. Thus $\mathrm{ab}\,\mathcal{T}_{E_\star}$ is equivalent to the full subcategory of objects in $\mathcal{T}_{E_\star}$ with square-zero augmentation ideal.

Any abelian group object $B \in \mathrm{ab}\,\mathcal{T}_{E_\star}$ has an underlying $\mathbb{Z}/2$-graded Γ-module $\overline{B}$, giving a forgetful functor $\mathcal{U}\colon\ \mathrm{ab}\,\mathcal{T}_{E_\star} \to \mathrm{Mod}_\Gamma^\star$. This forgetful functor actually lands in the full subcategory $(\mathrm{Mod}_\Gamma^\star)_{\frac{1}{2}\mathrm{Ft}} \subset \mathrm{Mod}_\Gamma^\star$ of $\frac{1}{2}$-Frobenius-trivial modules (5.7); this is an immediate consequence of the Frobenius congruence for $\mathbb{T}$-algebras, applied to a square-zero augmentation ideal.

The category of abelian group objects turns out to be equivalent to the category of $\mathbb{Z}/2$-graded Γ-modules, but not via the forgetful functor.

5.13. **Proposition.** *There exists an equivalence of categories $\mathcal{S}\colon \mathrm{Mod}_\Gamma^\star \to \mathrm{ab}\,\mathcal{T}_{E_\star}$ and a natural isomorphism between the composition of*

$$\mathrm{Mod}_\Gamma^\star \xrightarrow[\sim]{\mathcal{S}} \mathrm{ab}\,\mathcal{T}_{E_\star} \xrightarrow{\mathcal{U}} \mathrm{Mod}_\Gamma^\star$$

and the endofuntor $\omega^{1/2} \otimes -\colon \mathrm{Mod}_\Gamma^\star \to \mathrm{Mod}_\Gamma^\star$.

Sometimes we will abuse notation and write "$\omega^{-1/2} \otimes M$" for $\mathcal{S}^{-1}(M)$, where $M \in \mathrm{ab}\,\mathcal{T}_{E_\star}$. When the underlying $E_\star$-module of M is p-torsionfree, this notation is in fact unambiguous by (5.8), since $\mathcal{U}(M)$ is $\frac{1}{2}$-Frobenius trivial

We note that both $\mathrm{Mod}_\Gamma^\star \approx \mathrm{Mod}_\Gamma^{\mathrm{even}} \times \mathrm{Mod}_\Gamma^{\mathrm{odd}}$ and $\mathrm{ab}\,\mathcal{T}_{E_\star} \approx \mathrm{ab}\,\mathcal{T}_{E_\star}^{\mathrm{even}} \times \mathrm{ab}\,\mathcal{T}_{E_\star}^{\mathrm{odd}}$ can be separated into purely even and odd components. The functor $\omega^{1/2} \otimes -\colon \mathrm{Mod}_\Gamma^{\mathrm{even}} \to \mathrm{Mod}_\Gamma^{\mathrm{odd}}$ is an equivalence by construction, and thus we obtain an equivalence $\omega^{1/2} \otimes \mathcal{S}\colon \mathrm{Mod}_\Gamma^{\mathrm{even}} \to \mathrm{ab}\,\mathcal{T}_{E_\star}^{\mathrm{even}}$ whose composite with the forgetful functor $\mathcal{U}\colon\ \mathrm{ab}\,\mathcal{T}_{E_\star}^{\mathrm{even}} \to \mathrm{Mod}_\Gamma^{\mathrm{even}}$ is isomorphic to $\omega \otimes -$. If $M \in \mathrm{ab}\,\mathcal{T}_{E_\star}^{\mathrm{even}}$ has p-torsion free underlying $E_\star$-module, we will abuse notation and write "$\omega^{-1} \otimes M$" for the corresponding object of $\mathrm{Mod}_\Gamma^{\mathrm{even}}$.

6. Dependence on the formal group

All the structure we have discussed so far depends on a choice G_0/k of formal group of height $h \geq 1$ over a perfect field k of characteristic p. In this section we say a bit how the stucture varies as we change the formal group.

6.1. **The category of height h formal groups.** Let FmlGp_h denote the category whose objects are formal groups G_0/k of height h over a perfect field k of characteristic p, and whose morphisms $(j,\gamma)\colon G_0'/k' \to G_0/k$ are commutative squares

$$\begin{array}{ccc} G_0' & \xrightarrow{\gamma} & G_0 \\ \downarrow & & \downarrow \\ \operatorname{Spec} k' & \xrightarrow[\operatorname{Spec} j]{} & \operatorname{Spec} k \end{array}$$

such that the induced map $\widetilde{\gamma}\colon G_0' \to j^*G_0$ to the pullback of G_0 along $j\colon k \to k'$ is an isomorphism of formal groups over k'.

The Hopkins-Miller theorem asserts that Morava E-theory is realized in an essentially unique way as a functor of $(\infty,1)$-categories $\mathrm{FmlGp}_h \to \{\text{Commutative } S\text{-algebra}\}$, sending G_0/k to $E_{G_0/k}$.

6.2. **Dependence of** Def**.** It is immediate that a morphism $(j,\gamma)\colon G_0'/k' \to G_0/k$ induces functors on deformation categories of the form $(j,\gamma)_*\colon \mathrm{Def}_{G_0'/k'}(R) \to \mathrm{Def}_{G_0/k}(R)$, which on objects send (G,i,α) to $(G, i\circ j, i^*\widetilde{\gamma}\circ\alpha)$. These functors are represented by maps of affine graded-category schemes, so that in particular there are induced maps of rings

$$(j,\gamma)^*\colon A_{r,G_0/k} \to A_{r,G_0'/k'}$$

which commute with the structure maps s,t,μ of the graded-category scheme. Furthermore, the induced maps

$$A_{0,G_0'/k'} \otimes_{A_{0,G_0/k}} {}^s A_{r,G_0/k} \to A_{r,G_0'/k'}$$

are isomorphisms for all r.

There are special cases of special interest: extension of scalars, and automorphisms.

Extension of scalars. Suppose given G_0/k, and let $G_0' = (G_0)_{k'}$ be the base change along an inclusion $k \subset k'$. Then $A_{r,G_0'/k'} \approx \mathbb{W}k' \otimes_{Wk} A_{r,G_0/k}$, and the structure maps s,t,μ are also obtained by base change; for instance, s and t for G_0'/k' are given by

$$\mathrm{id}\otimes s\colon \mathbb{W}k' \otimes_{\mathbb{W}k} A_{0,G_0/k} \to \mathbb{W}k' \otimes_{\mathbb{W}k} A_{r,G_0/k}, \quad \widetilde{\phi}^r \otimes t\colon \mathbb{W}k' \otimes_{\mathbb{W}k} A_{0,G_0/k} \to \mathbb{W}k' \otimes_{\mathbb{W}k} A_{r,G_0/k},$$

where $\widetilde{\phi}\colon \mathbb{W}k' \to \mathbb{W}k'$ is the lift of the pth power map $\phi\colon k' \to k'$. The evident map $G_0'/k' \to G_0/k$ in FmlGp_h corresponds to the evident inclusions of rings $A_{r,G_0/k} \to A_{r,G_0'/k'}$.

Automorphisms. Fix a height h formal group $G_0/\overline{\mathbb{F}}_p$; in this context, it is usual to take G_0 to be the Honda formal group, although we won't assume this. Define

$$\mathbb{G} = \mathrm{FmlGp}_h(G_0/\overline{\mathbb{F}}_p, G_0/\overline{\mathbb{F}}_p).$$

There is an associated group extension

$$1 \to \mathbb{S} \to \mathbb{G} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1,$$

where the projection sends (σ,γ) to $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. The subgroup $\mathbb{S}$ is group of automorphisms of G_0 over $\overline{\mathbb{F}}_p$, i.e., the Morava stabilizer group of height h. (Recall that all height h formal groups over a separably closed field are isomorphic.)

The above extension admits a splitting, but the choice of splitting is not natural; rather such a splitting is determined by a model for G_0 over $\mathbb{F}_p$. More generally, suppose given a formal group $G_1/\mathbb{F}_{p^r}$, and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_{p^r})$ write $\iota_\sigma\colon (G_1)_{\overline{\mathbb{F}}_p} \to (G_1)_{\overline{\mathbb{F}}_p}$ for the tautological map of formal schemes covering $\sigma\colon \operatorname{Spec}\overline{\mathbb{F}}_p \to \operatorname{Spec}\overline{\mathbb{F}}_p$. Then we have a group homomorphism $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^r}) \to \mathrm{FmlGp}_h((G_1)_{\overline{\mathbb{F}}_p}, (G_1)_{\overline{\mathbb{F}}_p})$, defined by $\sigma \longmapsto (\sigma,\iota_\sigma)$. Then a choice of isomorphism $f\colon (G_1)_{\overline{\mathbb{F}}_p} \to G_0$ of formal groups over $\overline{\mathbb{F}}_p$ deterimes a homomorphism s of the form

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \supseteq \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^r}) \xrightarrow{s} \mathbb{G},$$

by sending $\sigma \longmapsto (\sigma, f\iota_\sigma f^{-1})$. The map s is a "partial section", in the sense that the composite $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^r}) \xrightarrow{s} \mathbb{G} \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the standard inclusion. The construction $(G_1, f) \mapsto s$ describes a bijection between the set of $\mathbb{F}_{p^r}$-isomorphism classes of height h formal groups over $\mathbb{F}_{p^r}$, and the set of $\mathbb{S}$-conjugacy classes of partial sections $s\colon \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^r}) \to \mathbb{G}$ (see [Frö68, §III.3]; a continuous section s up to conjugacy is the same thing as an element of the non-abelian cohomology $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^r}), \mathbb{S})$.)

The group $\mathbb{G}$ acts on the categories $\mathrm{Def}_{G_0/\overline{\mathbb{F}}_p}(R)$, and therefore acts on the rings $A_{r,G_0/\overline{\mathbb{F}}_p}$, compatibly with the structure maps s, t, μ. The action of $\mathbb{G}$ on $A_{0,G_0/\overline{\mathbb{F}}_p}$ is the usual action of the automorphisms of a formal group on the Lubin-Tate moduli space. The action of $\mathbb{G}$ on $A_{r,G_0/\overline{\mathbb{F}}_p}$ is compatible with its action on the cohomology of symmetric groups, via the quotient map $E^0B\Sigma_{p^r} \to E^0B\Sigma_{p^r}/I \approx A_{r,G_0/\overline{\mathbb{F}}_p}$ of Strickland's theorem.

6.3. **Dependence of** Mod_Γ**.** For each object G_0/k of FmlGp_h we have a category $\mathrm{Mod}_{\Gamma_{G_0/k}}$ of Γ-modules. Given a morphism $(j,\gamma)\colon G_0'/k' \to G_0/k$ in FmlGp_h, there is an evident functor $(j,\gamma)_*\colon \mathrm{Mod}_{\Gamma_{G_0/k}} \to \mathrm{Mod}_{\Gamma_{G_0'/k'}}$. Explicitly, this functor sends $(M, \{P_r\})$ to $(M', \{P_r'\})$, where

$$M' = A_{0,G_0'/k'} \otimes_{A_{0,G_0'}} M, \qquad P_r' = A_{0,G_0'/k'} \otimes_{A_{0,G_0/k}} P_r.$$

These constructions fit together to define a pseudofunctor $\mathrm{Mod}_\Gamma\colon \mathrm{FmlGp}_h \to \mathrm{Cat}$.

In particular, if $\mathbb{G} = \mathrm{FmlGp}_h(G_0/\overline{\mathbb{F}}_p, G_0/\overline{\mathbb{F}}_p)$, then we obtain an "action" of $\mathbb{G}$ on the category $\mathrm{Mod}_\Gamma = \mathrm{Mod}_{\Gamma_{G_0/\overline{\mathbb{F}}_p}}$. A **$\mathbb{G}$-equivariant Γ-module** is a Γ-module M together with for each $g \in \mathbb{G}$ a map $\alpha_g\colon g_*M \to M$ of Γ-modules, such that $g_*(\alpha_h) \circ \alpha_g = \alpha_{hg}$.

It is straightforward to show that a $\mathbb{G}$-equivariant Γ-module M is the same thing as a Γ-module $(M, \{P_r\})$, together with an action of $\mathbb{G}$ on the abelian group M, such that the A_0-module structure map $A_0 \otimes M \to M$ and the Γ-module structure maps $P_r\colon M \to A_r{}^s\!\otimes_{A_0} M$ are $\mathbb{G}$-equivariant, using the evident $\mathbb{G}$-action on the A_r.

We note that the usual $\mathbb{G}$-action on E-cohomology gives $\mathcal{O}_{G_{\mathrm{univ}}} \approx E^0\mathbb{CP}^\infty$ its tautological $\mathbb{G}$-equivariant Γ-ring structure, and thus gives $\omega \approx \widetilde{E}^0S^2$ its tautological $\mathbb{G}$-equivariant Γ-module structure.

7. The structure of Γ-modules

7.1. Γ **is quadratic.** The following result says that most of the rings A_r are superfluous for describing the category of Γ-modules; a Γ-module structure is determined by the map P_1, subject to a condition involving the ring A_2.

7.2. **Proposition.** *Let M be an A_0-module, and let $P\colon M \to A_1{}^s\!\otimes_{A_0} M$ be a map of A_0-modules (with the target module structure defined using $t_1\colon A_0 \to A_1$). There exists a dotted arrow in*

$$\begin{array}{ccc} M & \xrightarrow{\quad P \quad} & A_1{}^s\!\otimes_{A_0} M \\ \vdots & & \downarrow{\scriptstyle \mathrm{id}\otimes P} \\ A_2{}^s\!\otimes_{A_0} M & \xrightarrow{\mu\otimes\mathrm{id}} & A_1{}^s\!\otimes_{A_0}{}^tA_1{}^s\!\otimes_{A_0} M \end{array}$$

making the diagram commute if and only if there exists a Γ*-module structure* $\{P_r\}_{r\geq 0}$ *on* M *such that* $P_1 = P$*; furthermore, the* Γ*-module structure is unique if it exists.*

We will prove this below.

7.3. **The Koszul complex.** Let M and N be Γ-modules. We define the Koszul complex $\mathcal{C}^\bullet(M,N)$ as follows. Below we write "$\otimes$" as a shorthand for "${}^s\otimes_{A_0}{}^t$". For each $q \geq 0$, let B_q denote the image of

$$\bigoplus_{i=0}^{q-2} A_1^{\otimes i} \otimes A_2 \otimes A_1^{\otimes q-i-2} \xrightarrow{(\mathrm{id}\otimes\mu\otimes\mathrm{id})} A_1^{\otimes q}$$

inside $A_1^{\otimes q} = A_1 {}^s\otimes_{A_0}{}^t \cdots {}^s\otimes_{A_0}{}^t A_1$, and let

$$D_q = A_1^{\otimes q}/B_q.$$

We can regard D_q as both a right A_0-module (by $A_0 \xrightarrow{s} A_1 \xrightarrow{\text{rightmost factor}} A_1^{\otimes q} \to D_q$) and a left A_0-module (by $A_0 \xrightarrow{t} A_1 \xrightarrow{\text{leftmost factor}} A_1^{\otimes q} \to D_q$); these module structures commute. The induced quotient maps

$$\times\colon D_p \otimes_{A_0} D_q \to D_{p+q}$$

give $\bigoplus D_q$ the structure of an associative ring. In particular, since $D_1 = A_1$, there are maps

$$A_1 {}^s\otimes_{A_0} D_q \xrightarrow{\times} D_{q+1}, \qquad D_q \otimes_{A_0} {}^t A_1 \xrightarrow{\times} D_{q+1}.$$

Set

$$\mathcal{C}^q(M,N) \overset{\text{def}}{=} \mathrm{Hom}_{A_0}(M, D_q \otimes_{A_0} N),$$

with coboundary operator $d_q\colon \mathcal{C}^q(M,N) \to \mathcal{C}^{q+1}(M,N)$ given on $f\colon M \to D_q \otimes_{A_0} N$ by

$$d_q f = (\mathrm{id}_{D_q} \times P_N) \circ f - (-1)^q (\mathrm{id}_{A_1} \times f) \circ P_M.$$

That this defines a cochain complex follows from the fact that $(\mathrm{id}_{A_1} \times P_M) \circ P_M = 0$ for any Γ-module M.

7.4. **Proposition.** *If* M *is projective as an* A_0*-module, then*

$$H^q\mathcal{C}^\bullet(M,N) \approx \mathrm{Ext}^q_\Gamma(M,N).$$

Furthermore, $D_q \approx 0$ *for* $q > h$*, and thus for* A_0*-projective* M *we have* $\mathrm{Ext}^q_\Gamma(M,N) = 0$ *for* $q > h$*.*

We give a proof below.

7.5. **Duality for bimodules.** Let X be an A_0-bimodule. The "right-dual" of a bimodule X is

$$X^* \overset{\text{def}}{=} \mathrm{Hom}^{\text{right}}_{A_0}(X, A_0),$$

the group of right-A_0-module homomorphisms. Given $a, b \in A_0$ and $f \in X^*$, define $(a \cdot f \cdot b)(x) \overset{\text{def}}{=} a(f(bx)) = f(bxa)$. Because A_0 is commutative, $a \cdot f \cdot b \in X^*$, and it is straightforward to check that this operation makes X^* into an A_0-bimodule. Furthermore, the evaluation map

$$\mathrm{ev}_X\colon X^* \otimes_{A_0} X \to A_0, \qquad f \otimes x \mapsto f(x)$$

becomes a well-defined map of A_0-bimodules.

We use the evaluation map to define for A_0-modules M, N an abelian group homomorphism

$$\alpha_X\colon \mathrm{Hom}^{\text{left}}_{A_0}(M, X \otimes_{A_0} N) \to \mathrm{Hom}^{\text{left}}_{A_0}(X^* \otimes_{A_0} M, N),$$

sending $f\colon M \to X \otimes N$ to $\alpha(f) = (\mathrm{ev}_X \otimes \mathrm{id}_N) \circ (\mathrm{id}_{X^*} \otimes f)$. This map is an isomorphism of bimodules when X is finitely generated and free as a right A_0-module. Similarly, we have bimodule homomorphisms

$$\beta_k\colon X_k^* \otimes_{A_0} \cdots \otimes_{A_0} X_1^* \to (X_1 \otimes_{A_0} \cdots \otimes_{A_0} X_k)^*,$$

defined by

$$(f_k \otimes \cdots \otimes f_1) \mapsto \big(x_1 \otimes \cdots \otimes x_k \mapsto f_k(f_{k-1}(\cdots f_2(f_1(x_1)x_2)\cdots x_{k-1})x_k)\big)$$

which become isomorphisms when the X_i are finitely generated and free as right A_0-modules. We note that the β_ks are compatible with associativity in the evident way, e.g., $\beta_k \circ (\beta_{i_1} \otimes \cdots \otimes \beta_{i_k}) = \beta_{\sum i_j}$, and the β_ks are compatible with α, in the sense that the map

$$\mathrm{Hom}^{\mathrm{left}}_{A_0}(M, X_1 \otimes \cdots \otimes X_k \otimes N) \to \mathrm{Hom}^{\mathrm{left}}_{A_0}(X_k^* \otimes \cdots \otimes X_1^* \otimes M, N)$$

obtained by k applications of α_{X_i} coincides with $\mathrm{Hom}(\beta_k \otimes \mathrm{id}, \mathrm{id}) \circ \alpha_{X_1 \otimes \cdots \otimes X_k}$.

Finally, we note that there is a "left-dual" $X \mapsto X^\star = \mathrm{Hom}^{\mathrm{left}}_{A_0}(X, A_0)$ which satisfies analogous properties, and which behaves nicely on X which are finitely and free as left A_0-modules. There are evident maps $X \to (X^*)^\star$ and $Y \to (Y^\star)^*$, which are isomorphisms if X (resp. Y) are finitely generated and free as right (resp. left) A_0-modules.

7.6. **Proofs.** Recall (§4.3) that Γ-modules are in fact modules over the graded ring Γ, which is Koszul by [Rez11]. Thus, for any Γ-module M we obtain a Koszul complex [Rez11, Prop. 4.8], i.e., an augmented complex of Γ-modules $K_\bullet(M) \to M$ which in degree q is given by

$$K_q(M) = \Gamma \otimes_{A_0} C[q] \otimes_{A_0} M.$$

The A_0-bimodules $C[q] = H_q\overline{\mathcal{B}}(A_0, \Gamma, A_0) \approx \mathrm{Tor}^\Gamma_q(A_0, A_0)$, the homology of the reduced normalized bar construction of Γ. Explicitly, $C[q]$ is the kernel of

$$((-1)^i \,\mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes q-i-2})\colon \Gamma[1]^{\otimes q} \to \bigoplus_{i=1}^{q-2} \Gamma[1]^{\otimes i} \otimes \Gamma[2] \otimes \Gamma[1]^{\otimes q-i-2},$$

where $\mu\colon \Gamma[1] \otimes \Gamma[1] \to \Gamma[2]$ is multiplication.

The bimodules $C[q]$ are finitely generated and free as left A_0-modules, by [Rez11, Prop. 4.6], and the fact that the ranks of the $\Gamma[k]$ as free left A_0-modules are known from [Str98], so that we have the identity of Poincaré series

$$\sum_m \mathrm{rank}\,\Gamma[m] \cdot T^m = \left(\prod_{j=0}^{h-1} (1 - p^{j-1}T)\right)^{-1},$$

and hence

$$\sum_m \mathrm{rank}\, C[m] \cdot T^m = \left(\sum_m \mathrm{rank}\,\Gamma[m] \cdot (-T)^m\right)^{-1} = \prod_{j=0}^{h-1} (1 + p^{j-1}T).$$

From this we see that $C[q] \approx 0$ for $q > h$.

The boundary map of $K_\bullet(M)$ is obtained as the d_1 of the spectral sequence associated to a filtration of the bar complex $\mathcal{B}(\Gamma, \Gamma, M)$ as described in [Rez11, §4.7]. An explicit formula for the boundary map can be read off from this, and it is given as follows. There are evident "inclusion" maps

$$\ell\colon C[q] \to \Gamma[1] \otimes C[q-1], \qquad r\colon C[q] \to C[q-1] \otimes \Gamma[1],$$

coming from the inclusion $C[q] \subseteq \Gamma[1]^{\otimes q}$. The boundary operator

$$d_q \colon \Gamma \otimes_{A_0} C[q+1] \otimes_{A_0} M \to \Gamma \otimes_{A_0} C[q] \otimes_{A_0} M$$

is then given by

$$(\mathrm{mult} \otimes \mathrm{id}_{C[q]} \otimes \mathrm{id}_M) \circ (\mathrm{id}_\Gamma \otimes \ell \otimes \mathrm{id}_M) - (-1)^q (\mathrm{id}_\Gamma \otimes \mathrm{id}_{C[q]} \otimes \mathrm{act}) \circ (\mathrm{id}_\Gamma \otimes r \otimes \mathrm{id}_M),$$

where $\mathrm{mult}\colon \Gamma \otimes \Gamma[1] \to \Gamma$ and $\mathrm{act}\colon \Gamma[1] \otimes M \to M$ are the evident maps. It follows by [Rez11, Prop. 4.8] that if M is a flat A_0-module, then $K_\bullet(M) \to M$ is a quasi-isomorphism. In particular, if M is A_0-projective, then $K_\bullet(M)$ is a projective Γ-module resolution of M.

Proof of (7.4). The isomorphism of the proposition amounts to the statement that there is an isomorphism of complexes

$$\mathrm{Hom}_\Gamma(K_\bullet(M), N) \approx \mathcal{C}^\bullet(M, N).$$

The verification of this is entirely routine, using the identification $\Gamma[q] \approx A_q^*$ and the fact that the product maps $\Gamma[p] \otimes \Gamma[q] \to \Gamma[p+q]$ are dual to the coproduct maps $A_{p+q} \to A_q \otimes A_p$. From this we obtain an evident isomorphism $C[q] \approx D_q^*$, and thus

$$\mathrm{Hom}_\Gamma(K_q(M), N) = \mathrm{Hom}_{A_0}(C[q] \otimes M, N) \approx \mathrm{Hom}_{A_0}(M, D_q \otimes N) \approx \mathcal{C}^q(M, N).$$

The identification of the coboundary maps is straightforward. □

Proof of (7.2). In [Rez11] it is proved that Γ is Koszul, and in particular that it is quadratic [Rez11, Prop. 4.10]. That is, Γ is generated as a ring over $\Gamma[0] = A_0$ by $\Gamma[1] = \mathrm{Hom}_{A_0}({}^sA_1, A_0)$, with all relations generated by the A_0-sub-bimodule $Q = \mathrm{Ker}(\Gamma[1] \otimes \Gamma[1] \xrightarrow{\mathrm{mult}} \Gamma[2])$ of $\Gamma[1] \otimes_{A_0} \Gamma[1]$.

The statement of the proposition is a direct translation of these facts, together with the observation that the arguments of [Rez11] show that Q is a summand of $\Gamma[1] \otimes \Gamma[1]$ as an A_0-module. Taking duals, this implies that as a map of A_0-modules, $\mu\colon A_2 \to A_1{}^s\!\otimes_{A_0}{}^t\!A_1$ is split injective; therefore, $\mu \otimes \mathrm{id}_M$ is injective for any A_0-module M. This shows that the dotted arrow in the proposition is unique if it exists. □

8. The height 1 case

Suppose G_0/k is a formal group of height $h = 1$. We describe the nature of the theory in this case.

8.1. **The affine graded-category scheme $\{A_r\}$ for height 1.** We have the following.

- $A_0 = \mathbb{W}k$.
- For each $r \geq 0$, the map $s\colon A_0 \to A_k$ is an isomorphism. That is, any deformation of G_0 has a *unique* subgroup of rank p^r, corresponding to the kernel of $p^r\colon G_0 \to G_0$.
- Identify A_r with $A_0 = \mathbb{W}k$ using the isomorphism s. Then $t\colon A_0 \to A_r$ is identified with the lift $\tilde{\phi}^r\colon \mathbb{W}k \to \mathbb{W}k$ of the p^rth power map on k.
- The maps $\mu\colon A_{k+\ell} \to A_k{}^s\!\otimes_{A_0}{}^t\!A_\ell$ are uniquely determined by the above and the identities $\mu \circ s = \mathrm{id} \otimes s$ and $\mu \circ t = t \otimes \mathrm{id}$.
- The map $\Psi\colon A_0 \to A_0$ coincides with the map $\tilde{\phi}\colon \mathbb{W}k \to \mathbb{W}k$.

8.2. **Γ-modules for height 1.** By what we have just observed, we see that a Γ-module is precisely a a $\mathbb{W}k$-module M equipped with a $\mathbb{W}k$-module map

$$P_M : M \to {}^{\Psi}M.$$

That is, P_M is Ψ-linear, so $P_M(\alpha m) = \Psi(\alpha)m$ for $\alpha \in \mathbb{W}k$ and $m \in M$.

In fact, $P_M = P_1$ coincides with the operation Ψ_M of (4.15).

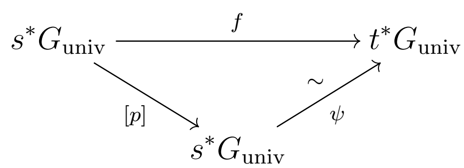
8.3. **Koszul complex for height 1.** For Γ-modules M and N, the Koszul complex $\mathcal{C}^\bullet(M, N)$ (§7.3) takes the form

$$\begin{aligned} \operatorname{Hom}_{\mathbb{W}k}(M, N) &\xrightarrow{d_0} \operatorname{Hom}_{\mathbb{W}k}(M, {}^{\Psi}N) \\ \gamma &\longmapsto P_N \circ \gamma - \gamma \circ P_M. \end{aligned}$$

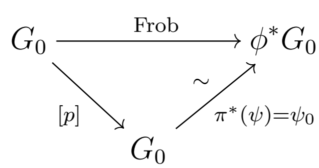
For example, if M and N are rank one A_0-modules with bases x of M and y of N, so that $P(x) = \alpha x$, $P(y) = \beta y$ for $\alpha, \beta \in \mathbb{W}k$, the complex becomes isomorphic to

$$\begin{aligned} \mathbb{W}k &\xrightarrow{d_0} \mathbb{W}k \\ f &\longmapsto \tilde{\phi}(f)\beta - f\alpha. \end{aligned}$$

8.4. **Invariant 1-forms for height 1.** Now let $f : s^*G_{\text{univ}} \to t^*G_{\text{univ}}$ be the universal example of a deformation of Frob, which is defined over A_1. There exists a commutative diagram of homomorphisms of formal groups over A_1 of the form



in which ψ is an isomorphism of formal groups. Modulo the maximal ideal in A_1, this becomes the commutative diagram



Pick a generator $u \in \omega$ of the invariant 1-forms on G_{univ}, and consider the pullbacks $s^*u \in \omega_{s^*G_{\text{univ}}}$ and $t^*u \in \omega_{t^*G_{\text{univ}}}$. If we write

$$f^*(t^*u) = b(s^*u) \quad \text{for some } b \in A_1 = \mathbb{W}k, \qquad \psi^*(t^*u) = \lambda(s^*u) \quad \text{for some } \lambda \in A_1^\times = (\mathbb{W}k)^\times,$$

then the identity $f = \psi \circ [p]$ implies that $b = p\lambda$.

Thus the Γ-module ω of invariant 1-forms is isomorphic to the free $\mathbb{W}_p k$-module on one generator u with $P(u) = bu = (p\lambda)u$, where $\lambda \in (\mathbb{W}_p k)^\times$.

In the special case that $k = \mathbb{F}_p$, then $s = t = \mathrm{id}_{\mathbb{Z}_p}$, and hence $\psi = [\lambda] : G_{\text{univ}} \to G_{\text{univ}}$, and thus $\psi_0 = [\lambda] : G_0 \to G_0$. Here are some examples.

- If G_{univ} is multiplicative group over $\mathbb{Z}_p$, then $[p]$ is a deformation of Frobenius, $\lambda = 1$, and $P(u) = pu$.
- If p is odd and G_{univ} is the formal group over $\mathbb{Z}_p$ given by the group law $x[+]y = (x+y)/(1-xy)$, then $[(-1)^{(p-1)/2}p]$ is a deformation of Frobenius, $\lambda = (-1)^{(p-1)/2}$, and $P(u) = (-1)^{(p-1)/2}pu$.

8.5. **On the element λ.** The definition of λ depends on both the formal group G_0/k and the choice of generator $u \in \omega$. Replacing u with αu for $\alpha \in A_0^\times$ changes λ to $\lambda\tilde{\phi}(\alpha)/\alpha$, and thus we get a well-defined element $\langle\lambda\rangle$ of

$$H(k) = \mathrm{Cok}\left[(\mathbb{W}k)^\times \xrightarrow{\alpha\mapsto\tilde{\phi}(\alpha)/\alpha} (\mathbb{W}k)^\times\right].$$

The element $\langle\lambda\rangle \in H(k)$ is an invariant of the isomorphism class of G_0 over k.

Over $k = \overline{\mathbb{F}}_p$, all height 1 formal groups are isomorphic to $\widehat{\mathbb{G}}_m$, and so in this case we can choose a basis $u \in \omega$ such that $\lambda = 1$.

Over $k = \mathbb{F}_{p^r}$, Hilbert's Satz 90 applied to the unramified extension $\mathbb{W}k \otimes \mathbb{Q}/\mathbb{Q}_p$ gives an isomorphism $H(k) \approx \mathbb{Z}_p^\times$, defined by $[\lambda] \mapsto \mathrm{Nm}(\lambda) = \prod_{k=0}^{r-1} \tilde{\phi}^k(\lambda)$. Explicitly, $\beta = \mathrm{Nm}(\lambda) \in \mathbb{Z}_p^\times$ is the unique map fitting in the commutative diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\mathrm{Frob}^r} & (\phi^r)^* G_0 = G_0 \\ & {\scriptstyle [p^r]}\searrow \quad \nearrow{\scriptstyle [\beta]}\ \sim & \\ & G_0 & \end{array}$$

The assignment $G_0/k \longmapsto \beta \in \mathbb{Z}_p^\times$ is a complete isomorphism invariant of height 1 formal groups over $\mathbb{F}_{p^r}$. See [Frö68, §III.3 Thm. 2] and the surrounding discussion. The formal multiplicative group $\widehat{\mathbb{G}}_m/\mathbb{F}_{p^r}$ has trivial invariant.

8.6. **Sample calculation.** Suppose that G_0/k is a height 1-formal group over a subfield k of $\overline{\mathbb{F}}_p$. Let $\det = (\mathbb{W}k)v$ denote the Γ-module defined by $P(v) = pv$.

8.7. **Proposition.** *We have*

$$\mathrm{Ext}^s_\Gamma(\omega^{-1} \otimes \det, \omega^m) \approx 0 \quad \text{if } s \neq 0, 1, \text{ or if } m \neq 0.$$

$$\mathrm{Hom}_\Gamma(\omega^{-1} \otimes \det, \mathbb{1}) \approx \begin{cases} \mathbb{Z}_p & \text{if } G_0 \approx \widehat{\mathbb{G}}_m \text{ over } k, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Ext}^1_\Gamma(\omega^{-1} \otimes \det, \mathbb{1}) \approx \begin{cases} \mathbb{Z}_p & \text{if } k \text{ finite and } G_0 \approx \widehat{\mathbb{G}}_m \text{ over } k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using §8.3, the complex for a given m is

$$\mathbb{W}k \xrightarrow{d_0} \mathbb{W}k$$
$$f \longmapsto \lambda^m p^m \tilde{\phi}(f) - \lambda^{-1} f.$$

For $m \geq 1$, the boundary map d_0 is an isomorphism, since modulo p it has the form $f \mapsto -\lambda^{-1} f$, and λ is a unit. In the case $m = 0$, the boundary map d_0 has non-trivial kernel if and only if $\lambda = \alpha/\tilde{\phi}(\alpha)$ for some $\alpha \in (\mathbb{W}k)^\times$, i.e., if $[\lambda] = 0$ in $H(k)$, which as noted above §8.5 happens if and only if $G_0 \approx \widehat{\mathbb{G}}_m$ over k. □

The above calculation is input for a spectral sequence computing the space of commutative S-algebra maps $\Sigma^\infty_+ \mathbb{Z} \to E$; the π_2 of this space is equal to the homotopy classes of commutative S-algebra maps $\Sigma^\infty_+ K(\mathbb{Z}, 2) \to E$. The dependence on G_0 being isomorphic to the multiplicative group is a reflection of Snaith's theorem (that the spectrum of complex K-theory is obtained by inverting the Bott class in $\Sigma^\infty_+ K(\mathbb{Z}, 2)$).

9. Supersingular elliptic curves and the height 2 case

Let C_0 be a supersingular elliptic curve over a perfect field k. Thus, the formal completion $\widehat{C}_0$ of C_0 at its identity element is a formal group of height 2 over k.

According to the theorem of Serre-Tate, the deformation theory of a supersingular elliptic curve is precisely the same as the deformation theory of its formal group. Thus, we may define a category $\mathrm{Def}_{C_0/k}(R)$ of deformations of C_0 to a complete local ring R, exactly as we did for the formal group G_0. The functor $\mathrm{Def}_{C_0/k}(R) \to \mathrm{Def}_{\widehat{C}_0/k}(R)$ to the category of deformations of the formal group $\widehat{C}_0$, defined by $(C, i, \alpha) \mapsto (\widehat{C}, i, \widehat{\alpha})$, is an equivalence of categories; this is the content of the Serre-Tate theorem applied to a supersingular elliptic curve.

We will now assume that our formal group $G_0 = \widehat{C}_0$ is the formal group of a supersingular curve. Thus, any deformation (G, i, α) of G_0 is the formal completion (uniquely up to canonical isomorphism) of a deformation of C_0, and any morphism $\overline{f}\colon G \to G'$ in $\mathrm{Def}^r(R)$ extends (uniquely up to canonicaly isomorphism) to an isogeny $f\colon C \to C'$ between elliptic curves of degree p^r, which itself a deformation of $\mathrm{Frob}^r\colon C_0 \to (\phi^r)^* C_0$.

9.1. **Dual isogenies.** For any isogeny of elliptic curves $f\colon C \to C'$ of rank p^r, there is an associated **dual isogeny** $\widehat{f}\colon C' \to C$, with the property that $\widehat{f} f = f \widehat{f} = [p^r]$. Observe that the dual of $\widehat{f}$ is f again.

If $f\colon (C, i, \alpha) \to (C', i', \alpha')$ is a deformation of Frob^1, then the identity $[p] = \widehat{f} f$ gives a commutative diagram in the deformation category of the form

$$\begin{array}{ccc} & (C, i, \alpha') & \\ {}^{f}\nearrow & & \searrow^{\widehat{f}} \\ (C, i, \alpha) & \xrightarrow{\quad [p] \quad} & (C, i \circ \phi^2, \alpha \circ \psi) \end{array}$$

Thus we obtain a ring homomorphism $w\colon A_1 \to A_1$ representing the operation

$$\big(f\colon (G, i, \alpha) \to (C, i', \alpha')\big) \mapsto \big(\widehat{f}\colon (C', i', \alpha') \to (C, i \circ \phi^2, \alpha \circ \psi)\big),$$

and which fits into a commutative diagram

$$\begin{array}{ccccc} A_0 & \xrightarrow{s} & A_1 & \xleftarrow{t} & A_0 \\ & {}_{t}\searrow & \downarrow{\scriptstyle w} & \swarrow_{s\Psi} & \\ & & A_1 & & \end{array}$$

Observe that $w^2\colon A_1 \to A_1$ is not generally the identity map, but rather we have that $w^2 s = s\Psi$ and $w^2 t = t\Psi$. In particular, w interacts in a complicated way with the A_0-module structures on A_1, which can be represented by the notation $w\colon {}^{t}A_1{}^{s} \to {}^{s\Psi}A_1{}^{t}$.

We will use the following notation in the remainder of the paper. If $f\colon M \to {}^{t}A_1{}^{s} \otimes_{A_0} N$ is an A_0-module homomorphism, we will write $w \times f\colon {}^{t}A_1{}^{s} \otimes_{A_0} M \to {}^{s\Psi}A_1{}^{s} \otimes_{A_0} N$ for the composite

$${}^{t}A_1{}^{s} \otimes_{A_0} M \xrightarrow{w \otimes f} {}^{s\Psi}A_1{}^{t} \otimes_{A_0} {}^{t}A_1{}^{s} \otimes_{A_0} N \xrightarrow{\text{multiply}} {}^{s\Psi}A_1{}^{s} \otimes_{A_0} N.$$

The resulting map is a map of left A_0-modules, using the indicated module structures. We note that $w \times \mathrm{id}\colon {}^{t}A_1{}^{s} \otimes_{A_0} {}^{t}A_1{}^{s} \to {}^{s\Psi}A_1{}^{s}$ is a ring homomorphism.

The identity $\widehat{f}f = [p]$ gives rise to a commutative square of ring homomorphisms

$$\begin{array}{ccc} A_2 & \xrightarrow{\mu} & A_1{}^s\otimes_{A_0}{}^tA_1 \\ {\scriptstyle [p]}\downarrow & & \downarrow{\scriptstyle w\times \mathrm{id}} \\ A_0 & \xrightarrow[s]{} & A_1 \end{array} \tag{9.2}$$

9.3. **Proposition.** *The diagram (9.2) is a pullback square of rings. Furthermore, for any A_0-module M, the induced diagram*

$$\begin{array}{ccc} {}^tA_2{}^s\otimes_{A_0} M & \xrightarrow{\mu\otimes\mathrm{id}} & {}^tA_1{}^s\otimes_{A_0}{}^tA_1{}^s\otimes_{A_0} M \\ {\scriptstyle [p]\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle (w\times\mathrm{id})\otimes\mathrm{id}} \\ {}^\Psi A_0\otimes_{A_0} M & \xrightarrow[s\otimes\mathrm{id}]{} & {}^{s\Psi}A_1{}^s\otimes_{A_0} M \end{array}$$

is a pullback square of modules. The induced map

$$D_2\otimes_{A_0} M \to (A_1/s(A_0))\otimes_{A_0} M$$

is an isomorphism, where $D_2 = \mathrm{Cok}[\mu\colon A_2 \to A_1{}^s\otimes_{A_0}{}^tA_1]$.

Proof. This is essentially the proof of part (4) of [Rez12, Thm. 1.6] given in §1.7 of that paper. ☐

9.4. **Proposition.** *Let M be an A_0-module, and let $P\colon M \to A_1{}^s\otimes_{A_0} M$ be a map of A_0-modules. There exists a dotted arrow in*

$$\begin{array}{ccc} M & \xrightarrow{P} & {}^tA_1{}^s\otimes_{A_0} M \\ \vdots & & \downarrow{\scriptstyle \mathrm{id}_{A_1}\otimes P} \\ \vdots & & {}^tA_1{}^s\otimes_{A_0}{}^tA_1{}^s\otimes_{A_0} M \\ \downarrow & & \downarrow{\scriptstyle w\times\mathrm{id}_{A_1\otimes M}} \\ {}^\Psi A_0\otimes_{A_0} M & \xrightarrow{s\otimes\mathrm{id}} & {}^{s\Psi}A_1{}^s\otimes_{A_0} M \end{array}$$

making the diagram commute if and only if there exists a Γ-module structure $\{P_r\}_{r\geq 0}$ on M such that $P_1 = P$. This Γ-module structure is unique if it exists. If it does exist, then the dotted arrow is precisely the operator $\Psi_M\colon M \to M$.

Proof. Immediate from (9.3) and (7.2). ☐

We can write the identity of the proposition in the form $s\otimes\Psi_M = (w\times P_M)\circ P_M$.

Thus, we arrive at the following. If $G_0 = \widehat{C}_0$ is the completion of a supersingular curve, then a Γ-module amounts to a pair (M, P), where M is an A_0-module, $P\colon M \to {}^tA_1{}^s\otimes_{A_0} M$ is an A_0-module map, and $(w\times P)\circ P\colon M \to A_1{}^s\otimes_{A_0} M$ lands in the image of $s\otimes\mathrm{id}\colon M = A_0\otimes_{A_0} M \to A_1{}^s\otimes_{A_0} M$.

9.5. **The Koszul complex.** In our setting, where $G_0 = \widehat{C}_0$, the Koszul complex has the form

$$\begin{aligned}\mathcal{C}^0(M,N) &= \operatorname{Hom}_{A_0}(M,N),\\ \mathcal{C}^1(M,N) &= \operatorname{Hom}_{A_0}(M, {}^tA_1{}^s\otimes_{A_0} N),\\ \mathcal{C}^2(M,N) &= \operatorname{Hom}_{A_0}(M, {}^{s\Psi}(A_1/A_0)^s\otimes_{A_0} N).\end{aligned}$$

The boundary maps are given by:

$$\begin{aligned}\phi \in \mathcal{C}^0(M,N), &\qquad d_0\phi\colon m \mapsto P_N(\phi(m)) - (\mathrm{id}\otimes\phi)(P_M(m)),\\ \psi \in \mathcal{C}^1(M,N), &\qquad d_1\psi\colon m \mapsto (w\times P_N)(\psi(m)) + (w\times\psi)(P_M(m)).\end{aligned}$$

(The formula given for d_1 produces an element in $A_1{}^s\otimes_{A_0} N$; the value of $(d_1\phi)(m)$ is the projection to $(A_1/s(A_0))^s\otimes_{A_0} N$.) This in fact defines a cochain complex; for $\phi\in\mathcal{C}^0(M,N)$ we have

$$\begin{aligned}d_1(d_0\phi)(m) &= (w\times P_N)((d_0\phi)(m)) + (w\times d_0\phi)(P_M(m))\\ &= (w\times P_N)(P_N(\phi(m))) - (w\times P_N)(\mathrm{id}\otimes\phi)(P_M(m))\\ &\qquad + (w\times P_N)(\mathrm{id}\otimes\phi)(P_M(m)) - (\mathrm{id}\otimes\phi)(w\times P_M)(P_M(m))\\ &= (s\otimes\Psi_N)(\phi(m)) - (\mathrm{id}\otimes\phi)(s\otimes\Psi_M)(m) \in (s\otimes\mathrm{id})(N).\end{aligned}$$

The last line uses the identity $s\otimes\Psi_M = (w\times P)\circ P$ of (9.4).

We note that this complex can also be arranged as a semi-cosimplicial object. That is, the cohomology of $\mathcal{C}^\bullet(M,N)$ is naturally isomorphic to the cohomology of the semi-cosimplicial abelian group

$$\operatorname{Hom}_{A_0}(M,N) \;\substack{\longrightarrow\\ \longrightarrow}\; \begin{array}{c}\operatorname{Hom}_{A_0}(M, {}^tA_1{}^s\otimes_{A_0} N)\\ \times\\ \operatorname{Hom}_{A_0}(M, {}^{\Psi}N)\end{array} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \operatorname{Hom}_{A_0}(M, {}^{s\Psi}A_1{}^s\otimes_{A_0} N)$$

with coface maps given by

$$\phi \;\begin{array}{c}\longmapsto (P_N\circ\phi,\ \Psi_N\circ\phi)\\ \longmapsto ((\mathrm{id}\otimes\phi)\circ P_M,\ \phi\circ\Psi_M)\end{array} \qquad (\psi,\phi)\;\begin{array}{c}\longmapsto (w\times P_N)\circ\psi\\ \longmapsto (s\otimes\mathrm{id})\circ\phi\\ \longmapsto (w\times\psi)\circ P_M\end{array}$$

This cosimplicial object is reminiscent of one considered in [MR09] and [Beh06], which are built using separable isogenies of elliptic curves, and which relate to stable homotopy rather than to commutative ring spectra.

9.6. **Γ-modules of rank 1.** Let $\beta\in A_1$ such that $w(\beta)\beta\in s(A_0)$. Then we can define a Γ-module L_β as follows. The underlying A_0-module of L_β is a free A_0-module on one generator x. The structure map $P\colon L_\beta\to A_1{}^s\otimes_{A_0} L_\beta$ is defined so that $P(x)=\beta\otimes x$; thus, $P(cx) = t(c)\beta\otimes x$ for $c\in A_0$. We verify that

$$(w\times P)P(cx) = w(t(c)\beta)\beta\otimes x = s\Psi(c)\, w(\beta)\beta\otimes x \in (s\otimes\mathrm{id})(A_0\otimes_{A_0} M),$$

and thus this P defines a valid Γ-module homomorphism. In particular, note that $\Psi_{L_\beta}(x) = w(\beta)\beta\, x\in A_0\otimes_{A_0} L_\beta$.

Let $\beta_1, \beta_2 \in A_1$ such that $w(\beta_1)\beta_1, w(\beta_2)\beta_2 \in s(A_0)$, and suppose that $\alpha \in A_0$ is such that

$$s(\alpha)\beta_1 = t(\alpha)\beta_2 \in A_1.$$

Then we can define a Γ-module homomorphism $f\colon L_{\beta_1} \to L_{\beta_2}$ by setting $f(x_1) = \alpha x_2$, so that $f(cx_1) = c\alpha x_2$ for $c \in A_0$. We verify that

$$P(f(x_1)) = P(\alpha x_2) = t(\alpha)\beta_2 \otimes x_2,$$
$$(\mathrm{id} \otimes f)(P(x_1)) = (\mathrm{id} \otimes f)(\beta_1 \otimes x_1) = \beta_1 \otimes \alpha x_2 = s(\alpha)\beta_1 \otimes x_2,$$

and thus f defines a valid Γ-module homomorphism.

We have thus done most of the work to prove the following.

9.7. **Proposition.**

(1) *The construction $\beta \mapsto L_\beta$ gives a bijective correspondence*

$$\frac{\{\, \beta \in A_1 \mid w(\beta)\beta \in s(A_0) \,\}}{\beta \sim t(\gamma)s(\gamma)^{-1}\beta \text{ for } \gamma \in A_0^\times} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{rank one } \Gamma\text{-modules} \end{array} \right\}.$$

(2) *We have*

$$\mathrm{Hom}_\Gamma(L_{\beta_1}, L_{\beta_2}) \approx \{\, \alpha \in A_0 \mid t(\alpha)\beta_2 = s(\alpha)\beta_1 \,\}.$$

(3) *We have $L_{\beta_1} \otimes L_{\beta_2} \approx L_{\beta_1\beta_2}$. The module L_β is $\otimes$-invertible as a Γ-module if and only if $\beta \in A_1^\times$.*

In particular, every $\beta \in \mathbb{Z}_p$ gives rise to a rank one Γ-module L_β. We note these examples.

- $\mathbb{1} = L_1$, the unit object in the symmetric monoidal category of Γ-modules.
- $\det = L_p$, the **determinant** module.
- $\mathrm{nul} = L_0$, the **null** module. Note that $\mathrm{Hom}_\Gamma(\mathrm{nul}, \mathrm{nul}) \approx A_0$, and thus $\mathrm{Hom}_\Gamma(M, N \otimes \mathrm{nul})$ has the structure of an A_0-module.

Note that if L_β is such that $\beta\gamma = p$ for some (necessarily unique) $\gamma \in A_1$, then we have an isomorphism $\det = L_\beta \otimes L_\gamma$. We may thus sometimes choose to write $L_\beta = L_\gamma^{-1} \otimes \det$, even when the module L_γ is not $\otimes$-invertible as a Γ-module

The invariant 1-form module ω is an example of a rank one Γ-module which is not in general described by an element of $\mathbb{Z}_p$, as we will see. If $u \in \omega$ is a basis, and we write $P(u) = b \otimes u$ with $b \in A_1$, then $\omega \approx L_b$. Because $[p] = \psi \circ \mathrm{Frob}^2$ on C_0 for some isomorphism $\psi\colon C_0 \to (\phi^2)^*C_0$, we have that $\Psi(u) = w(b)b \otimes u = \lambda p \otimes u$ for some $\lambda \in A_0^\times$. As a consequence, there exists a module $\omega^{-1} \otimes \det \approx L_{\lambda^{-1}w(b)}$.

9.8. **Standard supersingular curves.** We say that a supersingular elliptic curve C_0/k is **standard** if $k \subseteq \mathbb{F}_{p^2}$ and $\mathrm{Frob}^2 = [-p]$. Honda-Tate theory provides a standard supersingular elliptic curve over $k = \mathbb{F}_p$ for every prime p. In fact, we have the following.

9.9. **Proposition.** *Every supersingular elliptic curve over a finite field is isomorphic (over $\overline{\mathbb{F}}_p$) to a standard supersingular curve.*

Proof. [BGJGP05, Lemma 3.21]. □

For a standard curve C_0/k, we have that $\Psi\colon A_{0,C_0/k} \to A_{0,C_0/k}$ and $w^2\colon A_{1,C_0/k} \to A_{1,C_0/k}$ are identity maps.

9.10. *Remark.* It is important here that $k \subseteq \mathbb{F}_{p^2}$. If we extend to some larger field $k' \supset \mathbb{F}_{p^2}$, then neither Ψ nor w^2 are identity maps. In fact, on scalars $c \in \mathbb{W}_p k' \subseteq A_0$ we have $\Psi(c) = \tilde{\phi}^2(c)$, where $\tilde{\phi}\colon \mathbb{W}_p k' \to \mathbb{W}_p k'$ is the lift of the pth power map on k'.

We will now show, using the results of [Rez12], how to describe explicitly category of mod-p Γ-modules for a standard supersingular curve, and nearly explicitly describe the category of Γ-modules itself. (Similar results hold for curves which satisfy $\mathrm{Frob}^2 = [p]$.)

9.11. **Notation.** We let C_0/k be a standard supersingular curve, with $k \subseteq \mathbb{F}_{p^2}$. Let $\overline{C}_0 = (C_0)_{\overline{\mathbb{F}}_p}$ be the base change of C_0 to $\overline{\mathbb{F}}_p$.

We will use the following conventions when dealing with Γ-modules for $\overline{C}_0/\overline{\mathbb{F}}_p$. First, we write $A_r \subset \overline{A}_r$ for the rings $A_{r,C_0/k} \subset A_{r,\overline{C}_0/\overline{\mathbb{F}}_p}$; recall that $\overline{A}_r \approx \mathbb{W}\overline{\mathbb{F}}_p \otimes_{\mathbb{W}k} A_r$. Likewise, we write $\omega \subset \overline{\omega}$ for the Γ-modules of invariant 1-forms on C_0 and $\overline{C}_0$.

- We identify $\overline{A}_0$ with its image under the inclusion $s\colon \overline{A}_0 \to \overline{A}_1$, and similarly identify A_0 with the image of $s\colon A_0 \to A_1$.
- For any element $\beta \in \overline{A}_1$, we write β' for $w(\beta) \in \overline{A}_1$. This implies that for $\alpha \in \overline{A}_0$, we have $t(\alpha) = \alpha'$, and $\alpha'' = \Psi(\alpha)$.
- As a consequnce, if $\beta \in A_1$, we have $\beta'' = \beta$, and if $\alpha \in A_0$, we have $\alpha'' = \alpha = \Psi(\alpha)$.
- For $c \in \mathbb{W}_p\overline{\mathbb{F}}_p$, write $c^{(r)} = (\tilde{\phi}^r)(c)$, and for $f(x) = \sum c_I x^I \in \mathbb{W}_p k[\![x_1, \dots, x_n]\!]$, write $f^{(r)}(x) = \sum c_I^{(r)} x^I$. Note that $c' = c^{(1)}$.

9.12. **Structure of A_1.** The universal deformation of C_0 is defined over $A_0 \approx \mathbb{W}_p k[\![a]\!]$. We refer to any power series generator a of this ring as a **deformation parameter**. Thus, we *choose* a deformation parameter a, and write $a = s(a) \in A_1$ and $a' = t(a) \in A_1$.

9.13. **Proposition.** *The evident map $\mathbb{W}_p k[\![a, a']\!] \to A_1$ descends to a ring isomorphism*

$$k[\![a, a']\!]/\big((a^p - a')(a - a'^p)\big) \approx A_1/(p).$$

The map can$\colon A_1 \to A_0/(p)$ *classifying the canonical subgroup is given by $a \mapsto a$, $a' \mapsto a^p$. The maps $s, t\colon A_0/(p) \to A_1/(p)$ are given by $s(f(a)) = f(a)$ and $t(f(a)) = f^{(1)}(a')$.*

Proof. This is a special case of Proposition 3.15 of [Rez12]. □

Now *choose* a basis u of the module ω of invariant 1-forms. Then $P(u) = b \otimes u$ for some $b \in A_1$. Let $b' = w(b) \in A_1$. Since $k \subseteq \mathbb{F}_{p^2}$ and $\mathrm{Frob}^2 = [-p]$, we must have that $b'b = -p$.

9.14. **Proposition.** *The evident map $\mathbb{W}_p k[\![b, b']\!]/(bb' + p) \to A_1$ is an isomorphism of rings. Furthermore, there exists $e \in A_1^\times$ and $e' = w(e) \in A_1^\times$ such that*

$$b = e(a' - a^p) \quad \textit{and} \quad b' = e'(a - a'^p).$$

Proof. To demonstrate the isomorphism, it suffices to do so after reducing mod p, since both $W_p k[\![b, b']\!]/(bb' + p)$ and A_1 are p-complete and p-torsion free.

An isogeny $f\colon C \to C'$ of elliptic curves of rank p factors through Frobenius if and only if $f^*\colon \omega_{C'} \to \omega_C$ is the zero map. Therefore $\mathrm{Ker}(\mathrm{can}\colon A_1 \to A_0/(p)) = (b) \subseteq A_1$.

On the other hand, by (9.13) the evident map $k[\![a, a']\!]/((a^p - a')(a - a'^p)) \to A_1/(p)$ is an isomorphism of rings. The projection map $A_1 \to A_1/(p, a^p - a') \approx A_0/(p)$ exactly classifies Frobenius, and thus we must have that $b = e(a' - a^p)$ for some unit $\lambda \in A_1^\times$. Clearly this implies $b' = e'(a - a'^p)$, and that $k[\![b, b']\!]/(bb') \to A_1/(p)$ is an isomorphism. □

9.15. **Adapted parameters.** Given a basis u for ω, we say that a deformation parameter $a \in A_0$ is **adapted** to u if we have

$$a \equiv b' \mod bA_1,$$

where $b \in A_1$ is such that $P(u) = b \otimes u$, and $b' = w(b)$, so that $b'b = -p$. If a is adapted to u, then applying w to the above congruence gives

$$a' \equiv b \mod b'A_1.$$

Since $a \equiv a'^p \equiv b^p \mod b'A_1$, it also follows that

$$a \equiv b' + b^p \mod bA_1 \quad \text{and} \quad a \equiv b' + b^p \mod b'A_1.$$

The ring homomorphism $A_1/(p) \to A_1/(b) \times A_1/(b')$ is injective, and thus we have

$$a \equiv b' + b^p \mod pA_1$$

for any adapted parameter a.

9.16. **Proposition.** *For any generator $u \in \omega$, there exists a deformation parameter $\overline{a} \in A_0$ adapted to it.*

Proof. As noted above, for an arbitrary deformation parameter a we have that $b = \lambda(a' - a^p)$ for some unit $\lambda \in A_1^\times$. Thus

$$b' + b^p \equiv \lambda'(a - a'^p) \equiv \lambda'(a - a^{p^2}) \mod bA_1.$$

Because $s\colon A_0/(p) \to A_1/(b)$ is an isomorphism, we may choose $\overline{a} \in A_0$ which projects to $\lambda'(a - a^{p^2})$ modulo bA_1. Clearly, this $\overline{a}$ is a deformation parameter, and $\overline{a} \equiv b' \mod bA_1$, so it is adapted. $\square$

Now suppose that $a \in A_0$ is an adapted parameter. It will be convenient to use the evident isomorphisms $A_1/(b) \approx k[\![b']\!]$ and $A_1/(b') \approx k[\![b]\!]$, with respect to which the evident ring homomorphism $A_1 \to A_1/(b) \times A_1/(b')$ induces an isomorphism of rings

$$A_1/(p) \approx k[\![b']\!] \times_k k[\![b]\!] \subset k[\![b']\!] \times k[\![b]\!],$$

identifying $A_1/(p)$ with the set of pairs of power series $(g_1(b'), g_2(b))$ such that $g_1(0) = g_2(0)$. This isomorphism sends

$$a \mapsto (b', b^p), \qquad a' \mapsto (b'^p, b),$$
$$b \mapsto (0, b), \qquad b' \mapsto (b', 0).$$

With respect to this isomorphism we have

$$s(f(a)) = (f(b'), f(b^p)), \qquad t(f(a)) = (f^{(1)}(b'^p), f^{(1)}(b)),$$

and

$$w(g_1(b'), g_2(b)) = (g_2^{(1)}(b'), g_1^{(1)}(b)).$$

The map $A_1/(p, s(A_0)) \to k[\![b]\!]$ defined by

$$(g_1(b'), g_2(b)) \mapsto g_2(b) - g_1(b^p)$$

induces a bijection $A_1/(p, s(A_0)) \xrightarrow{\sim} b\, k[\![b]\!]$.

9.17. *Remark.* One would like to lift the congruence $a \equiv b' + b^p$ modulo bA_1 to an identity in A_1, so that $a = f(b, b')$ for some explicit polynomial f in b and b'. Armed with such an identity, one would then have an explicit description of the maps $s, t\colon A_0 \to A_1$ and $w\colon A_1 \to A_1$, and thus an explicit description of the category of Γ-modules. This has been done at the prime 2 [Rez08], and at the prime 3 [Zhu12].

9.18. **The Koszul complex for rank one modules.** Suppose that $M = L_\alpha$ and $N = L_\beta$ for some $\alpha, \beta \in A_1$ with $\alpha'\alpha, \beta'\beta \in A_0$. If we write $x \in M$ and $y \in N$ for the generators, we have

$$\begin{aligned} A_0 &\xrightarrow{\sim} \mathcal{C}^0(M,N) = \operatorname{Hom}_{A_0}(M,N), && f \mapsto (x \mapsto f\,y),\\ A_1 &\xrightarrow{\sim} \mathcal{C}^1(M,N) = \operatorname{Hom}_{A_0}(M, {}^tA_1{}^s \otimes_{A_0} N), && g \mapsto (x \mapsto g \otimes y),\\ A_1/A_0 &\xrightarrow{\sim} \mathcal{C}^2(M,N) = \operatorname{Hom}_{A_0}(M, {}^{s\Psi}(A_1/sA_0)^s \otimes_{A_0} N), && h \mapsto (x \mapsto h \otimes y). \end{aligned}$$

With respect to these identifications, the Koszul complex of §9.5 takes the form

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_1/A_0$$

with coboundary maps

$$d_0(f) = f'\beta - f\alpha, \qquad d_1(g) = g'\beta + g\,\alpha'.$$

(Verify: $d_1(d_0(f)) = (f'\beta - f\alpha)'\beta + (f'\beta - f\alpha)\alpha' = \Psi(f)\beta'\beta - f\alpha'\alpha \in A_0$.)

Suppose $g \in A_1$ is such that $g'\beta + g\alpha' \in s(A_0)$, and so corresponds to a 1-cocycle representing a class in $\operatorname{Ext}^1_\Gamma(L_\alpha, L_\beta)$. The corresponding extension $0 \to L_\beta \to E \to L_\alpha \to 0$ can be constructed as follows: set $E = A_0y \oplus A_0x$, with y the image of the standard generator of L_β, and so that x projects to the standard generator of L_α. Then the Γ-module structure on E is defined by

$$P(y) = \beta \otimes y, \qquad P(x) = g \otimes y + \alpha \otimes x.$$

The cocycle condition $g'\beta + g\alpha \in s(A_0)$ is exactly the condition that E is a Γ-module. In this case the map $\Psi\colon E \to E$ is given by

$$\Psi(y) = \beta'\beta\, y, \qquad \Psi(x) = (g'\beta + g\alpha')\, y + \alpha'\alpha\, x.$$

10. Calculation of $\operatorname{Ext}^*_\Gamma(\omega^m, \mathrm{nul})$

Fix a standard supersingular curve C_0 over $k \subseteq \mathbb{F}_{p^2}$. Recall that ω is the Γ-module of invariant differentials, and that nul is the Γ-module with "trivial" Γ action, defined in §9.6. We also use the notations introduced in §9.15.

10.1. **Theorem.** *We have that*

$$\operatorname{Ext}^s_\Gamma(\omega^m, \mathrm{nul}) = 0 \qquad \textit{for all } m \geq 0,\ s \neq 2,$$

and

$$\operatorname{Ext}^2_\Gamma(\omega^m, \mathrm{nul}) \approx A_1/(s(A_0), b'^m A_1).$$

Recall that $\operatorname{Hom}_\Gamma(\mathrm{nul}, \mathrm{nul}) \approx A_0$, so these Ext-groups are natuarlly A_0-modules.

Recall that $\omega^m \approx L_{b^m}$, and $\mathrm{nul} \approx L_0$. The Koszul complex $\mathcal{C}^\bullet(\omega^m, \mathrm{nul})$ thus takes the form

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_1/A_0$$

with

$$d_0(\alpha) = -\alpha b^m, \qquad d_1(\beta) = \beta b'^m.$$

That is, $\mathrm{Ext}^s_\omega(\omega^m, \mathrm{nul})$ is the cohomology of the complex

$$A_0 \xrightarrow{-b^m} A_1 \xrightarrow{b'^m} A_1/A_0.$$

We need two facts.

(1) Multiplication by p gives an injective map $A_1 \to A_1$.
(2) Multiplication by b^m gives an injective map $A_0/(p) \to A_1/(p)$.

Fact (1) is clear, since A_1 is a free A_0-module. Fact (2) follows using the identifications $A_0/(p) \approx k[\![a]\!]$ and $A_1/(p) \approx A_1/(p,b) \times_k A_1/(p,b') \approx k[\![b']\!] \times_k k[\![b]\!]$ described above; with respect to these, multiplication b^m is given by which we can use to identify the map with

$$f(a) \longmapsto (f(b')b^m, f(b^p)b^m) = \begin{cases} (f(b'), f(b^p)) & \text{if } m = 0, \\ (0, f(b^p)b^m) & \text{if } m \geq 1. \end{cases}$$

In either case the map is injective. (Note that $b\colon A_1/(p) \to A_1/(p)$ is *not* injective, since $b'b = -p$.)

10.2. **Case of $m = 0$.** In this case, the sequence $A_0 \xrightarrow{1} A_1 \xrightarrow{1} A_1/A_0$ is manifestly exact. Thus $\mathrm{Ext}^*_\Gamma(\mathbb{1}, \mathrm{nul}) = 0$.

10.3. **Case of $m \geq 1$.** Because p is not a zero-divisor in A_1, and $b'^m b^m = (-p)^m$, we have that $d_0\colon A_0 \to A_1$ is injective.

Now suppose $g \in \mathcal{C}^1 = A_1$ is such that $d_1(g) = 0$. That is, $b'^m g = f$ for some (necessarily unique) $f \in A_0$. Thus

$$b^m f = b^m b'^m g = (-p)^m g.$$

We claim that $f/p^m \in A_0$. It suffices to show that if $b^m f \in pA_1$, then $f \in pA_0$, in which case the claim is proved by induction on m. The statement to be proved is precisely fact (2) above. Thus we have shown that if $g \in \mathcal{C}^1$ is a cocycle, then $g = -b^m k = d_0(k)$ for some $k \in A_0$.

It is now clear that $\mathrm{Ext}^2_\Gamma(\omega^m, \mathrm{nul}) \approx A_1/(s(A_0), b'^m A_1)$.

11. Calculation of $\mathrm{Ext}^*_\Gamma(\det \otimes\, \omega^{-1}, \omega^m)$

We fix a standard supersingular curve C_0 over $k \subseteq \mathbb{F}_{p^2}$. We write $\overline{C}_0/\overline{\mathbb{F}}_p$ for its base change to the algebraic closure. The module $\omega^{-1} \otimes \det$ was defined in §9.6.

11.1. **Proposition.** *For $\overline{C}_0$ and $s, m \geq 0$, we have that*

$$\mathrm{Ext}^s_\Gamma(\omega^{-1} \otimes \det, \omega^m) \approx \begin{cases} \mathbb{Z}_p & \text{if } s = 1 = m, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathrm{Ext}^s_\Gamma(M, N) = H^s\mathcal{C}^\bullet(M, N)$ when M is projective over $\overline{A}_0$. In our case, each $\mathcal{C}^s(M, N)$ is a p-complete torsion free abelian group, and the coboundary maps are $\mathbb{Z}_p$-module maps. The proposition will follow once we show that

$$H^s(\mathcal{C}^\bullet(\omega^{-1} \otimes \det, \omega^m) \otimes \mathbb{Z}/p) \approx \begin{cases} \mathbb{Z}/p & \text{if } s = 1 = m, \\ 0 & \text{otherwise.} \end{cases}$$

In the remainder of this section, we give the proof.

Choose a basis $u \in \omega_{C_0}$ and an adapted parameter $a \in \mathbb{W}_p k[\![a]\!]$. We have $A_0 = \mathbb{W}_p k[\![a]\!]$ and $A_1 = \mathbb{W}_p k[\![b, b']\!]/(bb' + p)$, and thus $\overline{A}_0 = \mathbb{W}_p \overline{\mathbb{F}}_p[\![a]\!]$ and $\overline{A}_1 = \mathbb{W}_p \overline{\mathbb{F}}_p[\![b, b']\!]/(bb' + p)$.

Recall that $\det\otimes\omega^{-1}\approx L_{-b'}$, and $\omega^m\approx L_{b^m}$. The Koszul complex $\mathcal{C}^\bullet(\det\otimes\omega^{-1},\omega^m)$ thus takes the form

$$\overline{A}_0 \xrightarrow{d_0} \overline{A}_1 \xrightarrow{d_1} \overline{A}_1/\overline{A}_0$$

with

$$d_0(f)=f'b^m+fb', \qquad d_1(g)=g'b^m-gb.$$

Using the isomorphisms $A_1/(p)\approx\overline{\mathbb{F}}_p[\![b']\!]\times_{\overline{\mathbb{F}}_p}\overline{\mathbb{F}}_p[\![b]\!]$ and $A_1/(p,A_0)\approx b\overline{\mathbb{F}}_p[\![b]\!]$, the mod p reduced complex $\mathcal{C}^\bullet(\det\otimes\omega^{-1},\omega^m)\otimes\mathbb{Z}/p$ has the form

$$\overline{\mathbb{F}}_p[\![a]\!]\xrightarrow{d_0}\overline{\mathbb{F}}_p[\![b']\!]\times_{\overline{\mathbb{F}}_p}\overline{\mathbb{F}}_p[\![b]\!]\xrightarrow{d_1} b\overline{\mathbb{F}}_p[\![b]\!]$$

with differentials

$$d_0\colon f(a)\longmapsto (f(b')b', f^{(1)}(b)b^m)$$
$$d_1\colon (g_1(b'),g_2(b))\longmapsto g_1^{(1)}(b)b^m-g_2(b)b$$

when $m\geq 1$, and differentials

$$d_0\colon f(a)\longmapsto (f^{(1)}(b'^p)+f(b')b', f^{(1)}(b))$$
$$d_1\colon (g_1(b'),g_2(b))\longmapsto g_1^{(1)}(b)-g_2(b)b-g_2^{(1)}(b^p)$$

when $m=0$.

11.2. **Mod p calculation, $m=0$.** It is clear in this case that d_0 is injective, so $H^0(\mathcal{C}\otimes\mathbb{Z}/p)=0$.

Suppose $(g_1(b'),g_2(b))\in\overline{\mathbb{F}}_p[\![b']\!]\times_{\overline{\mathbb{F}}_p}\overline{\mathbb{F}}_p[\![b]\!]$ is a cocycle; i.e., $g_1(0)=g_2(0)$ in $\overline{\mathbb{F}}_p$ and $g_1^{(1)}(b)=g_2(b)b+g_2^{(1)}(b^p)$ in $b\overline{\mathbb{F}}_p[\![b]\!]=\mathcal{C}^2\otimes\mathbb{Z}/p$. We compute that

$$d_0\colon g_2^{(-1)}(a)\longmapsto (g_2(b'^p)+g_2^{(-1)}(b')b', g_2(b))=(g_1(b'),g_2(b)),$$

whence $H^1(\mathcal{C}\otimes\mathbb{Z}/p)=0$.

Let $h(b)\in b\overline{\mathbb{F}}_p[\![b]\!]$. We compute that

$$d_1\colon (h^{(-1)}(b'),0)\longmapsto h(b),$$

whence $H^2(\mathcal{C}\otimes\mathbb{Z}/p)=0$.

11.3. **Mod p calculation, $m=1$.** It is clear in this case that d_0 is injective, so $H^0(\mathcal{C}\otimes\mathbb{Z}/p)=0$.

Suppose $(g_1(b'),g_2(b))\in\overline{\mathbb{F}}_p[\![b']\!]\times_{\overline{\mathbb{F}}_p}\overline{\mathbb{F}}_p[\![b]\!]=\mathcal{C}^1\otimes\mathbb{Z}/p$ is a cocycle. That is, $g_1(0)=g_2(0)$ and $g_1^{(1)}(b)b=g_2(b)b$ in $b\overline{\mathbb{F}}_p[\![b]\!]=\mathcal{C}^2\otimes\mathbb{Z}/p$, which implies that $g_2(b)=g_1^{(1)}(b)$. If $g_1(0)=g_2(0)=0$, then we compute

$$d_0\colon g_1(a)/a\longmapsto (g_1(b'),g_1^{(1)}(b)).$$

Thus, every 1-cocycle in $\mathcal{C}^1\otimes\mathbb{Z}/p$ is cohomologous to one of the form (λ,λ) with $\lambda\in\overline{\mathbb{F}}_p$, and such an element is a cocycle if and only if $\lambda=\lambda^p$, i.e., if $\lambda\in\mathbb{F}_p$.

Thus $H^1\mathcal{C}\otimes\mathbb{Z}/p\approx\mathbb{Z}/p$. Note that this argument has constructed an explicit homomorphism

$$\rho\colon\ \operatorname{Ext}^1_\Gamma(\omega^{-1}\otimes\det,\omega)\to\mathbb{F}_p,$$

computed on cocycles by $\rho((g(b'),g^{(1)}(b))=g(0)$; see §11.5 for a geometric interpretation of ρ.

Let $h(b) \in b\overline{\mathbb{F}}_p[\![b]\!] \approx \mathcal{C}^2 \otimes \mathbb{Z}/p$. If $h(b) \in b^2\overline{\mathbb{F}}_p$, then we compute that

$$d_1 \colon (0, -h(b)/b) \longmapsto h(b).$$

Thus, every 2-cochain in $\mathcal{C}^2 \otimes \mathbb{Z}/p$ is cohomologous to one of the form μb with $\mu \in \overline{\mathbb{F}}_p$. Since for $\lambda \in \overline{\mathbb{F}}_p$ we have

$$d_1 \colon (\lambda, \lambda) \longmapsto (\lambda^p - \lambda)b.$$

We can choose $\lambda \in \overline{\mathbb{F}}_p$ so that $\lambda^p - \lambda = \mu$, and hence $H^2(\mathcal{C} \otimes \mathbb{Z}/2) = 0$.

11.4. **Mod p calculation, $m \geq 2$.** This is similar to (but easier than) the $m = 1$ case. Clearly d^0 is injective, so $H^0(\mathcal{C} \otimes \mathbb{Z}/p) = 0$.

If $(g_1(b'), g_2(b)) \in \mathcal{C}^1 \otimes \mathbb{Z}/p$ is a cocycle, then we have $g_2(b) = g_1^{(1)}(b)b^{m-1}$. This implies that $g_1(0) = g_2(0) = 0$. Thus

$$d_0 \colon g_1(a)/a \longmapsto (g_1^{(1)}(b'), g_1^{(1)}(b)b^{m-1}),$$

showing that $H^1(\mathcal{C} \otimes \mathbb{Z}/p) = 0$.

If $h(b) \in b\overline{\mathbb{F}}_p[\![b]\!] \approx \mathcal{C}^2 \otimes \mathbb{Z}/p$ is such that $h(b) \in b^2\overline{\mathbb{F}}_p[\![b]\!]$, then

$$d_1 \colon (0, -h(b)/b) \longmapsto h(b).$$

In addition, for $\mu \in \overline{\mathbb{F}}_p$ we have

$$d_1 \colon (\mu, -\mu + \mu^p b^{m-1}) \longmapsto \mu^p b^m - (-\mu + \mu^p b^{m-1})b = \mu b.$$

Thus $H^2(\mathcal{C} \otimes \mathbb{Z}/p) = 0$.

11.5. **More on** $\mathrm{Ext}^1_\Gamma(\omega^{-1} \otimes \det, \omega)$**.** According to our identification of the Koszul complex, a 1-cocycle in $\mathcal{C}^1 = \mathcal{C}^1(\omega^{-1} \otimes \det, \omega)$ is a homomorphism $g \colon \det \otimes\, \omega^{-1} \to {}^t\overline{A}_1{}^s \otimes_{\overline{A}_0} \omega$, given by

$$g(u^{-1}v) = \lambda \otimes u, \qquad \lambda \in \overline{A}_1 \quad \text{such that} \quad (\lambda' - \lambda)b \in s(A_0),$$

where $u \in \omega$ is our chosen basis, $v \in \det$ is the standard basis, and $P(u) = b \otimes u$. In particular, $\lambda \in \overline{A}_1$ such that $\lambda = \lambda'$ gives an example of a 1-cocycle. In particular, any $\lambda \in \mathbb{Z}_p \subseteq \overline{A}_1$ represents a 1-cocycle, and according to our calculation these cocycles represent all elements in Ext^1.

Note that the 1-cocycles we have just described depend on a choice of basis u for $\omega_{C_0} \subset \omega_{\overline{C}_0}$. The dependence of the cohomology class of g on the choice of basis element u seems difficult to unravel; it is hard to determine when explicitly given cocycles are cohomologous. However, modulo p we can say something.

Consider an extension $0 \to N \to E \to M \to 0$ of Γ-modules. Evaluating this extension at the Frobenius endomorphism $\mathrm{Frob} \colon \overline{C}_0 \to \phi^*\overline{C}_0$ of $\overline{C}_0/\overline{\mathbb{F}}_p$ gives a commuting diagram of $\overline{\mathbb{F}}_p$-vector spaces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\mathbb{F}}_p{}^{\phi\pi_0} \otimes_{\overline{A}_0} N & \longrightarrow & \overline{\mathbb{F}}_p{}^{\phi\pi_0} \otimes_{\overline{A}_0} E & \longrightarrow & \overline{\mathbb{F}}_p{}^{\phi\pi_0} \otimes_{\overline{A}_0} M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \underline{N}(\mathrm{Frob}_{\overline{C}_0})} & & \downarrow{\scriptstyle \underline{E}(\mathrm{Frob}_{\overline{C}_0})} & & \downarrow{\scriptstyle \underline{M}(\mathrm{Frob}_{\overline{C}_0})} & & \\
0 & \longrightarrow & \overline{\mathbb{F}}_p{}^{\pi_0} \otimes_{\overline{A}_0} N & \longrightarrow & \overline{\mathbb{F}}_p{}^{\pi_0} \otimes_{\overline{A}_0} E & \longrightarrow & \overline{\mathbb{F}}_p{}^{\pi_0} \otimes_{\overline{A}_0} M & \longrightarrow & 0
\end{array}$$

where $\pi_0 \colon \overline{A}_1 \to \overline{\mathbb{F}}_p$ classifies $\overline{C}_0/\overline{\mathbb{F}}_p$ together with its unique p-subgroup, and ϕ is the pth power map. (We are implicitly using the identification of Γ-modules with p-isogeny modules, as described in §4.5.)

In the case that both $\underline{N}(\mathrm{Frob}_{\overline{C}_0}) = 0$ and $\underline{M}(\mathrm{Frob}_{\overline{C}_0}) = 0$, then $\underline{E}(\mathrm{Frob}_{\overline{C}_0})$ factors uniquely through an $\overline{\mathbb{F}}_p$-vector space homomorphism

$$\rho(E)\colon {\overline{\mathbb{F}}_p}^{\phi\pi_0} \otimes_{\overline{A}_0} M \to {\overline{\mathbb{F}}_p}^{\pi_0} \otimes_{\overline{A}_0} N.$$

If such an extension is classified by a 1-cocycle $g\colon M \to {}^t\overline{A}_1{}^s \otimes_{\overline{A}_0} N$, then $\rho(E)$ is precisely the unique map fitting in the square

$$\begin{array}{ccc} M & \xrightarrow{\;g\;} & {}^t\overline{A}_1{}^s \otimes_{\overline{A}_0} N \\ {\scriptstyle \phi\pi_0\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi_0\otimes\mathrm{id}} \\ {\overline{\mathbb{F}}_p}^{\phi\pi_0} \otimes_{A_0} M & \xrightarrow[\rho(E)]{} & {\overline{\mathbb{F}}_p}^{\pi_0} \otimes_{\overline{A}_0} N. \end{array}$$

In our case that $M = \omega \otimes \det^{-1}$ and $N = \omega$, this invariant $E \mapsto \rho(E)$ coincides with the explicit homomorphism $\rho\colon \mathrm{Ext}^1_\Gamma(\omega^{-1} \otimes \det, \omega) \to \mathbb{F}_p$ described in §11.3.

For an elliptic curve C/S, there is a canonical extension of locally free coherent sheaves over S of the form

$$0 \to H^0(\Omega^1_{C/S}) \to H^1_{\mathrm{dR}}(C/S) \to H^1(\mathcal{O}_{C/S}) \to 0$$

associated to the "algebraic Hodge to de Rham spectral sequence", with $H^0(\Omega^1_{C/S}) \approx \omega_{C/S}$ and $H^1(\mathcal{O}_{C/S}) \approx \omega^{-1}_{C/S}$ (see [Kat73, §A1.2]). Since the spectral sequence is functorial with respect to maps of schemes over S, and hence with respect to isogenies, we may apply it to the case of the universal deformation of a supersingular curve C_0/k to obtain a canonical **Hodge extension**

$$0 \to \omega \to H^1_{\mathrm{dR}}(C_{\mathrm{univ}}/\operatorname{Spec} A_0) \to \omega^{-1} \otimes \det \to 0$$

of Γ-modules. The Γ-module structure on $H^1(\mathcal{O}_{C/S})$ can be deduced from the Serre duality map $H^0(\Omega^1_{C/S}) \otimes H^1(\mathcal{O}_{C/S}) \to H^1(\Omega^1_{C/S})$ and the observation that as a Γ-module, $H^1(\Omega^1_{C_{\mathrm{univ}}/\operatorname{Spec} A_0}) \approx H^2_{\mathrm{dR}}(C_{\mathrm{univ}}/\operatorname{Spec} A_0) \approx \det$.

According to [Kat77, Lemma 1], for an elliptic curve C/S over an $\mathbb{F}_p$-scheme S, the image of $\mathrm{Frob}^*\colon H^1_{dR}(\phi^*C/S) \to H^1_{dR}(C/S)$ is locally free of rank one, with cokernel also locally free of rank one. It follows that specialization of the "Hodge extension" at the Frobenius isogeny of C_0/k gives a non-trivial invariant $\rho \in \mathbb{Z}/p$. Thus, we may conclude that the "Hodge extension" presents a generator of $\mathrm{Ext}^1_\Gamma(\omega^{-1} \otimes \det, \omega)$.

11.6. **A calculation of Γ-ring maps.** It is convenient to give in this section a computation of the set of $\mathbb{T}$-algebra maps

$$\pi_\star(E \wedge \Sigma^\infty_+ \mathbb{Z})_{K(2)} \to \pi_\star E.$$

Since these are concentrated in even degree, and are p-torsion free, by (2.4) this amounts to computing maps of Γ-rings

$$A_0[t, t^{-1}]^\wedge_{\mathfrak{m}} \to A_0.$$

The Γ-ring structure on the left is given by $P(t) = t^p$.

Recall that for a perfect field k there is a bijection

$$\{\, c \in (\mathbb{W}k)^\times \mid \tilde{\phi}(c) = c^p \,\} \xrightarrow{\sim} k^\times.$$

Elements on the left-hand side are called **Teichmuller lifts** of units in k.

11.7. **Proposition.** *Elements $f \in A_0^\times$ such that $t(f) = s(f)^p$ are precisely the the Teichmuller lifts in $\mathbb{W}k^\times \subset A_0^\times$. Thus, there is a bijection*

$$\mathcal{T}(\pi_\star(E \wedge \Sigma^\infty_+ \mathbb{Z})_{K(2)}, \pi_\star E) \xrightarrow{\sim} k^\times.$$

Proof. It is clear that Teichmuller lifts $c \in \mathbb{W}k$ satisfy the condition, since then $t(c) = \tilde{\phi}(c) = c^p$.

Write $A_0 = \mathbb{W}k[\![a]\!]$, where $a \in A_0$ is an adapted coordinate. Suppose $f \in A_0^\times$ such that $t(f) = s(f)^p$. By replacing $f = f(a)$ with f/c, where c is a Teichmuller lift such that $c \equiv f(0)$ mod p, we reduce to the case that $f(0) \equiv 1 \mod p$.

Working modulo p, and using the usual identification $A_1/(p) \approx k[\![b']\!] \times_k k[\![b]\!]$, we have that

$$s\colon f(a) \longmapsto (f(b'), f(b^p)),$$
$$t\colon f(a) \longmapsto (f^{(1)}(b'^p), f^{(1)}(b)).$$

The condition $t(f) = s(f)^p$ implies

$$f^{(1)}(b'^p) = f(b')^p, \qquad f^{(1)}(b) = f(b^p)^p.$$

The first of these is always true, but the second amounts to $f^{(1)}(b) = f^{(1)}(b^{p^2})$, which cannot hold for any non-constant power series. Thus we have shown that $f(a) \equiv 1 \mod pA_0$.

Now assume $f(a) = 1 + p^k g(a)$ for some $k \geq 1$, $g(a) \in A_0$. The condition $t(f) = s(f)^p$ modulo $p^{k+1}A_0$ implies that

$$t(g(a)) \equiv 0 \mod pA_0,$$

and therefore, by using the formulas for t modulo p, we see that $g(a) \equiv 0 \mod pA_0$. Iterating the argument shows that $g(a) = 0$. □

12. Cohomology of augmented $\mathbb{T}$-algebras

In [Rez09], we defined a monad $\mathbb{T}$ on $\mathrm{Mod}_{E_\star}$, which encodes the algebraic structure in the homotopy of $K(h)$-local commutative E-algebras (see §7). We write $\mathcal{T}$ for the category of $\mathbb{T}$-algebras.

In this section, we will recall the homotopy theory of simplicial $\mathbb{T}$-algebras, an encoded in a Quillen model category structure on $s\mathcal{T}$, and we will define the cohomology $H^*_{\mathcal{T}_{E_\star}}(B, M)$ of augmented $\mathbb{T}$-algebras, with coefficients in an abelian group object M, which is computed using an appropriate cofibrant simplical resolution of B.

We will then describe a full subcategory of *analytically complete* $\mathbb{T}$-algebras. ...

Describe how above cohomology can sometimes be computed with analytically complete resolutions. Requires model category structure for analytically complete $\mathbb{T}$-algebras.

Set up composite functor spectral sequence. Handle case of analytic completion of smooth algebra.

12.1. **Homotopy theory of $\mathbb{T}$-algebras.** We recall some algebraic properties of the monad $\mathbb{T}$ on $\mathrm{Mod}_{E_\star}$.

12.2. **Proposition.** *We have the following.*

1. *The functor $\mathbb{T}$ commutes with filtered colimits.*
2. *The functor $\mathbb{T}$ commutes with reflexive coequalizers.*
3. *The functor $\mathbb{T}$ takes direct sums to tensor products. That is, the evident map $\mathbb{T}(M) \otimes \mathbb{T}(N) \to \mathbb{T}(M \oplus N)$ is an isomorphism.*

(4) *The category $\mathcal{T}$ of $\mathbb{T}$-algebras is complete and cocomplete. Limits, filtered colimits, and reflexive coequalizers are created by the forgetful functor $\mathcal{T} \to \mathrm{Mod}_{E_\star}$.*
(5) *Let $F_{\mathbb{T}}\colon \mathrm{Mod}_{E_\star} \to \mathcal{T}$ denote the free $\mathbb{T}$-algebra functor. If M is a free $E_\star$-module, then the underlying ring of $F_{\mathbb{T}}(M)$ is a free $\mathbb{Z}/2$-graded strongly commutative algebra.*

Let $s\mathcal{T}$ denote the category of simplicial objects in $\mathcal{T}$.

12.3. **Proposition.** *There is a cofibrantly generated, simplicial closed model category structure on $s\mathcal{T}$ with the following properties.*

(1) *Weak equivalences are maps which are weak equivalences on underlying simplicial sets,*
(2) *Fibrations are maps which are fibrations on underlying simplicial sets.*
(3) *Cofibrations are retracts of s-free maps.*
(4) *The model category structure is proper.*

Proof. This is largely consequence of [Qui67, §II.4, Thm. 4]. Because reflexive coequalizers in $\mathcal{T}$ are computed in the underlying category $\mathrm{Mod}_{E_\star}$, the collection$F_{\mathbb{T}}(E_\star), F_{\mathbb{T}}(\omega^{1/2})\}$ is what Quillen calls a set of small projective generator of $\mathcal{T}$, and thus condition (**) of his theorem is satisfied. Therefore, $s\mathcal{T}$ admits a simplical model category structure with the specified classes of morphisms. Cofibrant generation and the description of cofibrations is implicit in Quillen's construction of factorizations.

To show that $\mathcal{T}$ is proper, it suffices by [Rez02, Thm. 9.1] to observe that for any finitely generated free $E_\star$-module, the functor $s\mathcal{T} \to s\mathcal{T}$ defined by $B \mapsto B \coprod F_{\mathbb{T}}(M)$ (coproduct in $s\mathcal{T}$, where $F_{\mathbb{T}}(M)$ is regarded as a constant simplicial object), takes weak equivalences to weak equivalences. This is because coproducts in $\mathcal{T}$ are tensor products, and $F_{\mathbb{T}}(M)$ is a strongly commutative $\mathbb{Z}/2$-graded polynomial ring over $E_\star$, and hence flat as an $E_\star$-module. □

We immediately obtain a model category structure on the slice category $s\mathcal{T}_{E_\star}$, the category of simplicial objects in augmented $\mathbb{T}$-algebras.

12.4. **Cohomology of augmented $\mathbb{T}$-algebras.** Recall that there is a category $\mathrm{ab}\,\mathcal{T}_{E_\star}$ of abelian group objects in $\mathcal{T}_{E_\star}$, which may be identified with the full subcategory of augmented $\mathbb{T}$-algebras with square-zero augmentation ideal. It will be convenient notationally to indicate an abelian group object by specifying its augmentation ideal only, so that an object $M \in \mathrm{ab}\,\mathcal{T}_{E_\star}$ has associated $\mathbb{T}$-algebra $E_\star \rtimes M$.

There is an adjoint pair
$$Q\colon \mathcal{T}_{E_\star} \rightleftarrows \mathrm{ab}\,\mathcal{T}_{E_\star} : E_\star \rtimes -,$$
where Q takes $B \to E_\star$ to $\overline{B}/\overline{B}^2$, where $\overline{B} = \mathrm{Ker}[B \to E_\star]$.

12.5. **Proposition.** *There is a cofibrantly generated, simplical closed model category structure on $s(\mathrm{ab}\,\mathcal{T}_{E_\star})$, so that*

- *weak equivalences and fibrations are those on underlying simplicial sets, and*
- *the adjoint pair*
$$Q\colon s\mathcal{T}_{E_\star} \rightleftarrows s(\mathrm{ab}\,\mathcal{T}_{E_\star}) : E_\star \rtimes -$$
is a Quillen pair.

Proof. The model structure is immediate, since $\mathrm{ab}\,\mathcal{T}_{E_\star}$ is an abelian category with enough projectives. (Projective generators are given by $Q(F_{\mathbb{T}}(M) \to E_\star)$, where M is a free $E_\star$-module and the augmentations sends M to 0.) The existence of the Quillen pair is immediate. □

Given $M \in \mathrm{ab}\,\mathcal{T}_{E_\star}$ and $n \geq 0$, let $K(M,n)$ denote the Eilenberg-MacLane object in $s(\mathrm{ab}\,\mathcal{T}_{E_\star})$ with $\pi_n K(M,n) \approx M$. We then define the n**th cohomology group** of an augmented $\mathbb{T}$-algebra B with coefficients in an abelian group object M by

$$\begin{aligned} H^n_{\mathcal{T}_{E_\star}}(B,M) &\overset{\text{def}}{=} h(s\mathcal{T}_{E_\star})\big(B, E_\star \rtimes K(M,n)\big) \\ &\approx h(s(\mathrm{ab}\,\mathcal{T}_{E_\star}))\big(\mathbf{L}Q(B), K(M,n)\big). \end{aligned}$$

Here $\mathbf{L}Q$ denotes the total left derived functor of $Q\colon s\mathcal{T}_{E_\star} \to s\,\mathrm{ab}\,\mathcal{T}_{E_\star}$. Note that $E_\star \rtimes -\colon s\,\mathrm{ab}\,\mathcal{T}_{E_\star} \to s\mathcal{T}_{E_\star}$ computes its own total right derived functor, since all objects in $s\,\mathrm{ab}\,\mathcal{T}_{E_\star}$ are fibrant.

This description immediately implies a composite functor spectral sequence.

12.6. **Proposition.** *There is a first quadrant spectral sequence of the form*

$$\mathrm{Ext}^j_{\mathrm{ab}\,\mathcal{T}_{E_\star}}(\mathbf{L}_i Q(B), M) \Longrightarrow H^{i+j}_{\mathcal{T}_{E_\star}}(B,M),$$

where $\mathbf{L}_i Q(B) = \pi_i \mathbf{L}Q(B)$ denote the derived functors of the indecomposables functor Q.

12.7. **Analytic completion.** We need to incorporate analytic completion into our story.

We write $\mathcal{A}\colon \mathrm{Mod}_{E_\star} \to \mathrm{Mod}_{E_\star}$ for the analytic completion functor with respect to the the sequence $p, u_1, \dots, u_{n-1} \in E_0$, defined by

$$\mathcal{A}(M) \overset{\text{def}}{=} M[\![x_0,\dots,x_{n-1}]\!]/(x_0 - p, x_1 - u_1, \dots, x_{n-1} - u_{n-1})M[\![x_0,\dots,x_{n-1}]\!].$$

It comes with a natural unit map $\eta\colon M \to \mathcal{A}(M)$, and a natural comparison map $\mathcal{A}(M) \to M^\wedge_{\mathfrak{m}}$ to the $\mathfrak{m}$-adic completion of M, factoring the usual map $M \to M^\wedge_{\mathfrak{m}}$. By construction, the functor $\mathcal{A}$ is right-exact, and commutes with arbitrary products.

12.8. **Proposition.** *If M is regular for the sequence $p, u_1, \dots, u_{n-1}$, then the comparison map $\mathcal{A}(M) \to M^\wedge_{\mathfrak{m}}$ is an isomorphism.*

As a consequence, $\mathcal{A}$ is *isomorphic* to the 0th left derived functor of $\mathfrak{m}$-adic completion, typically denoted L_0.

Say that an $E_\star$-module is **analytic** if $\eta\colon M \to \mathcal{A}(M)$ is an isomorphism. Let $\widehat{\mathrm{Mod}}_{E_\star} \subset \mathrm{Mod}_{E_\star}$ denote the full subcategory of analytic modules.

12.9. **Proposition.** *The analytic completion functor $\mathcal{A}$ takes values in the full subcategory $\widehat{\mathrm{Mod}}_{E_\star}$ of analytic objects, and thus provides the left-half of an adjoint pair*

$$\overline{\mathcal{A}}\colon \mathrm{Mod}_{E_\star} \rightleftarrows \widehat{\mathrm{Mod}}_{E_\star} : incl.$$

The category $\widehat{\mathrm{Mod}}_{E_\star}$ has enough projectives, and is complete and cocomplete. Furthermore, the inclusion functor $\widehat{\mathrm{Mod}}_{E_\star} \to \mathrm{Mod}_{E_\star}$ commutes with finite colimits and arbitrary limits.

Say that $M \in \mathrm{Mod}_{E_\star}$ is **tame** if $\mathbf{L}_k\mathcal{A}(M) \approx 0$ for $k \geq 1$, where $\mathbf{L}_k\mathcal{A}$ denote the left-derived functors of $\mathcal{A}\colon \mathrm{Mod}_{E_\star} \to \mathrm{Mod}_{E_\star}$. (These coincide with the left-derived functors of $\overline{\mathcal{A}}\colon \mathrm{Mod}_{E_\star} \to \widehat{\mathrm{Mod}}_{E_\star}$, since the inclusion functor is exact.)

12.10. **Proposition.** *Flat $E_\star$-modules are tame. Analytic $E_\star$-modules are tame.*

Let $s\mathrm{Mod}_{E_\star}$ denote the category of simplicial $E_\star$-modules.

12.11. **Proposition.** *Let M be an object of $s\mathrm{Mod}_{E_\star}$ which is (i) degreewise tame, and (ii) is such that $\pi_* M$ is analytic. Then $\eta\colon M \to \mathcal{A}(M)$ is a weak-equivalence of simplicial $E_\star$-modules. flat.*

Proof. This is immediate from the evident spectral seqeunce $E_2^{i,j} = \mathbf{L}_i\mathcal{A}(\pi_j M) \Longrightarrow \pi_{i+j}\mathcal{A}(M)$, which is defined because M is degreewise tame. □

The connection to homotopy theory is given by the following.

12.12. **Proposition.** *Let $M \in \mathcal{M}$ be an E-module spectrum. Then M is $K(h)$-local if and only if $\pi_\star M$ is analytic.*

12.13. **Analytic $\mathbb{T}$-algebras.** Consider the natural map $\mathcal{A}\mathbb{T}\eta\colon \mathcal{A}\mathbb{T} \to \mathcal{A}\mathbb{T}\mathcal{A}$ of functors $\mathrm{Mod}_{E_\star} \to \mathrm{Mod}_{E_\star}$. The following says that the functor $\mathbb{T}$ is in some sense compatible with analytic completion.

12.14. **Proposition.** *The map $\mathcal{A}\mathbb{T}\eta\colon \mathcal{A}\mathbb{T} \to \mathcal{A}\mathbb{T}\mathcal{A}$ is an isomorphism.*

Proof. Proved by Barthel and Frankland [BF13]. □

Let $\widehat{\mathcal{T}} \subset \mathcal{T}$ denote the full subcategory of $\mathbb{T}$-algebras whose underlying $E_\star$-module is analytic.

12.15. **Proposition.** *There is an adjoint pair*

$$\overline{\mathcal{A}}_{\mathcal{T}}\colon \mathcal{T} \rightleftarrows \widehat{\mathcal{T}} : incl,$$

with the property that on underlying $E_\star$-modules, the left adjoint $\overline{\mathcal{A}}_{\mathcal{T}}$ coincides with analytic completion of $E_\star$-modules.

Proof. Given a $\mathbb{T}$-algebra $(B, \psi\colon \mathbb{T}B \to B)$, we define a $\mathbb{T}$-algebra $\overline{\mathcal{A}}_{\mathcal{T}}(B)$ to be $(\mathcal{A}B, \widehat{\psi}\colon \mathbb{T}\mathcal{A}B \to \mathcal{A}B)$, where $\widehat{\psi} = (\mathcal{A}\psi) \circ (\mathcal{A}\mathbb{T}\eta)^{-1} \circ \eta$. It is straightforward using (12.14) to show that this is in fact a $\mathbb{T}$-algebra, and that $\eta\colon B \to \mathcal{A}B$ defines a $\mathbb{T}$-algebra map. □

In particular, the analytic completion of a $\mathbb{T}$-algebra is canonically a $\mathbb{T}$-algebra. From now on we will write $\overline{\mathcal{A}}\colon \mathcal{T} \to \widehat{\mathcal{T}}$ for $\overline{\mathcal{A}}_{\mathbb{T}}$, and $\mathcal{A}\colon \mathcal{T} \to \mathcal{T}$ for the composite of $\overline{\mathcal{A}}$ with inclusion.

The above story descends to abelian group objects in augmented $\mathbb{T}$-algebras. Recall the indecomposable quotient functor $Q\colon \mathcal{T}_{E_\star} \to \mathrm{ab}\,\mathcal{T}_{E_\star}$.

12.16. **Proposition.** *The map $\mathcal{A}Q\eta\colon \mathcal{A}Q \to \mathcal{A}Q\mathcal{A}$ is an isomorphism.*

Proof. We have a diagram

$$\begin{array}{ccccccc} B \otimes B & \longrightarrow & B & \longrightarrow & Q(B) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle Q(\eta)} & & \\ \mathcal{A}B \otimes \mathcal{A}B & \longrightarrow & \mathcal{A}B & \longrightarrow & Q(\mathcal{A}B) & \longrightarrow & 0 \end{array}$$

with exact rows. After appyling $\mathcal{A}$ to this diagram, the rows remain exact. The map $\mathcal{A}(\eta)$ is clearly an isomorphism, and the map $\mathcal{A}(\eta \otimes \eta)$ is an isomorphism by [HS99, ???]. □

Let $\mathrm{ab}\,\widehat{\mathcal{T}}_{E_\star}$ denote the full subcategory of $\mathrm{ab}\,\mathcal{T}_{E_\star}$ whose underlying $E_\star$-module is analytic.

12.17. **Proposition.** *There is an (up to isomorphism) commutative square of adjoint pairs, whose left adjoints are*

$$\begin{array}{ccc} \mathcal{T}_{E_\star} & \xrightarrow{\overline{\mathcal{A}}_{\mathbb{T}}} & \widehat{\mathcal{T}}_{E_\star} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \widehat{Q}} \\ \mathrm{ab}\,\mathcal{T}_{E_\star} & \xrightarrow[\overline{\mathcal{A}}_{Q}]{} & \mathrm{ab}\,\widehat{\mathcal{T}}_{E_\star} \end{array}$$

12.18. **Homotopy theory of analytic $\mathbb{T}$-algebras.** Let $s\widehat{\mathcal{T}}$ denote the category of simplicial objects in $\widehat{\mathcal{T}}$, which may be identified as a full subcategory of $s\mathcal{T}$.

12.19. **Proposition.** *There is a simplicial closed model category structure on $s\widehat{\mathcal{T}}$ with the following properties.*

(1) *Weak equivalences are maps which are weak equivalences on underlying simplicial sets.*
(2) *Fibrations are maps which are fibrations on underlying simplicial sets.*
(3) *Cofibrations are maps which are retracts of s-free maps.*
(4) *The adjoint pair*
$$\overline{\mathcal{A}}_{\mathcal{T}} \colon s\mathcal{T} \rightleftarrows s\widehat{\mathcal{T}} : incl$$
is a Quillen pair.
(5) *The map of derived functors $\mathbf{L}\overline{\mathcal{A}}_{\mathbb{T}} \circ \mathbf{R}incl \to \mathrm{Id}$ is an isomorphism. Thus, $\mathbf{R}incl\colon h(s\widehat{\mathcal{T}}) \to h(s\mathcal{T})$ is fully faithful, with essential image the full subcategory of $h(s\mathcal{T})$ consisting of simplicial $\mathbb{T}$-algebras B such that $\pi_\star B$ is analytic.*

The above descends to abelian group objects. Thus, let $\mathrm{ab}\,\widehat{\mathcal{T}}_{E_\star}$ denote the category of abelian group objects in $\mathcal{T}_{E_\star}$ whose underlying $E_\star$-module is analytic.

13. Mapping space spectral sequence

Define a good resolution of a $K(h)$-local commutative E-algebra (i.e., simplicial resolution built from $K(h)$-localization of free algebras on free E-modules). Use to construct mapping space spectral sequence, and identify E_2-term as cohomology.

References

[BGJGP05] Matthew H. Baker, Enrique González-Jiménez, Josep González, and Bjorn Poonen, *Finiteness results for modular curves of genus at least 2*, Amer. J. Math. **127** (2005), no. 6, 1325–1387, available at `arXiv:math/0211394`.

[BF13] Tobias Barthels and Martin Frankland, *Completed algebraic approximation functors* (2013). In preparation.

[Beh06] Mark Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), no. 2, 343–402.

[Bou96] A. K. Bousfield, *On p-adic λ-rings and the K-theory of H-spaces*, Math. Z. **223** (1996), no. 3, 483–519, DOI 10.1007/PL00004271. MR1417857 (97i:55033)

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[Frö68] A. Fröhlich, *Formal groups*, Lecture Notes in Mathematics, No. 74, Springer-Verlag, Berlin, 1968.

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200.

[GM92] J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453, DOI 10.1016/0021-8693(92)90026-I. MR1172439 (93h:13009)

[HS99] Mark Hovey and Neil P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100. MR1601906 (99b:55017)

[Kat77] Nicholas M. Katz, *A result on modular forms in characteristic p*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 53–61. Lecture Notes in Math., Vol. 601.

[Kat73] ———, *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 69–190. Lecture Notes in Mathematics, Vol. 350.

[LT66] Jonathan Lubin and John Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France **94** (1966), 49–59.

[MR09] Mark Mahowald and Charles Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. **5** (2009), no. 2, 853–872. MR2508904

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.

[Rez02] Charles Rezk, *Every homotopy theory of simplicial algebras admits a proper model*, Topology Appl. **119** (2002), no. 1, 65–94. MR1881711 (2003g:55033)

[Rez08] ———, *Power operations for Morava E-theory of height* 2 *at the prime* 2 (2008), available at `arXiv:0812.1320(math.AT)`.

[Rez09] ———, *The congrugence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379, available at `arXiv:0902.2499`.

[Rez11] ———, *Rings of power operations for Morava E-theories are Koszul* (2011), available at `http://arxiv.org/abs/1204.4831`.

[Rez12] ———, *Modular isogeny complexes*, Algebraic & Geometric Topology **12** (2012), no. 3, 1373–1403, available at `arXiv:arXiv:1102.5022`.

[Rez13] ———, *Analytic completion* (2013), available at `http://www.math.uiuc.edu/~rezk/analytic-paper.pdf`.

[Str97] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), no. 2, 161–208.

[Str98] N. P. Strickland, *Morava E-theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779, available at `arXiv:math/9801125`.

[Zhu12] Yifei Zhu, *The power operation structure on Morava E-theory of height* 2 *at the prime* 3 (2012), available at `arXiv:1210.3730(math.AT)`.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL
E-mail address: `rezk@math.uiuc.edu`