An $\infty$-category is a gadget equipped with
- objects,
- 1-morphisms between objects,
- 2-morphisms between 1-morphisms,
- 3-morphisms between 2-morphisms,
- etc.

An $(\infty, n)$-category is one such that all $k$-morphisms are “invertible”, for $k > n$. 
Goal of this talk

I want to discuss an approach to $(\infty, n)$-categories, based on the following ideas:

- An $(\infty, 0)$-category (= an $\infty$-groupoid) is a space. ("Homotopy hypothesis".)
- An $(\infty, n)$-category should be more-or-less the same thing as a category enriched over $(\infty, n-1)$-categories.
- The collection of $(\infty, n)$-categories should have internal function objects, i.e., $(\infty, n)$-categories should be Cartesian closed, and thus be an example of some kind of $(\infty, n+1)$-category.
- We should avoid interpreting the above ideas too strictly.
Let $\text{Cat}_{\infty,1} =$ “category” of $(\infty, 1)$-categories.

- **equivalences**: class of morphisms in $\text{Cat}_{\infty,1}$
- $\text{Cat}_{\infty,1}$ is Cartesian closed:
  \[ C, D \in \text{Cat}_{\infty,1} \implies \{ C \times D \}, \text{ right adjoint to } \times \]
- $\text{Gpd}_{\infty} \subset \text{Cat}_{\infty,1}$ full subcategory of $\infty$-groupoids
  \[ C_{\text{gpd}} \subseteq C \text{ maximal sub-$\infty$-groupoid of } C \]
- classifying space functor $B : \text{Gpd}_{\infty} \to \text{Sp}$:
  \[
  \{ \text{groupoids up to equivalence} \} \iff \{ \text{spaces up to weak equivalence} \}
  \]

Can we understand $\text{Cat}_{\infty,1}$ using spaces?
Presheaf of spaces associated to an \((\infty, 1)\)-category

Given \(C \in \text{Cat}_{\infty, 1}\), let

\[
\mathcal{F} = \mathcal{F}_C : \text{Cat}^{\text{op}}_{\infty, 1} \to \text{Sp}
\]

\[
A \mapsto B(\{A, C\}^{\text{gpd}}) = \text{Map}(A, C)
\]

(representable space valued presheaf on \(\text{Cat}_{\infty, 1}\))

- Think of \(\mathcal{F}_C(\bullet) = B(C^{\text{gpd}})\) as the “moduli space” of objects of \(C\):

\[
B(C^{\text{gpd}}) \approx \coprod_{[X]} B\text{Aut}(X).
\]

(\(\bullet = \text{“freestanding object” category}\))

- Think of \(\mathcal{F}_C(A)\) as the “moduli space” of functors \(A \to C\)

- \(\text{Cat}_{\infty, 1} \iff \{\text{representable presheaves in } \text{Psh}(\text{Cat}_{\infty, 1}, \text{Sp})\}\)

Yoneda lemma!
Example: $C = \text{finite sets}$

$C = \text{category of finite sets}$

- “Size” is a complete isomorphism invariant of finite sets
- $\text{Aut}({1, \ldots, n}) = \Sigma_n$ symmetric group

$$
\mathcal{F}_C(\bullet) \approx \bigsqcup_{[S]} B\text{Aut}(S) \approx \bigsqcup_{n \geq 0} B\Sigma_n
$$
Example: $C =$ finite sets, continued

- Let $[1] = (\bullet \to \bullet)$

$\{[1], C\} =$ category of functors $[1] \to C$

**Objects:** morphisms $f : S_0 \to S_1$ in $C$

**Morphisms:** commutative diagrams

\[
\begin{array}{ccc}
S_0 \cong & T_0 & \\
\downarrow & \downarrow & \\
S_1 \cong & T_1 & 
\end{array}
\]

- $C =$ finite sets
  
  $p(f) = (p_0, p_1, p_2, \ldots)$ where $p_k =$ # of fibers of $f$ with size $k$

\[
\mathcal{F}_C([1]) \approx \bigsqcup_{[S_0 \overset{f}{\to} S_1]} B \text{Aut}(S_0 \overset{f}{\to} S_1) \approx \bigsqcup_p B \left( \prod_k \Sigma_k \wr \Sigma_{p_k} \right)
\]

- If $f$ is isomorphism, $p(f) = (0, n, 0, 0, \ldots)$, so $B \text{Aut}(f) \approx B \Sigma_n$
Properties of $\mathcal{F}_C$

General properties of $\mathcal{F}_C$

- $\mathcal{F}_C([1])_{\text{inv}} \overset{\text{def}}{=} \text{subspace of } \mathcal{F}_C([1]) \text{ of path components containing invertible maps}$
- $\mathcal{F}_C(\bullet) \to \mathcal{F}_C([1])$ factors through a weak equivalence

$$\mathcal{F}_C(\bullet) \sim \mathcal{F}_C([1])_{\text{inv}} \subseteq \mathcal{F}_C([1]).$$

- $\mathcal{F}_C(A)$ can always be recovered as a homotopy limit from diagrams involving the spaces $\mathcal{F}_C(\bullet)$ and $\mathcal{F}_C([1])$.

For instance

$$\mathcal{F}_C(0 \to 1 \to 2) \approx \lim (\mathcal{F}_C(0 \to 1) \to \mathcal{F}_C(1) \leftarrow \mathcal{F}_C(1 \to 2))$$

and similarly for $\mathcal{F}_C(0 \to 1 \to \cdots \to n)$. 
Recovering $C$ from $\mathcal{F}$

$\Delta \subset \text{Cat}$: full subcategory of categories of the form

$$[m] = (0 \to 1 \to 2 \to \cdots \to m)$$

Can recover $C$, up to equivalence, from the restriction of $\mathcal{F}_C$ to $\Delta$:

- $\pi_0 \mathcal{F}_C([0]) = \text{isomorphism classes of objects of } C$
- $\text{Map}_C(X, Y) \approx \text{hofiber}_{(X,Y)}[\mathcal{F}_C([1]) \to \mathcal{F}_C([0]) \times \mathcal{F}_C([0])]$
- composition is defined using

$$\text{Map}_C(X, Y) \times \text{Map}_C(Y, Z) \approx \text{hofiber}_{(X,Y,Z)}[\mathcal{F}_C([2]) \to \mathcal{F}_C([0]) \times \mathcal{F}_C([0]) \times \mathcal{F}_C([0])]$$

- associativity of composition uses fibers of $\mathcal{F}_C([3]) \to \mathcal{F}_C([0])^4$
Definition of complete Segal space

Complete Segal space: a functor $X : \Delta^{op} \to \text{Sp}$ satisfying the following.

- **Segal condition.** For all $k \geq 2$,

$$X([k]) \sim \lim \left( \begin{array}{cccc}
X([1]) & X([1]) & \cdots & X([1]) \\
X[0] & \cdots & X[0] & \cdots
\end{array} \right)$$

- **Completeness condition.**

  The map $X([0]) \to X([1])$ factors through a weak equivalence $X([0]) \to X([1])_{\text{inv}} \subseteq X([1])$.

  (If $X \in \text{Psh}(\Delta, \text{Sp})$ satisfies the Segal condition, $X([1])_{\text{inv}} \overset{\text{def}}{=} \text{union of components of } X([1]) \text{ which contain elements invertible in the “homotopy category” of } X.$)
A complete Segal space $X$ has

- “objects” $\iff$ points of $X([0])$
- “morphism spaces” for $a, b \in X([0])$

\[
\text{MAP}_X(a, b) \overset{\text{def}}{=} \text{hofiber}_{(a, b)}[X([1]) \to X([0]) \times X([0])].
\]

- a weakly defined “composition”

**Theorem (Bergner)**

\{complete Segal spaces\} $\iff$ \{categories enriched over spaces\}.

That is:

\{complete Segal spaces\} $\iff$ \{categories enriched over $(\infty, 0)$-categories\}.

Also equivalent to: **Segal categories** (Bergner), **quasicategories** (Joyal-Tierney).
A presentation \((C, S)\) consists of
- \(C = \) small category,
- \(S = \{s: S \rightarrow S'\} = \) set of morphisms in \(Psh(C, Sp)\).

An \(S\)-local presheaf is \(X \in Psh(C, Sp)\) such that for all \(s \in S\),

\[
\text{Map}(s, X): \text{Map}(S', X) \rightarrow \text{Map}(S, X)
\]

is weak equivalence of spaces. (\(\text{Map} = \) derived mapping space.)

\(Psh(C, Sp)_S \overset{\text{def}}{=} \) full subcategory of \(S\)-local presheaves in \(Psh(C, Sp)\).

\(\overline{S} \overset{\text{def}}{=} \) class of maps:
- \(f \in \overline{S} \) iff \(\text{Map}(f, X)\) is a weak equivalence for all \(S\)-local \(X\)
- (sometimes called \(S\)-local equivalences, or saturation of \(S\).)

Note: \(hPsh(C, Sp)_S \approx hPsh(C, Sp)[\overline{S}^{-1}]\).
Complete Segal spaces are presented by $(\Delta, S)$, where $S$ consists of

$$se_k : G[k] \to F[k] \quad (\text{for } k \geq 2),$$

$$cp : Z \to F[0].$$

- $F[k] = \text{presheaf represented by } [k] \in \text{ob} \Delta$
- $G[k] \subset F[k]$, e.g.:

- $Z = F[3]/\sim = \text{colim}(F[3] \leftarrow F[1] \amalg F[1] \to F[0] \amalg F[0]).$

$$\text{Map}(Z, X) \approx X([1])_{\text{inv}} \subseteq X([1]) \text{ if } X \text{ satisfies Segal condition}$$
Constructing elements of $\overline{S}$

An example of elements of $\overline{S}$.

- $se_2 \in \overline{S} \implies g \in \overline{S}$
- $g, se_3 \in \overline{S} \implies k \in \overline{S}$
Psh($C, Sp$) is **Cartesian closed**: internal function object $\{X, Y\}$.

\[ X \rightarrow \{Y, Z\} \iff X \times Y \rightarrow Z. \]

In what follows, $\{X, Y\} = \text{the derived version of function object}$.

**Definition**

A presentation $(C, S)$ is **Cartesian** if for all $X \in \text{Psh}(C, Sp)$,

\[ Y \in \text{Psh}(C, Sp)_S \implies \{X, Y\} \in \text{Psh}(C, Sp)_S. \]

$(C, S)$ Cartesian $\implies$ Psh($C, Sp$)$_S$ has a (derived) internal function object, which is **computed** as the function object between the underlying presheaves.
**Theorem (R.)**

The presentation \((\Delta, \mathcal{S})\) defining complete Segal spaces is Cartesian.

- To show that a presentation \((C, \mathcal{S})\) is Cartesian, check:

\[
(S \xrightarrow{s} S') \in \mathcal{S} \implies (S \times Fc \xrightarrow{s \times \text{id}} S' \times Fc) \in \overline{\mathcal{S}}
\]

for all \(c \in \text{ob} C\)

- \(Fc = \) presheaf represented by \(c\)

- To prove the Theorem, show that

\[
G[k] \times F[m] \xrightarrow{\text{se}_k \times \text{id}} F[k] \times F[m], \quad Z \times F[m] \xrightarrow{\text{cp}_m \times \text{id}} F[0] \times F[m]
\]

are in \(\overline{\mathcal{S}}\).

Want to show: $X$ complete Segal space $\implies$ \( \text{Map}(F[2] \times F[1], X) \to \text{Map}(G[2] \times F[1], X) \) is a weak equivalence
Idea of the proof, (continued)


\[
\begin{array}{ccc}
\mapsto & \mapsto & \mapsto \\
\mapsto & \mapsto & \mapsto \\
\mapsto & \mapsto & \mapsto \\
\end{array}
\]

\[
\text{Map} (\text{Black&Blue}, X) \to \text{Map} (\text{Blue}, X) \text{ is a weak equivalence.}
\]

if \( X \) is a complete Segal space
We want to base a definition of \((\infty, n)\)-categories on the following principles (here \(C, D \in \text{Cat}_{\infty,n}\)):

- function objects \(\{C, D\} \in \text{Cat}_{\infty,n}\)
- maximal sub-\(\infty\)-groupoid \(C^{\text{gpd}} \subseteq C\)
- \(\infty\)-groupoids are spaces
- these constructions invariant under equivalence

\[ \mapsto \text{functor} \]

\[ \mathcal{F} = \mathcal{F}_C : \text{Cat}^{\text{op}}_{\infty,n} \to \text{Sp} \]

\[ A \mapsto \{A, C\}^{\text{gpd}} \approx \text{Map}(A, C) \]

- To make this concrete, need a suitable small subcategory of \(\text{Cat}_{\infty,n}\)
The category $\Theta_n$

$\Theta_n$ introduced by Joyal; related to Batanin’s “pasting diagrams”
$\Theta_n$ is to $n$-categories as $\Delta = \Theta_1$ is to 1-categories

Definition (Vague)

$\Theta_n$ is the full subcategory of strict $n$-categories consisting of objects which “look like”

The name of this object (of $\Theta_2$) is $[4][2], [3], [0], [1])$.

$k$-cells in $\Theta_n$ for $0 \leq k \leq n$. Notation:

$O_0 = (\bullet)$, $O_1 = (\bullet \rightarrow \bullet)$, $O_2 = (\bullet \xrightarrow{\circlearrowleft} \bullet)$, ...
A $\Theta_n$-space is a functor $X : \Theta_n^{\text{op}} \to \mathcal{S}$ satisfying

- **Segal conditions.** $X(\theta) = \text{homotopy limit of } X(O_k)$'s:

$$X \left( \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\end{array} \right) \approx \lim \left[ X \left( \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\end{array} \right) \rightarrow X(\bullet) \leftarrow X \left( \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\end{array} \right) \right]$$

$$\approx \lim \left[ X(O_2) \downarrow \right. \quad \left. X(O_1) \rightarrow X(O_0) \leftarrow X(O_2) \right]$$

- **Completeness conditions.** $X(O_{k-1}) \rightarrow X(O_k)$ factors through a weak equivalence

$$X(O_{k-1}) \overset{\sim}{\rightarrow} X(O_k)^{\text{inv}} \subseteq X(O_k)$$

for $k = 1, \ldots, n$. 
“Composition”:
Morphism in $\Theta_n$

induces map of spaces

\[
X \left( \bullet \bigcirc \bullet \right) \leftarrow X \left( \bullet \bigcirc \bullet \right)
\]
The wreath category $\Theta C$

$C = \text{small category} \implies \text{category } \Theta C = \Delta \wr C$ (C. Berger):

**Objects of $\Theta C$**

Graphs like

\[
\begin{array}{cccc}
0 & \xrightarrow{c_1} & 1 & \xrightarrow{c_2} \rightarrow 2 & \xrightarrow{c_3} \rightarrow 3 \\
\end{array}
\]

where $c_i \in \text{ob}C$ (denoted $[3](c_1, c_2, c_3)$).

**Morphisms of $\Theta C$**

\[
\begin{array}{cccc}
0 & \xrightarrow{d_1} & \xleftarrow{c_1} 1 & \xrightarrow{d_2} \rightarrow 2 & \xrightarrow{d_3} \rightarrow 3 \rightarrow 4 \\
\end{array}
\]

consists of $\delta : [3] \rightarrow [4] \in \Delta$, $f_{ij} : c_i \rightarrow d_j \in C$.

Think of $[m](c_1, \ldots, c_m)$ as a $C$-enriched category.
Definition of $\Theta_n$

$\Theta_0 \overset{\text{def}}{=} 1$, \hspace{1cm} $\Theta_n \overset{\text{def}}{=} \Theta(\Theta_{n-1})$

- Inclusions $\Theta_1 \subset \Theta_2 \subset \cdots \subset \Theta_n$
- "Suspension" $\Theta_{n-1} \to \Theta_n$

\[
\theta \mapsto [1](\theta)
\]

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\quad \mapsto 
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet
\end{array}
\]

Definition of $\text{MAP}_X(a, b)$

$X \in \text{Psh}(\Theta_n, \text{Sp}), a, b \in X([0]) \implies \text{MAP}_X(a, b) \in \text{Psh}(\Theta_{n-1}, \text{Sp})$:

\[
\text{MAP}_X(a, b)(\theta) \overset{\text{def}}{=} \text{hofiber}_{(a,b)} \left[ X([1](\theta)) \to X([0]) \times X([0]) \right]
\]
A **Θ\textsubscript{n}-space** is a functor \( \Theta_n^{op} \to \text{Sp} \) satisfying the following.

- **Segal condition.** For all \( k \geq 2, \theta_1, \ldots, \theta_k \in \text{ob}\Theta_{n-1} \),
  
  \[
  X([k](\theta_1, \ldots, \theta_k)) \sim \lim \left( \begin{array}{cccc}
  X([1](\theta_1)) & X([1](\theta_2)) & \cdots & X([1](\theta_k)) \\
  \downarrow & \downarrow & \cdots & \downarrow \\
  X[0] & X[0] & \cdots & X[0]
  \end{array} \right)
  \]

- **Completeness condition.** \( X|_{\Theta_1} \) is a complete Segal space

- **Recursive condition.**
  \( \text{MAP}_X(a, b) \) is a \( \Theta_{n-1} \)-space for all \( a, b \in X([0]) \)

- A \( \Theta_0 \)-space is a space.
• Idea: $\Theta_n$-spaces model $(\infty, n)$-categories.

• $\Theta_n$-spaces are local objects for a presentation $(\Theta_n, \mathcal{T})$.

• Not the only model given by a presentation: $n$-fold complete Segal spaces, given by a presentation $(\Delta^n, \mathcal{T}')$. (Barwick, Lurie).

• These two presentations are different, but model the same underlying theory. (Underlying model categories are Quillen equivalent.)

• There are other models, not given by a presentation, e.g., $n$-fold Segal categories (Hirschowitz–Simpson).
\(\Theta_n\)-spaces are cartesian

\[(\Theta_n, \mathcal{T}) \overset{\text{def}}{=} \text{presentation for } \Theta_n\text{-spaces.}\]

**Theorem (R.)**

\[(\Theta_n, \mathcal{T}) \text{ is a Cartesian presentation.}\]

\[\implies \text{if } X, Y \in \text{Psh}(\Theta_n, \text{Sp}), \text{ and } Y \text{ is a } \Theta_n\text{-space, so is } \{X, Y\}.\]

- The presentation \((\Delta^n, \mathcal{T}')\) for \(n\)-fold complete Segal spaces is not Cartesian (though it comes close).
- The \(n\)-fold Segal category model (Hirschowitz–Simpson) gives a Cartesian model category, but isn’t given by a presentation.
Let \((C, S)\) be a Cartesian presentation. (Assume \(C\) has a terminal object.) There exists a presentation \((\Theta C, \mathcal{T})\), whose local objects \(X\) satisfy:

- **Segal condition.** For all \(k \geq 2, c_1, \ldots, c_k \in \text{ob} C\),

\[
X([k](c_1, \ldots, c_k)) \sim \lim_{\longrightarrow} \left( \begin{array}{cccc}
X([1](c_1)) & X([1](c_2)) & \cdots & X([1](c_k)) \\
X[0] & & & X[0]
\end{array} \right)
\]

- **Completeness condition.** \(X|_{\Theta 1}\) is a complete Segal space

- **Recursive condition.** \(\text{MAP}_X(a, b) \in \text{Psh}(C, Sp)\) is an \(S\)-model for all \(a, b \in X([0])\)

**Theorem**

\((\Theta C, \mathcal{T})\) is a Cartesian presentation if \((C, S)\) is
A more general construction, continued

- If $V = Psh(C, Sp)_S$, then

$$V-\Theta Sp \overset{\text{def}}{=} Psh(\Theta C, Sp)_T$$

should model "$(V, \times)$-enriched categories"

- Theorem says: $V$ Cartesian $\implies$ $V-\Theta Sp$ Cartesian.

- $\Theta_n$-spaces are obtained by iterating the $V \mapsto V-\Theta Sp$ construction, starting with $V = Sp$
The end

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