Power operations in Morava E-theory

a survey

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http://www.math.uiuc.edu/~rezk/midwest-2009-power-ops.pdf

What are power operations?

$h^* =$ multiplicative cohomology theory: $h^p(X) \otimes h^q(X) \to h^{p+q}(X)$.

m-th power map:

$$x \mapsto x^m \colon h^q(X) \to h^{mq}(X).$$

If h comes from a structured commutative ring spectrum, refine m-th power map to P^m:

$$\begin{array}{ccc} & & h^0(X \times B\Sigma_m) \\ & \overset{P^m}{\nearrow} & \big\downarrow {\scriptstyle [* \to B\Sigma_m]} \\ h^0(X) & \xrightarrow[x \mapsto x^m]{} & h^0(X) \end{array}$$

P_m is multiplicative, not additive.

Pairing with $\alpha \in h_0(B\Sigma_m)$ gives an operation $Q_\alpha \colon h^0(X) \to h^0(X)$.

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$R =$ commutative S-algebra.
$M =$ an R-module. Note: $[R, M]_R \approx [S, M]_S \approx \pi_0 M$.
Free commutative R-algebra on M:

$$\mathbb{P}_R M = \bigvee_{m \geq 0} \mathbb{P}_R^m M \approx \bigvee_{m \geq 0} \underbrace{(M \wedge_R \cdots \wedge_R M)}_{m \text{ times}}{}_{h\Sigma_m}$$

commutative R-algebra $A =$ algebra for the monad $\mathbb{P}_R$, determined by

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How to build a power operation

$A =$ commutative R-algebra.

- Choose $\alpha\colon S \to \mathbb{P}_R^m(R) \approx R \wedge B\Sigma_m^+$ (map of spectra).
- Represent $x \in \pi_0 A$ by $f_x\colon R \to A$.
- $$\mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f_x)} \mathbb{P}_R^m(A)$$

Remarks:

- $Q_\alpha\colon \pi_0 A \to \pi_0 A$ may not be additive or multiplicative.
- Can get $Q_\alpha\colon \pi_q A \to \pi_{q+r} A$ from

$$\alpha\colon \Sigma^{q+r} R \to \mathbb{P}_R^m(\Sigma^q R) \approx R \wedge B\Sigma_m^{qV_m}.$$

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Example 1: $H\mathbb{F}_2$

$H = H\mathbb{F}_2 =$ mod 2 Eilenberg Mac Lane spectrum.
$A =$ commutative H-algebra spectrum.
$\pi_* A$ is a graded commutative $\mathbb{F}_2$-algebra.

Operations on π_* of H-algebra

$Q^r : \pi_q A \to \pi_{q+r} A$ such that

- $Q^r(x+y) = Q^r(x) + Q^r(y)$.
- $Q^r Q^s(x) = \sum \epsilon^{i,j}_{r,s} Q^i Q^j(x)$ if $r > 2s$, where $i \leq 2j$.
- $Q^0(1) = 1$, $Q^r(1) = 0$ if $r \neq 0$.
- $Q^r(xy) = \sum Q^i(x) Q^{r-i}(y)$.
- $Q^r(x) = \begin{cases} x^2 & \text{if } r = q, \\ 0 & \text{if } r < q. \end{cases}$

$\pi_* \mathbb{P}_H(\Sigma^q H) \approx$ free gadget (with above structure) on one generator in dimension q. (See McClure in [BMMS 1986].)

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Example 1 (continued)

Special cases:

- **Cochains on a space.** $A = \mathrm{Func}(\Sigma^\infty_+ X, H\mathbb{F}_2) \rightsquigarrow$ power operations are **Steenrod operations** on $H^*(X, \mathbb{F}_2)$.
- **Chains on an infinite loop space.** $A = H\mathbb{F}_2 \wedge \Sigma^\infty_+ \Omega^\infty Y \rightsquigarrow$ power operations are **Kudo-Araki-Dyer-Lashof operations** on $H_*(\Omega^\infty Y, \mathbb{F}_2)$.

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Example 2: p-complete K-algebras [McClure]

K = complex K-theory spectrum.

p-**complete** K-**algebra**: commutative K-algebra A such that $A \approx A^\wedge_p$.

Operations on π_0 of p-complete K-algebra

$\psi^p\colon \pi_0 A \to \pi_0 A$ such that

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ψ^p and θ correspond to elements of $\alpha \in K^\wedge_0 B\Sigma_p$.

$$K^\wedge_q X \overset{\text{def}}{=} \pi_q \left((K \wedge X)^\wedge_p \right).$$

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ψ^p is the pth **Adams operation**.

Example 3: Morava E-theory ($n = 2, p = 2$)

- $C_0/\mathbb{F}_2$ = supersingular elliptic curve.
- $\widehat{C}_0$ = formal completion — formal group of height 2.
- E = Landweber exact spectrum associated to universal deformation of $\widehat{C}$.

$$\pi_* E \approx \mathbb{Z}_2[\![a]\!][u, u^{-1}], \qquad |a| = 0, |u| = 2.$$

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- $\theta \colon \pi_0 A \to \pi_0 A$ such that $Q_0(x) = x^2 + 2\,\theta(x)$

Example 3 (continued): The ring of power operations

The ring Γ of power operations

Associative ring containing $E_0 = \mathbb{Z}_2[\![a]\!]$ and generators Q_0, Q_1, Q_2, and subject to relations

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Γ has "admissible basis" as left $\mathbb{Z}_2[\![a]\!]$ module:

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Kashiwabara (1995): gives admissible basis for $\bar{\Gamma} = \mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![a]\!]} \Gamma$.
Problem: $\bar{\Gamma}$ is not a ring! (Kashiwabara knows this.)
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Example 3 (continued): Coproduct on Γ

"Cartan formula" is encoded by a coproduct.

Cocommutative coalgebra structure on Γ

$\epsilon\colon \Gamma \to E_0$ and $\Delta\colon \Gamma \to {}_{E_0}\Gamma \otimes {}_{E_0}\Gamma$ by

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(${}_{E_0}M \otimes {}_{E_0}N$ means tensor using left-module structures.)
Coproduct and product "commute".

Conclusion

Γ is a **twisted bialgebra** over E_0 (like a Hopf algebra, but E_0 isn't central).
Left Γ-modules have a symmetric monoidal tensor product: $M \otimes_{E_0} N$.

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Example 3: Summary

Definition

A Γ-**ring** is a commutative ring object in Γ-modules.

Definition

An **amplified** Γ-**ring** is a Γ-ring B equipped with $\theta\colon B \to B$ such that $Q_0(x) = x^2 + 2\,\theta(x)$ (together with formulas for $\theta(x+y)$, $\theta(xy)$, $\theta(ax)$).

In summary:

Proposition

For A a $K(2)$-local commutative E-algebra, $\pi_0 A$ naturally has the structure of an amplified Γ-ring.

$\pi_0 L_{K(2)} \mathbb{P}_E(E) \approx F^{\wedge}_{(2,a)}$, with $F =$ free amplified Γ-ring on one generator.

This can be extended to non-zero degrees:

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The general pattern

This is the general pattern for any Morava E-theory spectrum.

Power operations for Morava E-theory (height n, prime p)

π_* of a $K(n)$-local commutative E-algebra is a **graded amplified Γ-ring**:

- Γ is a certain twisted bialgebra over E_0.
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- $\pi_* L_{K(n)} \mathbb{P}_E(\Sigma^q E) \approx F^\wedge_{\mathfrak{m}}$,
  F = free graded amplified Γ-ring on one generator in dim. q.

Questions / topics

1. How does the formal group of E produce Γ? (Ando, Hopkins, Strickland)
2. Where does the "congruence" come from? (R.)
3. What is the algebraic structure of Γ? (quadratic? Koszul?) (R.)
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Formal group G_E of E

Formal scheme $G_E = \mathrm{Spf}(E^0\mathbb{CP}^\infty)$ over $\pi_0 E$.
Group law $G_E \times G_E \to G_E$ defined by

$$\mu^*\colon E^0\mathbb{CP}^\infty \to E^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \approx E^0\mathbb{CP}^\infty \widehat{\otimes}_{E_0} E^0\mathbb{CP}^\infty.$$

$\mu\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifies $\otimes$ of line bundles.

Additive and multiplicative transformation of functors:

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Topic 1: Deformations & Morava E-theory

Let G_0 = height n formal group over perfect field k, $\mathrm{char}\, k = p$, $n < \infty$.
Let R = complete local ring, $\pi\colon R \to R/\mathfrak{m}$.

Definition

A **deformation** of G_0 to R is (G, i, ψ):

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Theorem (Lubin-Tate)

There is a universal example of a deformation of G_0, defined over $E_0 \approx \mathbb{W}_p k[\![u_1, \dots, u_{n-1}]\!]$.

Theorem (Morava; Hopkins-Miller)

Given G_0/k, there is a corresponding even periodic commutative S-algebra $E = E_{G_0/k}$, whose formal group is the universal deformation of G_0.

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commute for some $r \geq 0$.
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$$E^0 X \xrightarrow{P^m} E^0(X \times B\Sigma_m)$$

Künneth isomorphism, if $E^0 B\Sigma_m$ is finite and flat over E_0 (true for Morava E-theory).

I is the "transfer ideal":

$$I = \sum_{0<i<m} \mathrm{Image}\left[E^0 B(\Sigma_i \times \Sigma_{m-i}) \xrightarrow{\text{transfer}} E^0 B\Sigma_m\right].$$

Proposition

$\tau P^m \colon E^0 X \to E^0 X \otimes_{E^0} E^0 B\Sigma_m / I$ is a ring homomorphism.

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Topic 1: The associated homomorphism

Let $(F_m)^0(X) = E^0 X \otimes_{E^0} E^0 B\Sigma_m / I$.

Ring homomorphisms:

- $s^*\colon E_0 \to (F_m)_0$, induced by $B\Sigma_m \to *$.
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- $t^*\colon E_0 \to (F_m)_0$, defined by $\tau P^m\colon E^0(*) \to E^0(*) \otimes_{E_0} E^0(B\Sigma_m)/I$.

The ring operation

$$E^0(X) \xrightarrow{\tau P^m} (F_m)^0(X)$$

produces a homomorphism of formal groups defined over $(F_m)_0$.

What is this homomorphism?

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Topic 1: The associated homomorphism

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Topic 1: Deformations of Frobenius, revisited

Let $m = p^r$, $r > 0$. Let $j\colon * \to B\Sigma_m$.

$$\begin{array}{ccccc}
E^0X & \xrightarrow{P^{p^r}} & E^0X \otimes_{E_0} E^0(B\Sigma_{p^r}) & \xrightarrow{\tau} & E^0X \otimes_{E_0} E^0B\Sigma_{p^r}/I \\
& \searrow^{x \mapsto x^{p^r}} & \downarrow{\scriptstyle \mathrm{id}\otimes j^*} & & \\
& & E^0X \otimes_{E_0} E^0(*) & &
\end{array}$$

Using the "double coset formula", have

$$E^0B\Sigma_{p^r}/(I + \mathrm{Ker}(j^*)) \approx E_0/p.$$

Thus

$$\pi\tau P^{p^r}(x) = x^{p^r} \qquad (\text{in } E^0X/(p)).$$

Conclusion

$(\tau P^{p^r})^*\colon s^*G_E \to t^*G_E$ is a deformation of Frobenius.

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Theorem (Strickland (1998))

The homomorphism $(\tau P_{p^r})^* \colon s^* G_E \to t^* G_E$ over $(F_{p^r})_0$ is the universal example of a deformation of Frob^r between deformations of G_0.

Remember: deformations of Frobenius correspond to finite subgroups of the domain.

Strickland actually proved the following statement:

Theorem (Strickland (1998))

The data $(s^* G_E, \mathrm{Ker}(\tau P_{p^r})^*)$ over $(F_{p^r})_0$ is the universal example of a pair (G, H) consisting of a deformation G of G_0 and a finite subgroup scheme $H \subset G$ of rank $m = p^r$.

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Topic 1: Descent (Ando-Hopkins-Strickland (mid 90s?))

$$\mathcal{D}(R) = \begin{cases} \text{Objects: deformations } (G, i, \phi) \text{ of } G_0/k \text{ to } R, \\ \text{Morphisms: deformations of Frobenius.} \end{cases}$$

$f\colon R \to R' \implies f^*\colon \mathcal{D}(R) \to \mathcal{D}(R')$.

Definition

A **sheaf of modules** M on $\mathcal{D} = \{\mathcal{D}(R)\}$ consists of

- functors $M_R\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathrm{Mod}_R$,
- natural isomorphisms $M_f\colon R' \otimes_R M_R \xrightarrow{\sim} M_{R'} \circ f^*$,

satisfying obvious "coherence" axioms.
$\implies$ symmetric monoidal category $\mathrm{Mod}_{\mathcal{D}}$ of sheaves of modules.

Let Γ = ring of **additive power operations** for E.
That is, $\Gamma \subset \bigoplus_{m\geq 0} E_0^{\wedge} B\Sigma_m$ consisting of α such that Q_α is additive.

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Example 3, revisited: Definition of Q_i

- $C_0/\mathbb{F}_2$ = elliptic curve in $\mathbb{P}^2$ defined by $Y^2Z + YZ^2 = X^3$.
  $\implies E = E_{\widehat{C_0/\mathbb{F}_2}}$.

Proposition

$$(F_2)_0 = E^0 B\Sigma_2/I \approx (\mathbb{Z}_2[\![a]\!])[d]/(d^3 - ad - 2).$$

- Write

$$E^0 X \xrightarrow{\tau P^2} (E^0 X)[d]/(d^3 - ad - 2)$$

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$$x \mapsto \tau P^2(x) = Q_0(x) + Q_1(x)\, d + Q_2(x)\, d^2.$$

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Example 3, revisited: Subgroups of order 2

- Universal deformation of $C_0/\mathbb{F}_2$:
  C/E_0 = elliptic curve defined over $E_0 = \mathbb{Z}_2[\![a]\!]$, by

$$Y^2Z + a\,XYZ + YZ^2 = X^3.$$

- Affine chart: $u = X/Y, v = Z/Y$.

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  (Basepoint is at $(u, v) = (0, 0)$.)
- Subgroup schemes of rank 2: "generated" by points P of C of form $(u(P), v(P)) = (d, -d^3)$ such that $d^3 - ad - 2 = 0$.
- These are also finite subgroups of the formal completion $\widehat{C}$, so

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- Given $P \in C$ with $(u(P), v(P)) = (d, -d^3)$, $d^3 - ad - 2 = 0$ $\Longrightarrow$ isogeny $\psi_P \colon C \to C'$ such that $\mathrm{Ker}(\psi_P) = \langle P \rangle$.

$$\{v + a\,uv + v^2 = u^3\} \to \{v' + (a^2 + 3d - ad^2)\,u'v' + v'^2 = u'^3\}$$

- Definition of ψ_P: if $Q' = \psi_P(Q)$, then

$$u'(Q') = -u(Q)u(Q+P), \qquad v'(Q') = v(Q)v(Q+P).$$

- By construction, ψ_P is a deformation of Frobenius: if $d = 0$, then $u'(Q') = u(Q)^2$ and $v'(Q') = v(Q)^2$.
- $\Longrightarrow$ computation of $t^* \colon E_0 \to (F_2)_0$:

$$t^*(a) = \tau P^2(a) = a^2 + 3d - ad^2,$$

- $\tau P^2(ax) = \tau P^2(a) \cdot \tau P^2(x) \Longrightarrow$

$$Q_0(ax) + Q_1(ax)\,d + Q_2(ax)\,d^2 = (a^2 + 3d - ad^2)\,(Q_0(x) + Q_1(x)\,d + Q_2(x)\,d^2).$$

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In Example 3, we have

Proposition

$$Q_0(x) \equiv x^2 \mod 2.$$

Remember:

$$\begin{array}{ccccc}
E^0X & \xrightarrow{P_{p^r}} & E^0X \otimes_{E_0} E^0(B\Sigma_{p^r}) & \xrightarrow{\tau} & E^0X \otimes_{E_0} E^0B\Sigma_{p^r}/I \\
& \searrow^{x \mapsto x^{p^r}} & \downarrow{\scriptstyle \mathrm{id}\otimes j^*} & & \downarrow{\scriptstyle \pi} \\
& & E^0X \otimes_{E_0} E^0(*) & \longrightarrow & E^0X \otimes_{E_0} E_0/p
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Formula:

$$(\tau P^2)(x) = Q_0(x) + Q_1(x)\,d + Q_2(x)\,d^2,$$

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Topic 2: The Frobenius congruence (Example 3)

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When $R \supset \mathbb{F}_p$, there is a relative Frobenius homomorphism

$$\mathrm{Frob}\colon G \to \phi^* G$$

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Observation

Universal example of $\mathrm{Frob}\colon G \to \phi^* G$ is determined by

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Topic 2: Frobenius is a deformation of Frobenius

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Topic 2: Frobenius congruence for sheaves

Definition

A sheaf of commutative rings B on $\mathcal{D}$ is a **Frobenius sheaf** if for every $R \supset \mathbb{F}_p$ and $G \in \mathcal{D}(R)$,

$$B_R(G) \xrightarrow{B_R(\mathrm{Frob})} B_R(\phi^* G) \approx R^\phi \otimes_R B_R(G)$$

is the relative Frobenius homomorphism of R-algebras.

Theorem (R.)

There is a functor

$$\{\text{amplified } \Gamma\text{-rings}\} \to \{\text{Frobenius sheaves on } \mathcal{D}\}$$

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Topic 3: Koszul algebras

$A = \bigoplus_{r \geq 0} A_r$ graded associative ring, $A_0 = R$ commutative.

Definition

A is **Koszul** if there exist R-modules C_r with $C_0 = R$, and an exact sequence (a "Koszul complex")

$$\cdots \xrightarrow{d} A \otimes_R C_3 \xrightarrow{d} A \otimes_R C_2 \xrightarrow{d} A \otimes_R C_1 \xrightarrow{d} A \otimes_R C_0 \xrightarrow{d} R \to 0$$

of left A-modules such that d raises degree by 1.

Fact

If A is Koszul, then

$$A \approx T_R(A_1)/(U), \qquad U \subset A_2$$

(i.e., A is "quadratic".)

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- Back to the example: $\Gamma \approx \bigoplus \Gamma_r \approx T_{E_0}(\Gamma_1)/(U)$, where $\Gamma_1 = E_0\{Q_0, Q_1, Q_2\}$, $U =$ Adem relations.
- **PBW Theorem** (Priddy (1970)): if Γ has a "nice" admissible basis, then Γ is Koszul.
- $\Longrightarrow$ Exact sequence.

$$0 \to \Gamma \otimes_{E_0} C_2 \to \Gamma \otimes_{E_0} C_1 \to \Gamma \to E_0 \to 0.$$

C_i are free modules over E_0: $\mathrm{rank}\, C_1 = 3$, $\mathrm{rank}\, C_2 = 2$.

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Topic 3: Is Γ always Koszul?

Conjecture (Ando-Hopkins-Strickland (mid 90s?))

For all $E = E_{G_0/k}$, the associated ring Γ of power operations is Koszul. The associated Koszul complex has the form

$$0 \to \Gamma \otimes_{E_0} C_n \to \cdots \to \Gamma \otimes_{E_0} C_1 \to \Gamma \to E_0 \to 0,$$

where $n =$ height of G_0.

- They developed a program to prove the result, using interesting ideas about a kind of "Bruhat-Tits building" formed using flags of certain finite subgroup schemes of G_E.
- I don't believe they ever completed their program; there is probably no obstruction to doing so, however.
- There is another proof, which avoids using formal group theory; it uses ideas related to the Whitehead conjecture (Kuhn, Mitchell, Priddy) and calculus (Arone-Mahowald, Arone-Dwyer).

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Topic 3: Linearization

Here are some of the ideas in the proof.

Definition

Given a (nonadditive) functor $F\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$, the **linearization** $\mathcal{L}[F]\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$ is

$$\mathcal{L}[F](M) = \mathrm{Cok}\left[\; F(M \oplus M) \underset{F(\pi_1)+F(\pi_2)}{\overset{F(\pi_1+\pi_2)}{\rightrightarrows}} F(M) \;\right].$$

$\mathcal{L}[F]$ is initial additive quotient functor of F.

In some cases, including ours, $\mathcal{L}[F \circ G] \to \mathcal{L}[F] \circ \mathcal{L}[G]$ is an isomorphism.

Topic 3: Linearization of the amplified Γ-ring monad

- $F\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$ the free amplified Γ-ring functor.
- For E-module M with $\pi_* M =$ free E_*-module concentrated in even degree,

$$F(\pi_0 M) \approx \bigoplus_{m \geq 0} \pi_0 L_{K(n)} \mathbb{P}_E^m(M).$$

- $$\mathcal{L}[F](E_0) = \Delta.$$

$$\mathcal{L}[F \circ \cdots \cdots \circ F](E_0) = \Delta \otimes_{E_0} \cdots \otimes_{E_0} \Delta.$$

  Δ is a ring, non-canonically isomorphic to Γ.
- Monadic bar construction $\mathcal{B}_\bullet(F, F, F)$.

$$\mathcal{L}\left[\mathcal{B}_\bullet(F, F, F)\right] \approx \mathcal{B}_\bullet(\Delta, \Delta, \Delta).$$

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(Priddy 1970):

- If Δ is a graded ring, filter $\mathcal{B}_\bullet(M, \Delta, N)$ according to grading on Δ.
- Δ is **Koszul** if $\mathrm{gr}_q \mathcal{B}_\bullet(E_0, \Delta, E_0)$ has homology concentrated in degree q.
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Topic 3: Partition poset

- $$\mathcal{B}_q(F,F,F)(E_0) \approx (F \circ \cdots \circ F)(E_0) \approx \bigoplus_{m \geq 0} E_0^\wedge(K_q(m)_{h\Sigma_m}).$$

  $K_\bullet(m)$ is the **partition complex**:

  $$K_\bullet(m) = \mathrm{nerve}\,\{\text{poset of partitions of } \{1,\ldots,m\}\}\,.$$

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Topic 3: The idea of the proof

- $\overline{K}_\bullet(m) = K_\bullet(m)/\sim$, associated to $\mathcal{B}_\bullet(E_0, \Delta, E_0) \approx \mathcal{B}_\bullet(\Delta, \Delta, \Delta)/\sim$.
- $Q_m(\overline{K}_\bullet(m)) = 0$ if $m \neq p^r$.
- Need to show $Q_{p^r}(\overline{K}_\bullet(p^r))$ has H_* concentrated in degree r.
- $$K_\bullet(p^r) \times \Sigma_{p^r}/(\Sigma_p \wr \cdots \wr \Sigma_p) \twoheadrightarrow K_\bullet(p^r),$$

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We return to the main example (height 2, prime 2).

- $\Psi \in \Gamma$ is element corresponding to the operation:

$$E^0X \xrightarrow{\tau P^4} E^0X \otimes_{E_0} (F_4)_0 \xrightarrow{\mathrm{id}\otimes\rho} E^0X \otimes_{E_0} E_0$$

where $\rho\colon (F_4)_0 \to E_0$ classifies $[-2]\colon G_E \to G_E$
(since $[-2](x) \equiv x^4 \mod (2, a)$, it is a deformation of Frob^2.)

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$$\frac{x^2\,\Psi(x)}{N(x)} \equiv 1 \mod 2.$$

- For any 2-complete amplified Γ-ring, get a homomorphism

$$\ell\colon B^\times \to B,$$
$$x \mapsto \frac{1}{2}\log\left[\frac{x^2\,\Psi(x)}{N(x)}\right].$$

- A = a $K(2)$-local commutative E-algebra, there is a map of spectra

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On π_0, this map is given by ℓ.

- This works in a similar way at all heights and primes.

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Topic 4: Hecke operators

Now E is a general Morava E-theory (height n, prime p).

- Elements $\tilde{T}(p^k) \in \Gamma$, given by

$$E^0 X \xrightarrow{\tau P^{p^r}} E^0 X \otimes_{E_0} (F_{p^r})_0 \xrightarrow{\text{Trace}} E^0 X.$$

  (First constructed by Ando (1992).)

- $\{\tilde{T}(p^k)\}$ generate a commuative subring $\mathbb{Z}_p[\tilde{T}_1, \ldots, \tilde{T}_n] \subset \Gamma$, where

$$\sum_{r=0}^{n} (-1)^r p^{r(r-1)/2} \tilde{T}_r \cdot U^r = \left(\sum_{k \geq 0} \tilde{T}(p^k) \cdot U^k \right)^{-1}$$

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- $\{\tilde{T}(p^k)\}$ generate a commuative subring $\mathbb{Z}_p[\tilde{T}_1, \ldots, \tilde{T}_n] \subset \Gamma$, where

$$\sum_{r=0}^{n} (-1)^r p^{r(r-1)/2} \tilde{T}_r \cdot U^r = \left(\sum_{k \geq 0} \tilde{T}(p^k) \cdot U^k \right)^{-1}$$

  in $\Gamma[\![U]\!]$.

Different construction of $\tilde{T}(p^k)$, due to Ganter.

G = finite group.

- The $K(n)$-local Tate homology of BG vanishes (Hovey-Strickland (1999)):
$$L_{K(n)}BG_+ \xrightarrow{\sim} \mathcal{F}(BG_+, L_{K(n)}S).$$
- $\implies L_{K(n)}BG_+$ is a **commutative Frobenius algebra** in the $K(n)$-local homotopy category (Strickland (2000)).
(analogy between $\mathcal{F}(BG_+, L_{K(n)}S)$ and representation ring RG.)
- Let $I_G \colon L_{K(n)}S \to L_{K(n)}BG_+$, dual to $L_{K(n)}BG_+ \to L_{K(n)}S$,
(analogous to $\frac{1}{|G|}\mathrm{Trace}\sum_{g\in G} g \colon RG \to \mathbb{Z}$.)

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- Define σ^m by

$$E^0X \xrightarrow{P^m} E^0X \otimes_{E_0} E^0B\Sigma_m \xrightarrow{\mathrm{id}\otimes I^*_{\Sigma_m}} E^0X \otimes_{E_0} E_0.$$

- $\sigma^m \colon B \to B$ are non-additive functions, analogous to symmetric powers of representations.

Theorem (Ganter (2004))

$$\exp\left(\sum_{k\geq 0} \frac{\tilde{T}(p^k)(x)}{p^k} \cdot U^{p^k}\right) = \sum_{m\geq 0} \sigma^m(x)\cdot U^m.$$

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Topic 4: Hecke operators on $K(n)$-local S-algebras

Let $R =$ a $K(n)$-local S-algebra.

- Ganter's operations σ^m are defined on $\pi_0 R$ for any $K(n)$-local S algebra. (They are defined using a homotopy class in $\pi_0 L_{K(n)} B\Sigma_m^+$.)
- $\implies$ Ganter's formula gives a definition of Hecke operators on $\pi_0 R$ for any $K(n)$-local S-algebra.
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The end

http://www.math.uiuc.edu/~rezk/midwest-2009-power-ops.pdf

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