Power operations in Morava E-theory

a survey

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May 2, 2009

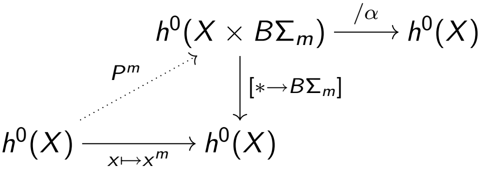
http://www.math.uiuc.edu/~rezk/midwest-2009-power-ops.pdf

What are power operations?

$h^* =$ multiplicative cohomology theory: $h^p(X) \otimes h^q(X) \to h^{p+q}(X)$.
m-th power map:

$$x \mapsto x^m : h^q(X) \to h^{mq}(X).$$

If h comes from a structured commutative ring spectrum, refine m-th power map to P^m:



P_m is multiplicative, not additive.
Pairing with $\alpha \in h_0(B\Sigma_m)$ gives an operation $Q_\alpha : h^0(X) \to h^0(X)$.

Power operations from commutative R-algebras

$R =$ commutative S-algebra.
$M =$ an R-module. Note: $[R, M]_R \approx [S, M]_S \approx \pi_0 M$.
Free commutative R-algebra on M:

$$\mathbb{P}_R M = \bigvee_{m \geq 0} \mathbb{P}_R^m M \approx \bigvee_{m \geq 0} \underbrace{(M \wedge_R \cdots \wedge_R M)}_{m \text{ times}}{}_{h\Sigma_m}$$

commutative R-algebra $A =$ algebra for the monad $\mathbb{P}_R$, determined by

$$\mu \colon \mathbb{P}_R A \to A.$$

$A =$ commutative R-algebra.

- Choose $\alpha\colon S \to \mathbb{P}_R^m(R) \approx R \wedge B\Sigma_m^+$ (map of spectra).
- Represent $x \in \pi_0 A$ by $f_x\colon R \to A$.
- $$R \xrightarrow{\alpha} \mathbb{P}_R^m(R) \xrightarrow{\mathbb{P}_R^m(f_x)} \mathbb{P}_R^m(A) \subset \mathbb{P}_R(A) \xrightarrow{\mu} A$$

  $Q_\alpha(x) \in \pi_0 A$ represented by composite.

Remarks:

- $Q_\alpha\colon \pi_0 A \to \pi_0 A$ may not be additive or multiplicative.
- Can get $Q_\alpha\colon \pi_q A \to \pi_{q+r} A$ from

  $$\alpha\colon \Sigma^{q+r} R \to \mathbb{P}_R^m(\Sigma^q R) \approx R \wedge B\Sigma_m^{qV_m}.$$

Example 1: $H\mathbb{F}_2$

$H = H\mathbb{F}_2 =$ mod 2 Eilenberg Mac Lane spectrum.
$A =$ commutative H-algebra spectrum.
$\pi_* A$ is a graded commutative $\mathbb{F}_2$-algebra.

Operations on π_* of H-algebra

$Q^r : \pi_q A \to \pi_{q+r} A$ such that

- $Q^r(x+y) = Q^r(x) + Q^r(y)$.
- $Q^r Q^s(x) = \sum \epsilon_{r,s}^{i,j} Q^i Q^j(x)$ if $r > 2s$, where $i \leq 2j$.
- $Q^0(1) = 1$, $Q^r(1) = 0$ if $r \neq 0$.
- $Q^r(xy) = \sum Q^i(x) Q^{r-i}(y)$.
- $Q^r(x) = \begin{cases} x^2 & \text{if } r = q, \\ 0 & \text{if } r < q. \end{cases}$

$\pi_* \mathbb{P}_H(\Sigma^q H) \approx$ free gadget (with above structure) on one generator in dimension q. (See McClure in [BMMS 1986].)

Example 1 (continued)

Special cases:

- **Cochains on a space.** $A = \mathrm{Func}(\Sigma_+^\infty X, H\mathbb{F}_2) \rightsquigarrow$ power operations are **Steenrod operations** on $H^*(X, \mathbb{F}_2)$.
- **Chains on an infinite loop space.** $A = H\mathbb{F}_2 \wedge \Sigma_+^\infty \Omega^\infty Y \rightsquigarrow$ power operations are **Kudo-Araki-Dyer-Lashof operations** on $H_*(\Omega^\infty Y, \mathbb{F}_2)$.

Example 2: p-complete K-algebras [McClure]

K = complex K-theory spectrum.

p-**complete** K-**algebra**: commutative K-algebra A such that $A \approx A_p^\wedge$.

Operations on π_0 of p-complete K-algebra

$\psi^p\colon \pi_0 A \to \pi_0 A$ such that

- $\psi^p(x+y) = \psi^p(x) + \psi^p(y)$.
- $\psi^p(1) = 1$.
- $\psi^p(xy) = \psi^p(x)\psi^p(y)$.
- $\theta\colon \pi_0 A \to \pi_0 A$ such that $\psi^p(x) = x^p + p\,\theta(x)$.

ψ^p and θ correspond to elements of $\alpha \in K_0^\wedge B\Sigma_p$.

$$K_q^\wedge X \stackrel{\text{def}}{=} \pi_q\left((K \wedge X)_p^\wedge\right).$$

ψ^p is the pth **Adams operation**.

Example 3: Morava E-theory ($n = 2, p = 2$)

- $C_0/\mathbb{F}_2$ = supersingular elliptic curve.
- $\widehat{C}_0$ = formal completion — formal group of height 2.
- E = Landweber exact spectrum associated to universal deformation of $\widehat{C}$.

  $$\pi_* E \approx \mathbb{Z}_2[\![a]\!][u, u^{-1}], \qquad |a| = 0, |u| = 2.$$

  Note: $K(2) \approx E/(2, a)$ (Morava K-theory).
- E is a commutative S-algebra (Hopkins-Miller Theorem).
- Power operations constructed by Ando (1992).

Next slide: calculation of the algebraic structure of power operations for $K(2)$-local commutative E-algebras (R., prefigured by Kashiwabara 1995).

Example 3 (continued): Formulas

$A = K(2)$-local commutative E-algebra ($\pi_0 A$ is an $E_0 = \mathbb{Z}_2[\![a]\!]$-algebra).

Operations on π_0 of $K(2)$-local E-algebra

$Q_0, Q_1, Q_2 \colon \pi_0 A \to \pi_0 A$ such that

- $Q_i(x+y) = Q_i(x) + Q_i(y)$
- $Q_0(ax) = a^2\, Q_0(x) - 2a\, Q_1(x) + 6\, Q_2(x)$
  $Q_1(ax) = 3\, Q_0(x) + a\, Q_2(x)$
  $Q_2(ax) = -a\, Q_0(x) + 3\, Q_1(x)$
- $Q_1 Q_0(x) = 2\, Q_2 Q_1(x) - 2\, Q_0 Q_2(x)$
  $Q_2 Q_0(x) = Q_0 Q_1(x) + a\, Q_0 Q_2(x) - 2\, Q_1 Q_2(x)$
- $Q_0(1) = 1,\ Q_1(1) = Q_2(1) = 0$
- $Q_0(xy) = Q_0 x Q_0 y + 2 Q_1 x Q_2 y + 2 Q_2 x Q_1 y$
  $Q_1(xy) = Q_0 x Q_1 y + Q_1 x Q_0 y + a Q_1 x Q_2 y + a Q_2 x Q_1 y + 2 Q_2 x Q_2 y$
  $Q_2(xy) = Q_0 x Q_2 y + Q_2 x Q_0 y + Q_1 x Q_1 y + a Q_2 x Q_2 y$
- $\theta \colon \pi_0 A \to \pi_0 A$ such that $Q_0(x) = x^2 + 2\,\theta(x)$

Example 3 (continued): The ring of power operations

The ring Γ of power operations

Associative ring containing $E_0 = \mathbb{Z}_2[\![a]\!]$ and generators Q_0, Q_1, Q_2, and subject to relations

$$Q_0\,a = a^2\,Q_0 - 2a\,Q_1 + 6\,Q_2$$
$$Q_1\,a = 3\,Q_0 + a\,Q_2$$
$$Q_2\,a = -a\,Q_0 + 3\,Q_1$$

$$Q_1 Q_0 = 2\,Q_2 Q_1 - 2\,Q_0 Q_2$$
$$Q_2 Q_0 = Q_0 Q_1 + a\,Q_0 Q_2 - 2\,Q_1 Q_2$$

Γ has "admissible basis" as left $\mathbb{Z}_2[\![a]\!]$ module:

$$Q_0^i Q_{j_1} \cdots Q_{j_r}, \qquad i \geq 0,\ j_k \in \{1, 2\}$$

Kashiwabara (1995): gives admissible basis for $\bar{\Gamma} = \mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![a]\!]} \Gamma$.
Problem: $\bar{\Gamma}$ is not a ring! (Kashiwabara knows this.)
He describes ring structure modulo indeterminacy.

Example 3 (continued): Coproduct on Γ

"Cartan formula" is encoded by a coproduct.

Cocommutative coalgebra structure on Γ

$\epsilon\colon \Gamma \to E_0$ and $\Delta\colon \Gamma \to {}_{E_0}\Gamma \otimes {}_{E_0}\Gamma$ by

$$\epsilon(Q_0) = 1, \qquad \epsilon(Q_1) = 0 = \epsilon(Q_2)$$

$$\begin{aligned}
\Delta(Q_0) &= Q_0 \otimes Q_0 + 2Q_1 \otimes Q_2 + 2Q_2 \otimes Q_1 \\
\Delta(Q_1) &= Q_0 \otimes Q_1 + Q_1 \otimes Q_0 + aQ_1 \otimes Q_2 + aQ_2 \otimes Q_1 + 2Q_2 \otimes Q_2 \\
\Delta(Q_2) &= Q_0 \otimes Q_2 + Q_2 \otimes Q_0 + Q_1 \otimes Q_1 + aQ_2 \otimes Q_2
\end{aligned}$$

(${}_{E_0}M \otimes {}_{E_0}N$ means tensor using left-module structures.)
Coproduct and product "commute".

Conclusion

Γ is a **twisted bialgebra** over E_0 (like a Hopf algebra, but E_0 isn't central).
Left Γ-modules have a symmetric monoidal tensor product: $M \otimes_{E_0} N$.

Example 3: Summary

Definition

A Γ-**ring** is a commutative ring object in Γ-modules.

Definition

An **amplified** Γ-**ring** is a Γ-ring B equipped with $\theta\colon B \to B$ such that $Q_0(x) = x^2 + 2\,\theta(x)$ (together with formulas for $\theta(x+y)$, $\theta(xy)$, $\theta(ax)$).

In summary:

Proposition

For A a $K(2)$-local commutative E-algebra, $\pi_0 A$ naturally has the structure of an amplified Γ-ring.
$\pi_0 L_{K(2)} \mathbb{P}_E(E) \approx F^{\wedge}_{(2,a)}$, with $F =$ free amplified Γ-ring on one generator.

This can be extended to non-zero degrees:
$\pi_* A$ is a **graded amplified** Γ-**ring**, etc.

The general pattern

This is the general pattern for any Morava E-theory spectrum.

Power operations for Morava E-theory (height n, prime p)

π_* of a $K(n)$-local commutative E-algebra is a **graded amplified Γ-ring**:

- Γ is a certain twisted bialgebra over E_0.
- $Q_0 \in \Gamma$ and θ such that $Q_0(x) = x^p + p\,\theta(x)$.
- $\pi_* L_{K(n)} \mathbb{P}_E(\Sigma^q E) \approx F^\wedge_{\mathfrak{m}}$,
  $F =$ free graded amplified Γ-ring on one generator in dim. q.

Questions / topics

1. How does the formal group of E produce Γ? (Ando, Hopkins, Strickland)
2. Where does the "congruence" come from? (R.)
3. What is the algebraic structure of Γ? (quadratic? Koszul?) (R.)
4. Logarithms and Hecke operators. (R., Ganter)

Topic 1: Formal groups and operations

E = even periodic ring spectrum $\Longrightarrow$ formal group G_E.

Formal group G_E of E

Formal scheme $G_E = \mathrm{Spf}(E^0\mathbb{CP}^\infty)$ over $\pi_0 E$.
Group law $G_E \times G_E \to G_E$ defined by

$$\mu^*\colon E^0\mathbb{CP}^\infty \to E^0(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \approx E^0\mathbb{CP}^\infty \widehat{\otimes}_{E_0} E^0\mathbb{CP}^\infty.$$

$\mu\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifies $\otimes$ of line bundles.

Additive and multiplicative transformation of functors:

$$E^0(X) \xrightarrow{\psi} F^0(X) \qquad \Longrightarrow \qquad g^* G_E \xleftarrow{\psi^*} G_F$$

ψ^* = homomorphism of formal groups over F_0,
where $g = \psi\colon E^0(*) \to F^0(*)$.

Topic 1: Deformations & Morava E-theory

Let $G_0 =$ height n formal group over perfect field k, $\mathrm{char}\, k = p$, $n < \infty$.
Let $R =$ complete local ring, $\pi\colon R \to R/\mathfrak{m}$.

Definition

A **deformation** of G_0 to R is (G, i, ψ):

- G a formal group over R,
- $i\colon k \to R/\mathfrak{m}$,
- $\psi\colon \pi^* G \xrightarrow{\sim} i^* G_0$ iso of formal groups over $R/\mathfrak{m}$.

Theorem (Lubin-Tate)

There is a universal example of a deformation of G_0, defined over $E_0 \approx \mathbb{W}_p k[\![u_1, \dots, u_{n-1}]\!]$.

Theorem (Morava; Hopkins-Miller)

Given G_0/k, there is a corresponding even periodic commutative S-algebra $E = E_{G_0/k}$, whose formal group is the universal deformation of G_0.

Topic 1: Deformations of Frobenius

Frobenius. $\phi\colon k \to k$ defined by $\phi(x) = x^p$.
Relative Frobenius. $\mathrm{Frob}\colon G_0 \to \phi^* G_0$.

Definition

A **deformation of Frobenius** $(G, i, \psi) \to (G', i', \psi')$ (of deformations of G_0 to R) is a homomorphism $f\colon G \to G'$ of formal groups over R, such that

$$\begin{array}{ccc} \pi^* G & \xrightarrow{\pi^*(f)} & \pi^* G' \\ {\scriptstyle\psi}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle\psi'} \\ i^* G_0 & \xrightarrow[i^*(\mathrm{Frob}^r)]{} & i'^* G_0 \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{i'} & R/\mathfrak{m} \\ {\scriptstyle\phi^r}\downarrow & \nearrow{\scriptstyle i} & \\ k & & \end{array}$$

commute for some $r \geq 0$.
($\pi\colon R \to R/\mathfrak{m}$.)

Remark: Deformations of Frobenius with domain (G, i, ψ) correspond *exactly* to finite subgroup schemes of G. ($f \rightsquigarrow \mathrm{Ker}(f) \subset G$.)

Topic 1: Formal groups and power operations

$E = E_{G_0/k}$. Power map:

$$E^0X \xrightarrow{P^m} E^0(X) \otimes_{E_0} E^0(B\Sigma_m) \xrightarrow{\tau} E^0X \otimes_{E_0} E^0B\Sigma_m/I$$

Künneth isomorphism, if $E^0B\Sigma_m$ is finite and flat over E_0 (true for Morava E-theory).
I is the "transfer ideal":

$$I = \sum_{0<i<m} \mathrm{Image}\left[E^0B(\Sigma_i \times \Sigma_{m-i}) \xrightarrow{\text{transfer}} E^0B\Sigma_m\right].$$

Proposition

$\tau P^m\colon E^0X \to E^0X \otimes_{E^0} E^0B\Sigma_m/I$ is a ring homomorphism.

Remark: $E^0B\Sigma_m/I = 0$ unless $m = p^r$.

Let $(F_m)^0(X) = E^0X \otimes_{E^0} E^0B\Sigma_m/I$.

Ring homomorphisms:

- $s^*\colon E_0 \to (F_m)_0$, induced by $B\Sigma_m \to *$.
- $t^*\colon E_0 \to (F_m)_0$, defined by $\tau P^m\colon E^0(*) \to E^0(*) \otimes_{E_0} E^0(B\Sigma_m)/I$.

The ring operation

$$E^0(X) \xrightarrow{\tau P^m} (F_m)^0(X) \qquad \Longrightarrow \qquad t^*G_E \xleftarrow{(\tau P^m)^*} s^*G_E$$

produces a homomorphism of formal groups defined over $(F_m)_0$.

What is this homomorphism?

Topic 1: Deformations of Frobenius, revisited

Let $m = p^r$, $r > 0$. Let $j\colon * \to B\Sigma_m$.

$$
\begin{array}{ccccc}
E^0X & \xrightarrow{P^{p^r}} & E^0X \otimes_{E_0} E^0(B\Sigma_{p^r}) & \xrightarrow{\tau} & E^0X \otimes_{E_0} E^0B\Sigma_{p^r}/I \\
 & \searrow^{x \mapsto x^{p^r}} & \big\downarrow \scriptstyle{\mathrm{id}\otimes j^*} & & \big\downarrow \scriptstyle{\pi} \\
 & & E^0X \otimes_{E_0} E^0(*) & \longrightarrow & E^0X \otimes_{E_0} E_0/p
\end{array}
$$

Using the "double coset formula", have

$$E^0B\Sigma_{p^r}/(I + \mathrm{Ker}(j^*)) \approx E_0/p.$$

Thus

$$\pi\tau P^{p^r}(x) = x^{p^r} \qquad (\text{in } E^0X/(p)).$$

Conclusion

$(\tau P^{p^r})^*\colon s^*G_E \to t^*G_E$ is a deformation of Frobenius.

Theorem (Strickland (1998))

The homomorphism $(\tau P_{p^r})^*\colon s^*G_E \to t^*G_E$ over $(F_{p^r})_0$ is the universal example of a deformation of Frob^r between deformations of G_0.

Remember: deformations of Frobenius correspond to finite subgroups of the domain.

Strickland actually proved the following statement:

Theorem (Strickland (1998))

The data $(s^*G_E, \mathrm{Ker}(\tau P_{p^r})^*)$ over $(F_{p^r})_0$ is the universal example of a pair (G, H) consisting of a deformation G of G_0 and a finite subgroup scheme $H \subset G$ of rank $m = p^r$.

Topic 1: Descent (Ando-Hopkins-Strickland (mid 90s?))

$$\mathcal{D}(R) = \begin{cases} \text{Objects: deformations } (G, i, \phi) \text{ of } G_0/k \text{ to } R, \\ \text{Morphisms: deformations of Frobenius.} \end{cases}$$

$$f\colon R \to R' \qquad \Longrightarrow \qquad f^*\colon \mathcal{D}(R) \to \mathcal{D}(R').$$

Definition

A **sheaf of modules** M on $\mathcal{D} = \{\mathcal{D}(R)\}$ consists of

- functors $M_R\colon \mathcal{D}(R)^{\text{op}} \to \mathrm{Mod}_R$,
- natural isomorphisms $M_f\colon R' \otimes_R M_R \xrightarrow{\sim} M_{R'} \circ f^*$,

satisfying obvious "coherence" axioms.

$\Longrightarrow$ symmetric monoidal category $\mathrm{Mod}_{\mathcal{D}}$ of sheaves of modules.

Let $\Gamma =$ ring of **additive power operations** for E.

That is, $\Gamma \subset \bigoplus_{m\geq 0} E_0^\wedge B\Sigma_m$ consisting of α such that Q_α is additive.

Theorem

Equivalence $\mathrm{Mod}_{\mathcal{D}} \approx \mathrm{Mod}_\Gamma$ of symmetric monoidal categories.

Example 3, revisited: Definition of Q_i

- $C_0/\mathbb{F}_2$ = elliptic curve in $\mathbb{P}^2$ defined by $Y^2Z + YZ^2 = X^3$.
  $\implies E = E_{\widehat{C_0}/\mathbb{F}_2}$.

Proposition

$$(F_2)_0 = E^0 B\Sigma_2/I \approx (\mathbb{Z}_2[\![a]\!])[d]/(d^3 - ad - 2).$$

- Write
$$E^0X \xrightarrow{\tau P^2} (E^0X)[d]/(d^3 - ad - 2)$$
  as
$$x \mapsto \tau P^2(x) = Q_0(x) + Q_1(x)\,d + Q_2(x)\,d^2.$$

- τP^2 is a ring homomorphism $\implies$ Cartan formulas.

- Universal deformation of $C_0/\mathbb{F}_2$:
  C/E_0 = elliptic curve defined over $E_0 = \mathbb{Z}_2[\![a]\!]$, by

$$Y^2Z + a\,XYZ + YZ^2 = X^3.$$

- Affine chart: $u = X/Y, v = Z/Y$.

$$v + a\,uv + v^2 = u^3.$$

  (Basepoint is at $(u, v) = (0, 0)$.)

- Subgroup schemes of rank 2: "generated" by points P of C of form $(u(P), v(P)) = (d, -d^3)$ such that $d^3 - ad - 2 = 0$.
- These are also finite subgroups of the formal completion $\widehat{C}$, so

$$(F_2)_0 = E^0B\Sigma_2/I \approx (\mathbb{Z}_2[\![a]\!])[d]/(d^3 - ad - 2).$$

Example 3, revisited: The homomorphism

- Given $P \in C$ with $(u(P), v(P)) = (d, -d^3)$, $d^3 - ad - 2 = 0$ $\implies$ isogeny $\psi_P \colon C \to C'$ such that $\mathrm{Ker}(\psi_P) = \langle P \rangle$.

$$\{v + a\, uv + v^2 = u^3\} \to \{v' + (a^2 + 3d - ad^2)\, u'v' + v'^2 = u'^3\}$$

- Definition of ψ_P: if $Q' = \psi_P(Q)$, then

$$u'(Q') = -u(Q)u(Q+P), \qquad v'(Q') = v(Q)v(Q+P).$$

- By construction, ψ_P is a deformation of Frobenius: if $d = 0$, then $u'(Q') = u(Q)^2$ and $v'(Q') = v(Q)^2$.
- $\implies$ computation of $t^* \colon E_0 \to (F_2)_0$:

$$t^*(a) = \tau P^2(a) = a^2 + 3d - ad^2,$$

- $\tau P^2(ax) = \tau P^2(a) \cdot \tau P^2(x) \implies$

$$Q_0(ax) + Q_1(ax)\, d + Q_2(ax)\, d^2$$
$$= (a^2 + 3d - ad^2)\,(Q_0(x) + Q_1(x)\, d + Q_2(x)\, d^2).$$

Topic 2: The Frobenius congruence (Example 3)

In Example 3, we have

Proposition

$$Q_0(x) \equiv x^2 \mod 2.$$

In the example:

$$\begin{array}{ccccc}
E^0X & \xrightarrow{P^2} & E^0X \otimes_{E_0} E^0(B\Sigma_2) & \xrightarrow{\tau} & (E^0X)[d]/(d^3 - ad - 2) \\
& \searrow^{x \mapsto x^2} & \downarrow{\scriptstyle \mathrm{id}\otimes j^*} & & \downarrow{\scriptstyle d \mapsto 0} \\
& & E^0X \otimes_{E_0} E^0(*) & \longrightarrow & E_0X/(2)
\end{array}$$

Formula:

$$(\tau P^2)(x) = Q_0(x) + Q_1(x)\, d + Q_2(x)\, d^2,$$

pass to $E_0/2$:

$$x^2 \equiv Q_0(x) \mod 2.$$

Topic 2: Frobenius is a deformation of Frobenius

$(G, i, \psi) =$ deformation of G_0/k to R.
When $R \supset \mathbb{F}_p$, there is a relative Frobenius homomorphism

$$\mathrm{Frob}\colon G \to \phi^* G$$

$(G, i, \psi) \to (\phi^* G, i\phi, \phi^*(\psi))$ in $\mathcal{D}(R)$.

Observation

Universal example of $\mathrm{Frob}\colon G \to \phi^* G$ is determined by

$$\pi\colon E^0 B\Sigma_p / I \to E_0/p.$$

Topic 2: Frobenius congruence for sheaves

Definition

A sheaf of commutative rings B on $\mathcal{D}$ is a **Frobenius sheaf** if for every $R \supset \mathbb{F}_p$ and $G \in \mathcal{D}(R)$,

$$B_R(G) \xrightarrow{B_R(\mathrm{Frob})} B_R(\phi^* G) \approx R^\phi \otimes_R B_R(G)$$

is the relative Frobenius homomorphism of R-algebras.

Theorem (R.)

There is a functor

$$\{\textit{amplified } \Gamma\textit{-rings}\} \to \{\textit{Frobenius sheaves on } \mathcal{D}\}$$

which restricts to an equivalence between the full subcategories of p-torsion free objects.

Topic 3: Koszul algebras

$A = \bigoplus_{r \geq 0} A_r$ graded associative ring, $A_0 = R$ commutative.

Definition

A is **Koszul** if there exist R-modules C_r with $C_0 = R$, and an exact sequence (a "Koszul complex")

$$\cdots \xrightarrow{d} A \otimes_R C_3 \xrightarrow{d} A \otimes_R C_2 \xrightarrow{d} A \otimes_R C_1 \xrightarrow{d} A \otimes_R C_0 \xrightarrow{d} R \to 0$$

of left A-modules such that d raises degree by 1.

Fact

If A is Koszul, then

$$A \approx T_R(A_1)/(U), \qquad U \subset A_2$$

(i.e., A is "quadratic".)

- Back to the example: $\Gamma \approx \bigoplus \Gamma_r \approx T_{E_0}(\Gamma_1)/(U)$, where $\Gamma_1 = E_0\{Q_0, Q_1, Q_2\}$, $U =$ Adem relations.
- **PBW Theorem** (Priddy (1970)): if Γ has a "nice" admissible basis, then Γ is Koszul.
- $\Longrightarrow$ Exact sequence.

$$0 \to \Gamma \otimes_{E_0} C_2 \to \Gamma \otimes_{E_0} C_1 \to \Gamma \to E_0 \to 0.$$

C_i are free modules over E_0: $\mathrm{rank}\, C_1 = 3$, $\mathrm{rank}\, C_2 = 2$.

Topic 3: Is Γ always Koszul?

Theorem (Ando-Hopkins-Strickland(?), R.)

For all $E = E_{G_0/k}$, the associated ring Γ of power operations is Koszul. The associated Koszul complex has the form

$$0 \to \Gamma \otimes_{E_0} C_n \to \cdots \to \Gamma \otimes_{E_0} C_1 \to \Gamma \to E_0 \to 0,$$

where $n =$ height of G_0.

- They developed a program to prove the result, using interesting ideas about a kind of "Bruhat-Tits building" formed using flags of certain finite subgroup schemes of G_E.
- I don't believe they ever completed their program; there is probably no obstruction to doing so, however.
- There is another proof, which avoids using formal group theory; it uses ideas related to the Whitehead conjecture (Kuhn, Mitchell, Priddy) and calculus (Arone-Mahowald, Arone-Dwyer).

Topic 3: Linearization

Here are some of the ideas in the proof.

Definition

Given a (nonadditive) functor $F\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$, the **linearization** $\mathcal{L}[F]\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$ is

$$\mathcal{L}[F](M) = \mathrm{Cok}\left[\; F(M \oplus M) \underset{F(\pi_1)+F(\pi_2)}{\overset{F(\pi_1+\pi_2)}{\rightrightarrows}} F(M) \;\right].$$

$\mathcal{L}[F]$ is initial additive quotient functor of F.

In some cases, including ours, $\mathcal{L}[F \circ G] \to \mathcal{L}[F] \circ \mathcal{L}[G]$ is an isomorphism.

Topic 3: Linearization of the amplified Γ-ring monad

- $F\colon \mathrm{Mod}_{E_0} \to \mathrm{Mod}_{E_0}$ the free amplified Γ-ring functor.
- For E-module M with $\pi_* M =$ free E_*-module concentrated in even degree,

$$F(\pi_0 M) \approx \bigoplus_{m \geq 0} \pi_0 L_{K(n)} \mathbb{P}_E^m(M).$$

- $$\mathcal{L}[F](E_0) = \Delta.$$

$$\mathcal{L}[F \circ \cdots \cdots \circ F](E_0) = \Delta \otimes_{E_0} \cdots \otimes_{E_0} \Delta.$$

  Δ is a ring, non-canonically isomorphic to Γ.
- Monadic bar construction $\mathcal{B}_\bullet(F, F, F)$.

$$\mathcal{L}\left[\mathcal{B}_\bullet(F, F, F)\right] \approx \mathcal{B}_\bullet(\Delta, \Delta, \Delta).$$

(Priddy 1970):

- If Δ is a graded ring, filter $\mathcal{B}_\bullet(M, \Delta, N)$ according to grading on Δ.
- Δ is **Koszul** if $\mathrm{gr}_q \mathcal{B}_\bullet(E_0, \Delta, E_0)$ has homology concentrated in degree q.
- Koszul complex "is" the spectral sequence associated to this filtration on $\mathcal{B}_\bullet(M, \Delta, N)$; $E_1^{p,q} =$ chain complex.

- $$\mathcal{B}_q(F,F,F)(E_0) \approx (F \circ \cdots \circ F)(E_0) \approx \bigoplus_{m \geq 0} E_0^{\wedge}(K_q(m)_{h\Sigma_m}).$$

  $K_\bullet(m)$ is the **partition complex**:

  $$K_\bullet(m) = \mathrm{nerve}\,\{\text{poset of partitions of } \{1,\ldots,m\}\}\,.$$

- $$\mathcal{B}_q(\Delta,\Delta,\Delta) \approx \mathcal{L}[\mathcal{B}_q(F,F,F)](E_0) \approx \bigoplus_{m \geq 0} Q_m(K_q(m))$$

  where

  $$Q_m(X) = \mathrm{Cok}\left[\bigoplus_{0<i<m} E_0^{\wedge}(X_{h(\Sigma_i \times \Sigma_{m-i})}) \to E_0^{\wedge}(X_{h\Sigma_m})\right],$$

  X is a set with Σ_m action.

- $\overline{K}_\bullet(m) = K_\bullet(m)/\sim$, associated to $\mathcal{B}_\bullet(E_0, \Delta, E_0) \approx \mathcal{B}_\bullet(\Delta, \Delta, \Delta)/\sim$.
- $Q_m(\overline{K}_\bullet(m)) = 0$ if $m \neq p^r$.
- Need to show $Q_{p^r}(\overline{K}_\bullet(p^r))$ has H_* concentrated in degree r.
- $$K_\bullet(p^r) \times \Sigma_{p^r}/(\Sigma_p \wr \cdots \wr \Sigma_p) \twoheadrightarrow K_\bullet(p^r),$$
  where
  $$U_\bullet(p^r) = \bigcup_{\substack{A \subset \Sigma_{p^r} \\ \text{max. ab. subgp.}}} (K_\bullet(p^r) \times \Sigma_{p^r}/(\Sigma_p \wr \cdots \wr \Sigma_p))^A.$$
- Reduce to showing $Q_{p^r}(\overline{U}_\bullet(p^r))$ is chain homotopy equivalent to a complex concentrated in degree r.
- Claim: There is a Σ_{p^r}-equivariant homotopy equivalence $\overline{U}_\bullet(p^r) \approx X_+ \wedge S^r$, where X is a Σ_{p^r}-set.

Topic 3: $U_\bullet(p^r)$ and the Tits building for $GL(r, \mathbb{F}_p)$

- $A \subset \Sigma_{p^r}$ maximal abelian subgroup:

  $$K_\bullet(p^r)^A = \mathrm{nerve}\,\{\text{ poset of subgroups of } A\ \}\,.$$

  For $A \approx (\mathbb{Z}/p)^r$, the quotient $\overline{K}_\bullet(p^r)^A$ is (a 2-fold suspension of) the Tits building for $GL(r, \mathbb{F}_p)$.

- $$\overline{K}_\bullet(p^r)^A \approx \begin{cases} \bigvee S^r & \text{if } A \approx (\mathbb{Z}/p)^r, \\ * & \text{otherwise.} \end{cases}$$

  $A = (\mathbb{Z}/p)^r$ result is theorem of Solomon-Tits (1969).

- Show $\overline{U}_\bullet(p^r) \approx X_+ \wedge S^r$ by the same "shellability" argument that Solomon-Tits use for $\overline{K}_\bullet(p^r)^{(\mathbb{Z}/p)^r}$.

Topic 4: The operation Ψ (Example 3)

We return to the main example (height 2, prime 2).

- $\Psi \in \Gamma$ is element corresponding to the operation:

$$E^0 X \xrightarrow{\tau P^4} E^0 X \otimes_{E_0} (F_4)_0 \xrightarrow{\mathrm{id}\otimes\rho} E^0 X \otimes_{E_0} E_0$$

  where $\rho\colon (F_4)_0 \to E_0$ classifies $[-2]\colon G_E \to G_E$
  (since $[-2](x) \equiv x^4 \mod (2, a)$, it is a deformation of Frob^2.)

- $\Psi = Q_0 Q_0 + a\, Q_0 Q_1 - 2\, Q_1 Q_1 + a^2\, Q_0 Q_2 - 2a\, Q_1 Q_2 + 4\, Q_2 Q_2.$

- $\Psi\colon B \to B$ is a ring homomorphism.

Topic 4: The operation N (Example 3)

- $N\colon B \to B$ corresponds to the operation:

$$E^0X \xrightarrow{\tau P^2} E^0X \otimes_{E_0} (F_2)_0 \xrightarrow{\text{Norm}} E^0X.$$

(*N* is a "multiplicative Hecke operator".)

- $$N(x) = (Q_0x)^3 + 2a\,(Q_0x)^2 Q_2x - a\,Q_0x(Q_1x)^2 + a^2\,Q_0x(Q_2x)^2 \\ - 6\,Q_0\,Q_1x\,Q_2x + 2\,(Q_1x)^3 - 2a\,Q_1x(Q_2x)^2 + 4\,(Q_2x)^3.$$

- $N(xy) = N(x)\,N(y)$, but N is not additive.
- $N(x) \equiv x^2\,\Psi(x) \mod 2$.

Topic 4: A logarithmic operation (Example 3)

- If $x \in B^\times$, then $N(x) \in B^\times$, so $N(x) \equiv x^2\,\Psi(x) \mod 2$ implies

$$\frac{x^2\,\Psi(x)}{N(x)} \equiv 1 \mod 2.$$

- For any 2-complete amplified Γ-ring, get a homomorphism

$$\ell\colon B^\times \to B,$$
$$x \mapsto \frac{1}{2}\log\left[\frac{x^2\,\Psi(x)}{N(x)}\right].$$

- $A =$ a $K(2)$-local commutative E-algebra, there is a map of spectra

$$\mathrm{gl}_1(A) \to A.$$

  On π_0, this map is given by ℓ.

- This works in a similar way at all heights and primes.

Topic 4: Hecke operators

Now E is a general Morava E-theory (height n, prime p).

- Elements $\tilde{T}(p^k) \in \Gamma$, given by

$$E^0 X \xrightarrow{\tau P_{p^r}} E^0 X \otimes_{E_0} (F_{p^r})_0 \xrightarrow{\text{Trace}} E^0 X.$$

(First constructed by Ando (1992).)

- $\{\tilde{T}(p^k)\}$ generate a commuative subring $\mathbb{Z}_p[\tilde{T}_1, \dots, \tilde{T}_n] \subset \Gamma$, where

$$\sum_{r=0}^{n} (-1)^r p^{r(r-1)/2}\, \tilde{T}_r \cdot U^r = \left(\sum_{k \geq 0} \tilde{T}(p^k) \cdot U^k \right)^{-1}$$

in $\Gamma[\![U]\!]$.

Topic 4: $K(n)$-localization of BG

Different construction of $\tilde{T}(p^k)$, due to Ganter.
G = finite group.

- The $K(n)$-local Tate homology of BG vanishes (Hovey-Strickland (1999)):
$$L_{K(n)}BG_+ \xrightarrow{\sim} \mathcal{F}(BG_+, L_{K(n)}S).$$
- $\Longrightarrow$ $L_{K(n)}BG_+$ is a **commutative Frobenius algebra** in the $K(n)$-local homotopy category (Strickland (2000)).
(analogy between $\mathcal{F}(BG_+, L_{K(n)}S)$ and representation ring RG.)
- Let $I_G\colon L_{K(n)}S \to L_{K(n)}BG_+$, dual to $L_{K(n)}BG_+ \to L_{K(n)}S$,
(analogous to $\frac{1}{|G|}\mathrm{Trace}\sum_{g\in G} g\colon RG \to \mathbb{Z}$.)

Topic 4: Ganter's symmetric powers

- Define σ^m by

$$E^0 X \xrightarrow{P^m} E^0 X \otimes_{E_0} E^0 B\Sigma_m \xrightarrow{\mathrm{id}\otimes I^*_{\Sigma_m}} E^0 X \otimes_{E_0} E_0.$$

- $\sigma^m\colon B \to B$ are non-additive functions, analogous to symmetric powers of representations.

Theorem (Ganter (2004))

$$\exp\left(\sum_{k\geq 0} \frac{\tilde{T}(p^k)(x)}{p^k} \cdot U^{p^k}\right) = \sum_{m\geq 0} \sigma^m(x) \cdot U^m.$$

- $\Longrightarrow$

$$\sum_{k\geq 0} \tilde{T}(p^k)(x) \cdot U^{p^k} = \frac{d}{dU} \log\left(\sum_{m\geq 0} \sigma^m(x) \cdot U^m\right).$$

Topic 4: Hecke operators on $K(n)$-local S-algebras

Let $R =$ a $K(n)$-local S-algebra.

- Ganter's operations σ^m are defined on $\pi_0 R$ for any $K(n)$-local S algebra. (They are defined using a homotopy class in $\pi_0 L_{K(n)} B\Sigma_m^+$.)
- $\Longrightarrow$ Ganter's formula gives a definition of Hecke operators on $\pi_0 R$ for any $K(n)$-local S-algebra.
- By "suspension", get Hecke operators acting on $\pi_q R$ for $q \geq 0$ as well.

The end

http://www.math.uiuc.edu/~rezk/midwest-2009-power-ops.pdf

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