WHEN ARE HOMOTOPY COLIMITS COMPATIBLE WITH HOMOTOPY BASE CHANGE?

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Abstract. The goal of this note is to understand how to prove things about geometric realizations of pullbacks, without using the dreaded "π_*-Kan condition".

Contents

1. Realization-fibrations

I'll write Spaces for the category of spaces, by which I probably mean simplicial sets.

For this section, fix a small category $\mathbf{I}$. An **I-presheaf** is a functor $\mathbf{I}^{\mathrm{op}} \to$ Spaces. I write PSh($\mathbf{I}$) for the category of $\mathbf{I}$-presheaves.

I will write $|X|_{\mathbf{I}}$ for the homotopy colimit $\operatorname{hocolim}_{\mathbf{I}^{\mathrm{op}}} X$ of an $\mathbf{I}$-presheaf X, and call it the **realization** of the presheaf X.

If $\mathbf{J}$ is a small category, a **J-diagram** is a functor $Y : \mathbf{J} \to$ Spaces. I'll write $\operatorname{hocolim}_{\mathbf{J}} Y$ for the homotopy colimit of such functors.

Definition 1.1. A map $p\colon E \to B$ of $\mathbf{I}$-presheaves is said to be a **realization fibration (RF)**, if for every homotopy pullback square

(1.2)
$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \longrightarrow & B \end{array}$$

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of **I**-presheaves, the square

$$\begin{array}{ccc} |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow \\ |B'|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}}, \end{array} \tag{1.3}$$

obtained by applying $|\quad|_{\mathbf{I}}$ to each corner of the square, is a homotopy pullback of spaces.

We make the following elementary remarks about this definition:

(1) The property of being a realization-fibration is closed under weak equivalence; if p is a realization-fibration, then any map weakly equivalent to p is also a realization-fibration.
(2) The property of being a realization-fibration is stable under homotopy base change; if p is a realization-fibration, and p' is a map fitting into a homotopy pullback square as in (1.2), then p' is a realization-fibration.
(3) Every weak equivalence of I-presheaves is a realization-fibration.

2. Local-to-global principle

Our main theorems for recognizing realization-fibrations encode a kind of "local-to-global" principle.

2.1. **Descent.** Let $\mathbf{J}$ be a small category, let E, B be functors $\mathbf{J} \to \mathbf{M}$ where $\mathbf{M}$ is some model category, and let $p\colon E \to B$ be a natural transformation. We say that p is **J-equifibered** (or simply **equifibered**) if for every morphism $\alpha\colon J_1 \to J_2$ in $\mathbf{J}$, the square

$$\begin{array}{ccc} E(J_2) & \xrightarrow{E(\alpha)} & E(J_1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B(J_2) & \xrightarrow[U(\alpha)]{} & B(J_1) \end{array}$$

is a homotopy pullback of spaces.

Remark 2.2. A map $p\colon E \to B$ of functors $\mathbf{I}\times\mathbf{J} \to \mathbf{M}$ from a product category can potentially be equifibered in only one of the input variables. Thus we might say that such p is "**J**-equifibered", by which we really mean that the tautologically equivalent map of functors $\mathbf{J} \to \mathbf{M}^{\mathbf{I}}$ is an equifibered map of $\mathbf{J}$-diagrams.

We will use the principle of descent.

Proposition 2.3 (Descent)**.**
Let $\mathbf{J}$ *be a small category.*

(1) *Suppose that* $p\colon E \to B$ *is a fibration in* Spaces*, that* $V\colon \mathbf{J} \to$ Spaces *is a functor, and that* $h\colon$ $\operatorname{hocolim}_{\mathbf{J}} V \to B$ *is a weak equivalence. Let* $U\colon \mathbf{J} \to$ Spaces *be defined by* $U(J) = V(J) \times_B E$*. Then the evident map* $\operatorname{hocolim}_{\mathbf{J}} U \to E$ *is a weak equivalence.*

(2) *Suppose that* $f: U \to V$ *is an equifibered map in* Spaces^J. *Then for each object* $J \in \mathbf{J}$, *the evident square*

$$\begin{CD} U(J) @>>> \operatorname{hocolim}_{\mathbf{J}} U \\ @VVV @VVV \\ V(J) @>>> \operatorname{hocolim}_{\mathbf{J}} V \end{CD}$$

is a homotopy pullback of spaces.

In particular, we note the following consequence of descent.

Proposition 2.4. *Let* $\mathbf{J}$ *be a small category, and let*

$$\begin{CD} X' @>>> X \\ @VVV @VV{f}V \\ Y' @>>> Y \end{CD}$$

be a homotopy pullback square in $\mathrm{Spaces}^{\mathbf{J}}$. *If* f *is equifibered, the commutative square of spaces applied by applying* $\operatorname{hocolim}_{\mathbf{J}}$ *at each corner of the above diagram is a homotopy pullback.*

In particular, this means that any equifibered map of $\mathbf{I}$-presheaves is a realization-fibration.

2.5. **The first local-to-global principle.** The next theorem tells us that we can sometimes "glue together" realization-fibrations to get another realization-fibration.

Theorem 2.6. *Let* $\mathbf{J}$ *be a small category, let* V *and* W *be functors* $\mathbf{J} \to \mathrm{PSh}(\mathbf{I})$, *and let* $h: W \to V$ *be a natural transformation. Suppose that* h *is an equifibered map of functors* $\mathbf{J} \to \mathrm{PSh}(\mathbf{I})$, *and that for each object* $J \in \mathbf{J}$ *the map* $h(J): W(J) \to V(J)$ *of* $\mathbf{I}$*-presheaves is a realization-fibration of* $\mathbf{I}$*-presheaves. Then* $\operatorname{hocolim}_{\mathbf{J}} h: \operatorname{hocolim}_{\mathbf{J}} W \to \operatorname{hocolim}_{\mathbf{J}} V$ *is a realization-fibration of* $\mathbf{I}$*-presheaves.*

Proof. Let $B = \operatorname{hocolim}_{\mathbf{J}} V$, and choose a factorization $\operatorname{hocolim}_{\mathbf{J}} W \xrightarrow{i} E \xrightarrow{p} B = \operatorname{hocolim}_{\mathbf{J}} V$ so that i is a weak equivalence and p is a fibration of $\mathbf{I}$-presheaves. Let $U: \mathbf{J} \to \mathrm{PSh}(\mathbf{I})$ be defined by $U(J) = V(J) \times_B E$. Since h is equifibered, descent implies that the map $W \to U$ of functors $\mathbf{J} \to \mathrm{PSh}(\mathbf{I})$ is a weak equivalence. Thus the maps $f(J): U(J) \to V(J)$ are realization-fibrations of $\mathbf{I}$-presheaves. There are weak equivalences $\operatorname{hocolim}_{\mathbf{J}} W \to \operatorname{hocolim}_{\mathbf{J}} U \to E$ of $\mathbf{I}$-presheaves, so to prove that $\operatorname{hocolim}_{\mathbf{J}} h$ is a realization-fibration, it suffices to prove that p is a realization-fibration of $\mathbf{I}$-presheaves.

Let $i: B' \to B$ be a map of $\mathbf{I}$-presheaves. Define an $\mathbf{I}$-presheaf E', and functors $U', V': \mathbf{J} \to \mathrm{PSh}(\mathbf{I})$, by forming the homotopy pullback of each of the objects in the square

$$\begin{CD} U(J) @>>> E \\ @V{f(J)}VV @VV{p}V \\ V(J) @>>> B \end{CD}$$

along i. The homotopy pullback square of $\mathbf{I}$-presheaves

(2.7)
$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \longrightarrow & B \end{array}$$

is obtained by applying $\mathrm{hocolim}_{\mathbf{J}}$ to each corner of the homotopy pullback square

(2.8)
$$\begin{array}{ccc} U' & \longrightarrow & U \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V' & \longrightarrow & V \end{array}$$

of functors $\mathbf{J} \to \mathrm{PSh}(\mathbf{I})$. That is is the case is an application of (2.4).

Since $f(J)$ is a realization fibration for all J, applying $|\ \ |_{\mathbf{I}}$ to each corner of (2.8) gives a homotopy pullback of $\mathbf{J}$-diagrams. Thus, commuting colimits with colimits shows that applying $|\ \ |_{\mathbf{I}}$ to each corner of (2.7) gives a homotopy pullback of spaces. □

2.9. **Local realization-fibrations.** For each object I of $\mathbf{I}$, let H_I denote the $\mathbf{I}$-presheaf represented by I, i.e.,

$$H_I(I') = \hom_{\mathbf{I}}(I', I).$$

These fit together to give the **Yoneda functor** $H\colon \mathbf{I} \to \mathrm{PSh}(\mathbf{I})$.

Given maps $p\colon E \to B$ and $b\colon H_I \to B$ of $\mathbf{I}$-presheaves, let $\mathrm{Fib}(p, b)$ denote the $\mathbf{I}$-presheaf which is the homotopy pullback of p along b. Given $f\colon I' \to I$ in $\mathbf{I}$, there is a diagram

$$\begin{array}{ccccc} \mathrm{Fib}(p, b \circ H_f) & \longrightarrow & \mathrm{Fib}(p, b) & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ H_{I'} & \xrightarrow[H_f]{} & H_I & \xrightarrow[b]{} & B \end{array}$$

obtained by taking homotopy pullbacks. Note that if p is itself a fibration of $\mathbf{I}$-diagrams, then we may assume that $\mathrm{Fib}(p, b)$ is actually the pullback of p along b.

Definition 2.10. A map $p\colon E \to B$ of $\mathbf{I}$-presheaves is said to be a **local realization-fibration (LRF)**, if for every $f\colon I' \to I$ in $\mathbf{I}$ and every $b\colon H_I \to B$, the induced map

$$|\mathrm{Fib}(p, b \circ H_f)|_{\mathbf{I}} \to |\mathrm{Fib}(p, b)|_{\mathbf{I}}$$

is a weak equivalence of spaces.

We make the following elementary remarks about this definition:

(1) The property of being a local realization-fibration is closed under weak equivalence.
(2) The property of being a local realization-fibration is stable under homotopy base change.

(3) Every realization-fbration is a local realization-fibration. In fact, if p is a realization-fibration, then

$$\begin{array}{ccc} |\mathrm{Fib}(p, b \circ H_f)|_{\mathbf{I}} & \longrightarrow & |\mathrm{Fib}(p, b)|_{\mathbf{I}} \\ \downarrow & & \downarrow \\ |H_{I'}|_{\mathbf{I}} & \longrightarrow & |H_I|_{\mathbf{I}} \end{array}$$

is necessarily a homotopy pullback of spaces. But the bottom map $|H_f|_{\mathbf{I}}$ is a weak equivalence, since both $|H_I|_{\mathbf{I}}$ and $|H_{I'}|_{\mathbf{I}}$ are contractible, whence the top map is a weak equivalence.

Lemma 2.11. *Let $p\colon E \to B$ be a map of $\mathbf{I}$-presheaves, such that $B = H_I$ for some $I \in \mathbf{I}$. Then p is a realization-fibration if and only if p is a local realization-fibration.*

Proof. We know that in general RF implies LRF, so we need only deal with the converse. Without loss of generality, we may assume that p is a fibration of $\mathbf{I}$-presheaves which is LRF. Let $\mathcal{C}(p)$ denote the collection of maps $f\colon B' \to B$ of simplicial spaces such that

$$\begin{array}{ccc} |B' \times_B E|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow \\ |B'|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}} \end{array}$$

is a homotopy pullback of spaces. To show that p is RF, it suffices to show that all maps to B are in $\mathcal{C}(p)$, and this in turn will follow from the following three statements.

(a) If $B'' \xrightarrow{g} B' \xrightarrow{f} B$ are such that g is a weak equivalence, and $f \in \mathcal{C}(p)$, then $fg \in \mathcal{C}(p)$.
(b) If $V\colon \mathbf{J} \to \mathrm{PSh}(\mathbf{I})/B$ is a functor from a small category $\mathbf{J}$ to the slice category of presheaves over B, and if each map $V(J) \to B \in \mathcal{C}(p)$ for every object J in $\mathbf{J}$, then $\mathrm{hocolim}_{\mathbf{J}}\, V \to B$ is in $\mathcal{C}(p)$.
(c) Every map of the form $g\colon H_{I'} \to B$ is in $\mathcal{C}(p)$.

The conclusion follows from the observation that every object in $\mathrm{PSh}(\mathbf{I})/B$ is weakly equivalent to one obtained as homotopy colimit of a small diagram objects of the form $H_{I'} \to B$.

Statement (a) is clear. Statement (b) is a straightforward consequence of (a) and our first local-to-global principle (2.6).

To prove statement (c) note that since $B = H_I$, the map g is of the form H_f for some $f\colon I \to I' \in \mathbf{I}$. The claim is then immediate from the fact that p is LRF, where in the definition of LRF we take b to be the identity map of H_I. □

2.12. **Second local-to-global principle.** Our second local-to-global principle is the following.

Theorem 2.13. *Let $p\colon E \to B$ be a map of $\mathbf{I}$-presheaves. Then p is RF if and only if it is LRF.*

Proof. We have already noted that RF implies LRF.

Suppose that p is LRF, and without loss of generality suppose that p is a fibration. We can find a small category $\mathbf{J}$ and a functor $V\colon \mathbf{J} \to \mathrm{PSh}(\mathbf{I})/B$ such that $\mathrm{hocolim}_{\mathbf{J}}\, V \to B$ is a weak equivalence, and such that each $V(J)$ is weakly equivalent to an H_I for some $I \in \mathbf{I}$.

Let $U\colon \mathbf{J} \to \mathrm{PSh}(\mathbf{I})/E$ be defined by $U(J) = V(J) \times_B E$, and let $f\colon U \to V$ be the natural transformation induced by pulling back p. Then $\mathrm{hocolim}_{\mathbf{J}} f$ is weakly equivalent to p, and is a realization-fibration by (2.6). □

3. Approximating a map by a realization-fibration

Although a map $p\colon E \to B$ of $\mathbf{I}$-presheaves might fail to be a realization-fibration, it turns out that there is a "maximal subobject" $B_{\mathrm{RF}(p)}$ of B such that the restriction of p over $B_{\mathrm{RF}(p)}$ is a realization-fibration. Furthermore, for any map $B' \to B$ which factors through the subobject $B_{\mathrm{RF}(p)}$, realization $|\ \ |_{\mathbf{I}}$ preserves the resulting homotopy pullback.

3.1. **The** lrf **sub-presheaf.** Given a $\mathbf{I}$-presheaf X, let $\pi_0 X\colon \mathbf{I}^{\mathrm{op}} \to \mathrm{Set}$ denote the $\mathbf{I}$-presheaf of *sets* defined by $(\pi_0 X)(I) = \pi_0(X(I))$. I'll typically think of this as a *discrete* presheaf of spaces.

Given a map $p\colon E \to B$ of $\mathbf{I}$-presheaves, let $\mathrm{lrf}(p) \subseteq \pi_0 B$ denote the subpresheaf defined as follows: $\mathrm{lrf}(p)(I) \subseteq \pi_0 B(I)$ consists of all elements of $\pi_0 B(I)$ represented by maps $b\colon H_I \to B$ such that $\mathrm{Fib}(p, b) \to H_I$ is a realization-fibration.

In particular,

Proposition 3.2. *$p\colon E \to B$ is RF if and only if $\mathrm{lrf}(p) = \pi_0 B$.*

3.3. **The RF-approximation to a map.** Let $B_{\mathrm{RF}(p)}$ denote the $\mathbf{I}$-diagram of spaces defined by $B_{\mathrm{RF}(p)}(I) = B(I) \times_{\pi_0 B(I)} \mathrm{lrf}(p)$. We can think of $B_{\mathrm{RF}(p)}$ as a subobject of B, where each $B_{\mathrm{RF}(p)}(I)$ is isomorphic to a union of some of the path components of $B(I)$. Let $\mathrm{RF}(p)\colon E_{\mathrm{RF}(p)} \to B_{\mathrm{RF}(p)}$ denote the restriction of p to the subobject $B_{\mathrm{RF}(p)}$; we call it the **RF-approximation** to p.

Theorem 3.4. *Let*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow[f]{} & B \end{array}$$

be a homotopy pullback square of $\mathbf{I}$-presheaves. Then p' is a realization-fibration if and only if i factors through $B_{\mathrm{RF}(p)} \subseteq B$. In particular, $\mathrm{RF}(p)\colon E_{\mathrm{RF}(p)} \to B_{\mathrm{RF}(p)}$ is a realization-fibration.

Proof. The equivalence between RF and LRF shows that if p' is a realization-fibration, then i must factor through $B_{\mathrm{RF}(p)}$.

Conversely, suppose i factors through $B_{\mathrm{RF}(p)}$. Then it is clear that p' is LRF, and therefore is RF by (2.6). □

4. Application to sifted categories

Let $\mathbf{D}$ be a small category. Recall that $\mathbf{D}^{\mathrm{op}}$ is **sifted** if

(i) $|1|_{\mathbf{D}} \approx 1$ (i.e., the realization of the terminal object is contractible), and
(ii) for any pair of objects D_1, D_2 in $\mathbf{D}$, we have $|H_{D_1} \times H_{D_2}|_{\mathbf{I}} \approx 1$ (i.e., the realization of a product of representable presheaves is contractible).

An elementary consequence of the definition is that for any $X, Y \in \mathrm{PSh}(\mathbf{D})$, the evident map $|X \times Y|_{\mathbf{D}} \to |X|_{\mathbf{D}} \times |Y|_{\mathbf{D}}$ is a weak equivalence; i.e., realization commutes with finite products.

The most significant example of a sifted category for our purposes is $\mathbf{D}^{\mathrm{op}} = \Delta^{\mathrm{op}}$, the indexing category for simplicial objects.

Although for such $\mathbf{D}$, realization commutes with products, it does not follow that realization commutes with homotopy pullbacks. Thus, we will apply the theory of the previous sections in this context, and see what we get.

4.1. **Local projection maps.** In what follows, we give a sufficient condition for a map of presheaves on a sifted category to be a realization-fibration: namely, a map is RF whenever it looks "locally" like a projection.

In the following, fix a small category $\mathbf{D}$ such that $\mathbf{D}^{\mathrm{op}}$ is sifted.

Say that a map $p\colon E \to B$ of $\mathbf{D}$-presheaves is a **weak projection** if it is weakly equivalent in $\mathrm{PSh}(\mathbf{D})/B$ to a projection map $p'\colon B \times C \to B$, where C is some $\mathbf{D}$-presheaf.

The class of weak projection maps in $\mathrm{PSh}(\mathbf{D})$ is clearly closed under weak equivalence, is stable under homotopy base change, and includes all weak equivalences.

Say that a map $p\colon E \to B$ of $\mathbf{D}$-presheaves is a **local projection** if for every object D of $\mathbf{D}$ and every homotopy pullback square of the form

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ H_D & \xrightarrow[b]{} & B \end{array}$$

the map q is a weak projection map.

The class of local projection maps in $\mathrm{PSh}(\mathbf{D})$ is closed under weak equivalence, is stable under homotopy base change, and includes all weak equivalences. Every weak projection map is a local projection map.

Recall that if X is a $\mathbf{D}$-presheaf of spaces, we have a $\mathbf{D}$-presheaf of sets $\pi_0 X$, obtained by $(\pi_0 X)(D) = \pi_0(X(D))$. I'll think of this as a discrete object in $\mathrm{PSh}(\mathbf{D})$.

Given a map $p\colon E \to B$, let $\mathrm{lproj}(p) \subseteq \pi_0 B$ denote the subobject defined as follows: $\mathrm{lproj}(p)(D) \subseteq \pi_0 B(D)$ consists of all elements of $\pi_0 B(D)$ represented by maps $b\colon H_D \to B$ such that the pullback of p along b is a weak projection map.

Proposition 4.2. *Let $p\colon E \to B$ and $f\colon B' \to B$ be maps of $\mathbf{D}$-presheaves. The pullback of p along f is a local projection map if and only if $f(\pi_0 B') \subseteq \mathrm{lproj}(p)$.*

In particular, p is a local projection map if and only if $\mathrm{lproj}(p) = \pi_0 B$.

Our interest in local projection maps comes from the following result, which makes use of the fact that $\mathbf{D}$ is sifted for the first time.

Lemma 4.3. *If $p\colon B \times C \to B$ is a projection map between $\mathbf{D}$-presheaves for a sifted category $\mathbf{D}$, then p is RF.*

Proof. If $i\colon B' \to B$ is any map, then the homotopy pullback of p along i is weakly equivalent to the projection map $B' \times C \to B'$. When we realize at each corner, the siftedness of $\mathbf{D}$ implies that we get a homotopy pullback, as desired. $\square$

Theorem 4.4. *Let $p\colon E \to B$ be a map of $\mathbf{D}$-presheaves for a sifted $\mathbf{D}$. Then* $\mathrm{lproj}(p) \subseteq \mathrm{lrf}(p)$. *In particular, all local projections in* $\mathrm{PSh}(\mathbf{D})$ *are realization-fibrations.*

Proof. It suffices to show that the projection map $p\colon H_D \times C \to H_D$ is a realization fibration for all $\mathbf{D}$-presheaves C. This is immediate from (4.3). □

5. Application to simplicial spaces

5.1. **Identifying local projection maps in simplicial spaces.** We now consider the problem of identifying $\mathrm{lproj}(p) \subseteq \pi_0 B$ in the case when $\mathbf{D} = \Delta$, and thus $\mathrm{PSh}(\mathbf{D})$ is simplicial spaces.

Note that if B is a simplicial space, then $\pi_0 B$ is a simplicial set. We first note that the subobject $\mathrm{lproj}(p) \subseteq \pi_0 B$ contains all path components of the simplicial set $\pi_0 B$ not in the image of p.

Lemma 5.2. *Let $p\colon E \to B$ be a map of simplicial spaces, and let $V(p) \subseteq \pi_0 B$ denote the union of all path compenents of of the simplicial set $\pi_0 B$ which are not in the image the map $\pi_0(p)\colon \pi_0 E \to \pi_0 B$. Then $V(p) \subseteq \mathrm{lproj}(p)$.*

Proof. If $\tilde{b}\colon H_{[m]} \to B$ is any map whose effect on π_0 is a map $H_{[m]} \to \pi_0 B$ which lands in a path component not in the image of $\pi_0(p)$, then the pullback of p along $\tilde{b}$ is the empty simplicial space; thus, this pullback is trivially a projection map, and so b represents a point in $\mathrm{lproj}(p)$. □

Next, we note that the simplicial set $\mathrm{lproj}(p)$ contains all vertices of $\pi_0 B$.

Lemma 5.3. *Let $p\colon E \to B$ be a map of simplicial spaces. Then every 0-simplex of $\pi_0 B$ is contained in* $\mathrm{lproj}(p)$.

Proof. This is a consequence of the fact that $H_{[0]}$ is the terminal object in simplical spaces, and so every map $p\colon E \to H_{[0]}$ is a projection. □

This gives an immediate easy consequence.

Proposition 5.4. *Let $p\colon E \to B$ be a map of simplicial spaces such that $\pi_0 B$ is discrete (i.e., such that each map $B([m]) \to B([0])$ in the simplicial space is an isomorphism on π_0). Then p is a realization fibration.*

Proof. Immediate from (5.3). □

5.5. ***H*-group objects.** An ***H*-group** in a homotopy theory $\mathbf{M}$ consists of: an object X, a map $u\colon 1 \to X$ from the terminal object, and a map $m\colon X \times X$, which (i) satisfy the axioms for a commutative monoid in the homotopy category $h\mathbf{M}$, and (ii) are such that the "shearing map"

$$s = (\pi_1, m)\colon X \times X \to X \times X$$

is a weak equivalence.

A map of H-groups is a map $f\colon E \to B$ objects in $\mathbf{M}$ such that f commutes with the structure maps e and m on the nose (i.e., not just up to homotopy). (Up to a suitable weakly-equivalent replacement, this commutativity condition is equivalent to a condition which can be stated purely in terms of homotopy commutativity, but I don't want to bother.)

Let $p\colon E \to B$ be a map of H-group objects in simplicial spaces. Let $G = \mathrm{Ker}(p)$, i.e., the homotopy pullback of p along the inclusion $u_B\colon 1 \to B$ of the identity. Observe that G is itself naturally an H-group object.

Lemma 5.6. *Let $p\colon E \to B$ be a map of H-group objects in simplicial spaces which is a fibration, with $G = \mathrm{Ker}(p)$. Let $b\colon T \to B$ be a map of simplicial spaces. If there exists a map $e\colon T \to E$ such that $pe = b$, then there exists a commutative diagram*

$$\begin{array}{ccc} T \times G & \xrightarrow{\ \phi\ } & T \times_B E \\ & {\scriptstyle proj}\searrow \quad \swarrow & \\ & T & \end{array}$$

such that ϕ is a weak equivalence of simplicial spaces.

Proof. There is a universal example of this, namely $b = p\colon E \to B$ and $e = \mathrm{id}_E\colon E \to E$; the general case is obtained by pullback. In the universal case, it is straightforward to see that ϕ is obtained as the homotopy pullback of the shearing equivalence $s_E\colon E \times E \to E \times E$ along $E \times_B E \to E \times E$. □

Theorem 5.7. *Let $p\colon E \to B$ be a map of H-group objects in simplicial spaces. Let $F(p) \subseteq \pi_0 B$ be the image of the map $\pi_0(p)\colon \pi_0 E \to \pi_0 B$ of simplicial sets, and let $V(p) \subseteq \pi_0 B$ is the union of path components of the simplicial set $\pi_0 B$ which do not touch $F(p)$. Then*

$$\mathrm{lproj}(p) = F(p) \cup V(p),$$

Proof. By (5.6), $F(p) \subseteq \mathrm{lproj}(p)$. By (5.2), $V(p) \subseteq \mathrm{lproj}(p)$. Thus $F(p) \cup V(p) \subseteq \mathrm{lproj}(p)$.

Now suppose $b \in \mathrm{lproj}(p)$, represented by $b\colon H_{[m]} \to B$. By definition, the homotopy pullback of p along b is equivalent to a projection map $H_{[m]} \times C \to H_{[m]}$. If $C = \varnothing$, then $b \in V(p)$. If $C \neq \varnothing$, then the projection map admits a section, and thus $b \in F(p)$. Thus $\mathrm{lproj}(p) \subseteq F(p) \cup V(p)$, as desired. □

Corollary 5.8. *Let $p\colon E \to B$ is a map of H-group objects in simplical spaces such that $\pi_0(p)\colon \pi_0 E \to \pi_0 B$ is surjective onto all components which touch its image. Then p is a local projection, and thus a realization-fibration.*

5.9. **The second local-to-global criterion for simplicial spaces.**

Proposition 5.10. *A map $p\colon E \to B$ of simplicial spaces is a realization-fibration if and only if for all $b\colon H_{[n]} \to B$ and all $f\colon [0] \to [n] \in \Delta$, the induced map $|\mathrm{Fib}(p, b \circ H_f)|_\Delta \to |\mathrm{Fib}(p, b)|_\Delta$ is a weak equivalence of spaces.*

Proof. This is essentially (2.13), together with the fact that $[0]$ is weakly initial in Δ. Let $b\colon H_{[n]} \to B$. Given $u\colon [m] \to [n]$ in Δ, choose any map $i\colon [0] \to [m]$ in Δ, so that we obtain

$$H_{[0]} \xrightarrow{H_i} H_{[m]} \xrightarrow{H_u} H_{[n]} \xrightarrow{b} B, \qquad H_{ui} = H_u \circ H_i\colon H_{[0]} \to H_{[n]}$$

Pulling back p along these maps, we obtain

$$\mathrm{Fib}(p, b \circ H_{ui}) \xrightarrow{g} \mathrm{Fib}(p, b \circ H_u) \xrightarrow{h} \mathrm{Fib}(p, b), \qquad hg\colon \mathrm{Fib}(p, b \circ H_{ui}) \to \mathrm{Fib}(p, b)$$

By hypothesis, $|g|_\Delta$ and $|hg|_\Delta$ are weak equivalences, and therefore so is $|h|_\Delta$, as desired. $\square$

6. Appendix: The π_*-Kan condition

I recall for comparison the Bousfield-Friedlander criterion based on the π_*-Kan condition.

Recall that $H_{[m]}$ denotes the functor Δ^{op} to spaces corepresented by $[m] \in \Delta$. Let $H_{\Lambda^i[m]} \subseteq H_{[m]}$ denote the subfunctor of $H_{[m]}$ with represents the "ith horn".

Let X be a simplical space. For $t \geq 1$, let $v\colon \underline{\pi}_t X \to X$ denote the map of simplicial spaces, defined so that for each $[m] \in \Delta$ the map

$$v([m])\colon (\underline{\pi}_t X)([m]) \to X([m])$$

is a covering map whose fiber over $a \in X([m])$ is $\pi_t(X([m]), a)$.

We say that X satisfies the **π_*-Kan condition** if for all $m, t \geq 1$, and all $0 \leq i \leq m$, a dotted arrow exists in every square of the form

$$\begin{array}{ccc} H_{\Lambda^i[m]} & \longrightarrow & \underline{\pi}_t X \\ \downarrow & \nearrow & \downarrow v \\ H_{[m]} & \longrightarrow & X \end{array}$$

If all $X([m])$ are connected, then X necessarily satisfies the π_*-Kan condition.

Say that a space K is **simple** if in each component, π_1 is abelian and acts trivially on π_t for all $t \geq 2$.

Proposition 6.1 ([BF78, B.3.1]). *Let X be is a simplical space with $X([m])$ simple for all m. Then X satisfies the π_*-Kan condition if and only if the map $[S^t, X] \to \pi_0 X$ is a fibration of simplicial sets. In particular, if X is a group object, then X satisfies the π_*-Kan condition.*

The Bousfield-Friedlander criterion is the following.

Theorem 6.2 ([BF78, Thm. B.4]). *Let $f\colon X \to Y$ be a map of simplicial spaces. If X and Y satisfy the π_*-Kan condition, and if $\pi_0 X \to \pi_0 Y$ is a fibration of simplicial sets, then f is RF.*

References

[BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130.

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