PROOF OF THE BLAKERS-MASSEY THEOREM

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Abstract. An exposition of some proofs of the Freudenthal suspension theorem and the Blakers-Massey theorem. These are meant to be reverse engineered versions of proofs in homotopy type theory due to Lumsdaine, Finster, and Licata. The proof of Blakers-Massey given here is based on a formalization given by Favonia.

1. Introduction

This proof is a reverse engineered version of the homotopy type theoretic proof given by Lumsdaine, Finster, and Licata (stated in [TUFP13, Theorem 8.10.1]), as formalized by Favonia (at `http://github.com/HoTT/HoTT-Agda/tree/1.0`).

I've written everything in (the homotopy theory of) spaces, but I expect that things go through in an arbitrary ∞-topos. "Pushout/pullback" really means "homotopy pushout/pullback", etc.

The first section gives some preliminaries on n-truncated and n-connected maps. The second section gives a reverse engineered version of the proof of the Freudenthal suspension theorem given in [TUFP13, Theorem 8.6.4]. The third section gives the reverse engineered proof of Blakers-Massey. The fourth section presents some allegedly helpful pictures.

1.1. **Truncated and connected.** Recall that a space X is n**-truncated** if $X \to \mathrm{Map}(S^{n+1}, X)$ is a weak equivalence. We say that a map $f\colon X \to Y$ is n-truncated if its homotopy fibers are n-truncated, or equivalently if $X \to \mathrm{Map}(S^{n+1}, Y) \times^h_{\mathrm{Map}(S^{n+1},Y)} Y$ is a weak equivalence.

We say that a space A is n**-connected** if $\mathrm{Map}(A, X)$ is contractible for any n-truncated space X; equivalently, A is n-connected if n-truncated map $f\colon X \to Y$ the map $\mathrm{Map}(A, f)\colon \mathrm{Map}(A, Y) \to \mathrm{Map}(A, X)$ is a weak equivalence.

More generally, we say that a map $j\colon A \to B$ is n-connected if for any n-truncated map $f\colon X \to Y$ the map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times^h_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a weak equivalence. That is, j is n-connected if and only if the (derived) space of solutions to the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

is contractible for every n-connected f.

The following properties of n-connected maps are immediate.

1.2. **Lemma.** *The composite of two n-connected maps is n-connected.*

1.3. **Lemma.** *The class of n-connected maps is closed under homotopy colimits. That is, given any diagram $\{f_i\colon A_i \to B_i\}$ such that each f_i is n-connected, then $\mathrm{colim}\, f_i\colon \mathrm{colim}\, A_i \to \mathrm{colim}\, B_i$ is n-connected.*

We have the following characterization of n-connected maps of spaces.

Date: January 19, 2015.

1.4. **Lemma.** *A map $j\colon X\to Y$ of spaces is n-connected if and only if its homotopy fibers are n-connected.*

This is a special case of the following.

1.5. **Lemma.** *Consider a homotopy pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

If f is n-connected, then f' is n-connected. If f' is n-connected and g is (-1)-connected, then f is n-connected.

Proof. To see that n-connected maps are closed under pullback, note that "pullback along g" admits (homotopically) a right adjoint, which necessarily preserves n-truncated maps.

If g is (-1)-connected then Y is equivalent to the realization of the Cech complex of g. The pullback g' is also (-1)-connected, so X is equivalent to the realization of the Cech complex of g'. That f is n-connected follows easily using (1.3). □

1.6. *Remark.* Our use of an "n-connected map" differs from the usual convention in topology by an offset of 1. Under this definition, $X\to *$ is n-connected iff X is an n-connected space in the usual sense. A space is (-1)-connected if and only if it is non-empty, and thus a map is (-1)-connected if and only if it is surjective on path components.

We also have the following.

1.7. **Lemma.** *If $X\xrightarrow{f} Y\xrightarrow{g} Z$ is such that g and gf are n-connected, then f is $(n-1)$-connected.*

Proof. By looking at homotopy fibers, it suffices to show that if X and Y are n-connected spaces, then any map $f\colon X\to Y$ is $(n-1)$-connected. □

As a consequence of the definition of truncation, we have that if $X\to *$ is n-truncated, then $X\to \operatorname{Map}(S^k,X)$ is $(n-k-1)$-truncated. More generally, if $X\to Y$ is n-truncated, then $X\to \operatorname{Map}(S^k,X)\times_{\operatorname{Map}(S^k,Y)} Y$ is $(n-k-1)$-truncated. More generally, we have the following.

1.8. **Proposition.** *If $j\colon A\to B$ is k-connected and $f\colon X\to Y$ is n-truncated, then $\operatorname{Map}(B,X)\to \operatorname{Map}(A,X)\times^h_{\operatorname{Map}(A,Y)}\operatorname{Map}(B,Y)$ is $(n-k-2)$-truncated.*

Using the lifting criterion and cartesian closedness, we obtain the following.

1.9. **Lemma** (Join connectivity)**.** *If $A\to X$ is m-connected and $B\to Y$ is n-connected, then $(A\times Y)\cup_{A\times B}(X\times B)\to X\times Y$ is $(m+2+n)$-connected.*

1.10. **Corollary.** *If X and Y are pointed, and X is m-connected and Y is n-connected, then $X\vee Y\to X\times Y$ is $(m+n)$-connected.*

Proof. If $X\to *$ is m-connected, then $*\to X$ is $(m-1)$-connected by (1.7). □

1.11. **Truncation and fiberwise truncation.** For every X, there exists an ***n*-truncation** $i\colon X\to |X|_n$, where $|X|_n$ is n-truncated and i is n-connected. A space X is n-truncated if and only if $|X|_n\approx *$.

More generally, given $f\colon X\to Y$, there exists a **fiberwise *n*-truncation**, which is a factorization

$$X\xrightarrow{i}|f|_n\xrightarrow{j}Y$$

where i is n-connected and j is n-truncated. This map is characterized by the fact that for each $y \in Y$, the induced map on homotopy fibers over y,

$$X_y \xrightarrow{i_y} (|f|_n)_y$$

presents $(|f|_n)_y$ as an n-truncation of X_y. It is essentially unique, characterized by being a factorization of f into an n-connected map followed by an n-truncated map.

1.12. **n-equifibered squares.** Say that a commutative square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[g]{} & Y' \end{array} \tag{1.13}$$

is **n-equifibered** if the induced square

$$\begin{array}{ccc} |f|_n & \longrightarrow & |f'|_n \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

involving fiberwise n-truncations is a homotopy pullback. Equivalently, it is n-equifibered if for each $y \in Y$ the maps $X_y \to X'_{g(y)}$ between homotopy fibers become equivalences after applying n-truncation.

Note: "n-equifibered" is similar to, but distinct from, "n-cartesian", which asserts that each of the maps $X_y \to X'_{g(y)}$ between homotopy fibers are $(n \pm \epsilon)$-connected. (I don't want to bother figuring out what ϵ is under the conventions I am using.)

The property "n-equifibered" (unlike n-cartesian) is not diagonally symmetric; it is really a condition on a map between maps $f \Rightarrow f'$.

We have the following result for recognizing n-equifibered squares.

1.14. **Lemma.** *A commutative square as in* (1.13) *is n-equifibered if there exists a factorization of it into two commutative squares*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \longrightarrow & Z' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ Y & \longrightarrow & Y' \end{array}$$

such that g and g' are n-connected and $h \Rightarrow h'$ is n-equifibered.

Proof. If $Z \xrightarrow{i} |h|_n \xrightarrow{j} Y$ is fiberwise n-truncation of h, then $X \xrightarrow{ig} |h|_n \xrightarrow{j} Y$ presents a fiberwise n-truncation of f, because g is n-connected. □

The following observation, though trivial, is crucial to our arguments.

1.15. **Lemma.** *If $f \Rightarrow f'$ is n-equifibered and f' is n-connected, then f is also n-connected.*

For instance, in the proof of the Freudenthal suspension theorem given below, we will show that $\sigma\colon X \to \Omega\Sigma X$ is $(2n)$-connected (for X n-connected) by constructing a $(2n)$-equifibered square

$$\sigma \Rightarrow \rho,$$

where $\rho\colon Y \to P$ is actually a map between contractible spaces. A similar technique is used in the proof of homotopy excision which follows.

To use this idea, we will use the following "patching" result for n-equifibered squares.

1.16. **Lemma.** *Consider a diagram of n-equifibered squares*

$$f_1 \overset{g_1}{\Leftarrow} f_0 \overset{g_2}{\Rightarrow} f_2,$$

and let f_{12} be the pushout of g_1 along g_2. Then the evident squares $f_1 \Rightarrow f_{12} \Leftarrow f_2$, as well as the composite square $f_0 \Rightarrow f_{12}$ are n-equifibered.

Proof. Apply fiberwise n-truncations to each f_i for $i = 0, 1, 2$. The usual descent property implies that the squares relating these fiberwise n-truncations to their pushout are pullback squares. Now use (1.3) to identify the pushout of fiberwise n-truncations as the fiberwise n-truncation of f_{12}. □

Finally, we use the following result on composition of n-equifibered squares.

1.17. **Lemma.** *Consider squares*

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ Y_1 & \xrightarrow[g]{} & Y_2 & \longrightarrow & Y_3 \end{array}$$

(1) *Suppose $f_2 \Rightarrow f_3$ is n-equifibered. Then $f_1 \Rightarrow f_2$ is n-equifibered iff $f_1 \Rightarrow f_3$ is n-equifibered.*
(2) *Suppose g is (-1)-connected, and $f_1 \Rightarrow f_2$ and $f_1 \Rightarrow f_3$ are n-equifibered. Then $f_2 \Rightarrow f_3$ is n-equifibered.*

Proof. Proved just as for pullback squares. □

2. The Freudenthal suspension theorem

2.1. **Theorem** (Freudenthal)**.** *Suppose X is n-connected and pointed. Then $\sigma\colon X \to \Omega\Sigma X$ is $(2n)$-connected.*

I give the proof below. First note that if $n < 0$, then the statement is vacuous (every map is (-2)-connected), so we may assume $n \geq 0$.

Write $G = \Omega\Sigma X$. I am going to pretend that G is actually a monoid.

Consider the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{(\mathrm{id},*)} & X \vee X & \xrightarrow{(\mathrm{id},\mathrm{id})} & X \\ {\scriptstyle \sigma}\downarrow & & {\scriptstyle \tau}\downarrow & & {\scriptstyle \sigma}\downarrow \\ G & \xleftarrow{\pi} & G \times X & \xrightarrow{\mu} & G \end{array} \tag{2.2}$$

Here σ is the standard unit map, π is projection, and μ is "multiplication", i.e., $\mu(\gamma, x) = \gamma \cdot \sigma(x)$. The map τ is given on the left summand by $(\sigma, *)\colon X \to G \times X$, and on the right summand by $(*, \mathrm{id})\colon X \to G \times X$.

Take homotopy colimits in each row to get a map $\rho\colon Y \to P$. We observe that

(1) Y is obviously contractible.
(2) P is contractible, since it presents a decomposition of the path space fibration $P \to \Sigma X$, where ΣX is assembled as the colimit of $* \leftarrow X \to *$.

We now claim that both squares in (2.2), i.e., the maps $\sigma \Leftarrow \tau \Rightarrow \sigma$, are $(2n)$-equifibered. As a consequence, it will follow from (1.16) that (either) square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\rho} \\ G & \longrightarrow & P \end{array}$$

is $(2n)$-equifibered. Then (1.15) implies that σ is $(2n)$-connected, because ρ is an equivalence and thus $(2n)$-connected.

The left-hand square of (2.2)*.* Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \vee X & \xrightarrow{(\mathrm{id},*)} & X \\ \| & & {\scriptstyle f}\downarrow & & \| \\ X & \xrightarrow{(\mathrm{id},*)} & X \times X & \xrightarrow{\pi_1} & X \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\sigma\times\mathrm{id}}\downarrow & & {\scriptstyle\sigma}\downarrow \\ G & \xrightarrow{(\mathrm{id},*)} & G \times X & \xrightarrow{\pi} & G \end{array} \tag{2.3}$$

where f is the wedge inclusion. The two lower squares of (2.3) are manifestly homotopy pullbacks. The map f is $(2n)$-connected by (1.10). It follows that the two tall rectangles in (2.3) are $(2n)$-equifibered by (1.14). The right-hand tall rectangle of (2.3) is precisely the left-hand square of (2.2).

The right-hand square of (2.2)*.* Consider

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \vee X & \xrightarrow{(\mathrm{id},\mathrm{id})} & X \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\sigma} \\ G & \xrightarrow{(\mathrm{id},*)} & G \times X & \xrightarrow{\mu} & G \end{array} \tag{2.4}$$

The left-hand square in (2.4) is the tall left-hand rectangle in (2.3), and so is $(2n)$-equifibered. The composite rectangle of (2.4) is a pullback (the composite horizontal maps are identities), so is certainly $(2n)$-equifibered. Because X is 0-connected, $(\mathrm{id}, *)\colon G \to G \times X$ is (-1)-connected, so statement (2) of (1.17) applies to show that the right-hand square in (2.4) is $(2n)$-equifibered, and this square is precisely the right-hand square of (2.2).

We are done.

3. The homotopy excision theorem

3.1. **Theorem** (Blakers-Massey)**.** *Consider a homotopy pushout square*

$$\begin{array}{ccc} Q & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & P \end{array}$$

Let $R := X \times^h_P Y$ denote the homotopy pullback. If f is m-connected and g is n-connected, and $m, n \geq -1$, then the tautological map $Q \to R$ is $(m + n)$-connected.

Because connectivity (1.5), pushouts, and pullbacks are preserved under (homotopy) pullbacks, to prove the conclusion of the theorem it suffices to prove it after (homotopy) pullback along any map $* \to P$. That is, we can immediately reduce to the following special case.

3.2. **Proposition.** *Let* $X \xleftarrow{f} Q \xrightarrow{g} Y$ *be maps such that* f *is* m*-connected,* g *is* n*-connected,* $m, n \geq -1$*, and the homotopy pushout of* f *along* g *is contractible. Then* $(f,g)\colon Q \to X \times Y$ *is* $(m+n)$*-connected.*

Without loss of generality, we can assume that $(f,g)\colon Q \to X \times Y$ is a fibration. I will sometimes use the following notation: if $(x,y) \in X \times Y$, then $Q(x,y)$ denotes the fiber of (f,g) over (x,y). I will also write $Q(X,y)$ for the pullback of $(f,g)\colon Q \to X \times Y$ along $X \times \{y\} \to X \times Y$, and similarly $Q(x,Y)$ for the pullback of (f,g) along $\{x\} \times Y \to X \times Y$.

Note that $Q(X,y)$ is precisely the fiber of g over y, and thus the hypotheses of the theorem assert that $Q(X,y)$ is n-connected. Likewise, $Q(x,Y)$ is precisely the fiber of f over x, so the hypotheses of the theorem assert that $Q(x,Y)$ is m-connected.

Consider the pullback square

$$\begin{array}{ccc} Q \times_X Q & \xrightarrow{(q_{00},q_{01}) \mapsto q_{01}} & Q \\ {\scriptstyle \sigma_Y := \left((q_{00},q_{01}) \mapsto (q_{00}, g(q_{01}))\right)} \big\downarrow & & \big\downarrow {\scriptstyle (f,g)} \\ Q \times Y & \xrightarrow[f \times 1_Y]{} & X \times Y \end{array}$$

Because f is (-1)-connected, so is $f \times 1_Y$, and thus to prove the result it suffices suffices to show that σ_Y is $(m+n)$-connected (1.5).

Let J be the homotopy pushout

$$Q \times_Y Q \leftarrow Q \rightarrow Q \times_X Q$$

along diagonal inclusions. Let $j_X\colon Q \times_Y Q \to J$ and $j_Y\colon Q \times_X Q \to J$ denote the tautological maps.

Now we consider the following commutative diagram.

$$\begin{array}{ccccc} Q \times_Y Q & \xleftarrow{p_X} & J & \xrightarrow{p_Y} & Q \times_X Q \\ {\scriptstyle \sigma_X} \big\downarrow & & {\scriptstyle \tau} \big\downarrow & & {\scriptstyle \sigma_Y} \big\downarrow \\ Q \times X & \xleftarrow[1_Q \times f]{} & Q \times Q & \xrightarrow[1_Q \times g]{} & Q \times Y \end{array} \tag{3.3}$$

where the maps are defined as follows.

$$\sigma_X(q_{00}, q_{10}) = (q_{00}, f(q_{10})),$$
$$\sigma_Y(q_{00}, q_{01}) = (q_{00}, g(q_{01})).$$

The map σ_Y is precisely the one we need to show is $(m+n)$-connected.

$$\begin{aligned} p_X j_X(q_{0\_}, q_{1\_}) &= (q_{0\_}, q_{1\_}), & p_X j_Y(q_{\_0}, q_{\_1}) &= (q_{\_0}, q_{\_0}), \\ p_Y j_X(q_{0\_}, q_{1\_}) &= (q_{0\_}, q_{0\_}), & p_Y j_Y(q_{\_0}, q_{\_1}) &= (q_{\_0}, q_{\_1}), \\ \tau j_X(q_{0\_}, q_{1\_}) &= (q_{0\_}, q_{1\_}), & \tau j_Y(q_{\_0}, q_{\_1}) &= (q_{\_0}, q_{\_1}). \end{aligned}$$

3.4. **Lemma.** *The map* μ *induced by taking homotopy colimits along rows in* (3.3) *is equivalent to the identity map of* Q*.*

Proof. For the top row, consider the commutative square

$$\begin{array}{ccc} J & \xrightarrow{p_Y} & Q \times_X Q \\ {\scriptstyle p_X} \big\downarrow & & \big\downarrow {\scriptstyle (q_{00},q_{10}) \mapsto q_{00}} \\ Q \times_Y Q & \xrightarrow[(q_{00},q_{01}) \mapsto q_{00}]{} & Q \end{array}$$

To show this is a homotopy pushout, it suffices to show that the square of fibers over any point $\{q_{00}\} \in Q$ is a homotopy pushout. The square of the fibers has the form

$$\begin{array}{ccc} Q(X,y_0) \vee_{\{q_{00}\}} Q(x_0,Y) & \xrightarrow{(*,\mathrm{id})} & Q(x_0,Y) \\ \downarrow{\scriptstyle (\mathrm{id},*)} & & \downarrow \\ Q(X,y_0) & \longrightarrow & * \end{array}$$

which is clearly a homotopy pushout.

The rest of the proof is straightforward, using the fact that the assumption that the pushout of $X \xleftarrow{f} Q \xrightarrow{g} Y$ is contractible to show that the homotopy pushout of the bottom row is also equivalent to Q. □

We define maps $d, d' \colon J \to Q \times_X Q \times_Y Q$ as follows.

$$dj_X(q_{0\_}, q_{1\_}) = (q_{0\_}, q_{0\_}, q_{1\_}), \qquad dj_Y(q_{\_0}, q_{\_1}) = (q_{\_0}, q_{\_1}, q_{\_1}),$$
$$d'j_X(q_{0\_}, q_{1\_}) = (q_{1\_}, q_{1\_}, q_{0\_}), \qquad d'j_Y(q_{\_0}, q_{\_1}) = (q_{\_1}, q_{\_0}, q_{\_0}).$$

3.5. **Lemma.** *The maps d and d' are $(m+n)$-connected.*

Proof. We actually have that $d' = di$, where $i \colon J \to J$ is the involution of J defined by $ij_X(q_{0\_}, q_{1\_}) = (q_{1\_}, q_{0\_})$ and $ij_Y(q_{\_0}, q_{\_1}) = (q_{\_1}, q_{\_0})$. Thus, it suffices to show that d is $(m+n)$-connected.

The map d is induced by the commutative square

$$\begin{array}{ccc} Q & \longrightarrow & Q \times_Y Q \\ \downarrow & & \downarrow{\scriptstyle (q_{0\_},q_{1\_}) \mapsto (q_{0\_},q_{0\_},q_{1\_})} \\ Q \times_X Q & \xrightarrow[(q_{\_0},q_{\_1}) \mapsto (q_{\_0},q_{\_1},q_{\_1})]{} & Q \times_X Q \times_Y Q \end{array} \tag{3.6}$$

To show that d is $(m+n)$-connected it suffices to show (by (1.5)) that for any point $q_{01} \in Q(x_0, y_1) \subseteq Q$, the pullback of d along the inclusion

$$k \colon Q(x_0, Y) \times Q(X, y_1) = Q \times_X \{q_{01}\} \times_Y Q \to Q \times_X Q \times_Y Q,$$

is $(m+n)$-connected. The pullback of (3.6) along k is the square

$$\begin{array}{ccc} \{q_{01}\} & \longrightarrow & Q(X,y_1) \\ \downarrow & & \downarrow \\ Q(x_0,Y) & \longrightarrow & Q(x_0,Y) \times Q(X,y_1) \end{array}$$

The pullback of d along k is a the map

$$Q(x_0,Y) \times \{q_{01}\} \cup_{\{(q_{01},q_{01})\}} \{q_{01}\} \times Q(X,y_1) \to Q(x_0,Y) \times Q(X,y_1),$$

which is $(m+n)$-connected by (1.10). □

3.7. **Lemma.** *Each of the two squares in* (3.3) *is $(m+n)$-equifibered.*

Proof. For the right-hand square, consider the commutative square

$$\begin{array}{ccc}
J & \xrightarrow{p_Y} & Q \times_X Q \\
\downarrow{\scriptstyle d} & & \| \\
Q \times_X Q \times_Y Q & \xrightarrow[(q_{00},q_{01},q_{11})\mapsto(q_{00},q_{01})]{} & Q \times_X Q \\
{\scriptstyle (q_{00},q_{01},q_{11})\mapsto(q_{00},q_{11})}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\
Q \times Q & \xrightarrow[1_Q \times g]{} & Q \times Y
\end{array}$$

in which the the composite of the left-hand column is τ, the lower square is a pullback, and and d is $(m+n)$-connected by (3.5). The result follows by (1.14).

For he left-hand square, consider the commutative square

$$\begin{array}{ccc}
Q \times_Y Q & \xleftarrow{p_X} & J \\
\| & & \downarrow{\scriptstyle d'} \\
Q \times_Y Q & \xleftarrow[(q_{00},q_{10})\leftarrow\!\shortmid(q_{11},q_{10},q_{00})]{} & Q \times_X Q \times_Y Q \\
{\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle (q_{11},q_{10},q_{00})\mapsto(q_{00},q_{11})} \\
Q \times X & \xleftarrow[1_Q \times f]{} & Q \times Q
\end{array}$$

in which the composite of the right-hand column is τ, the lower square is a pullback, and d' is $(m+n)$-connected by (3.5). The result follows by (1.14). □

Now we can finish the proof of (3.2), and thus of the homotopy excision theorem. As noted earlier, it suffies to show that σ_Y is $(m+n)$-connected. By (3.7), each of the two squares in (3.3) is $(m+n)$-equifibered. By (1.16) it follows that the square $\sigma_Y \Rightarrow \mu$ relating σ_Y to the homotopy pushout of the rows of (3.3) is $(m+n)$-equifibered. By (3.4) the map μ is equivalent to the identity map of Q, and thus in particular is $(m+n)$-connected. It follows that σ_Y is $(m+n)$-connected by (1.15).

4. Graphic remarks on the proof of homotopy excision

Fix a fibration $(f,g)\colon Q \to X \times Y$, and consider the diagram

(4.1)
$$\begin{array}{ccccc}
Q \times_Y Q & & & & Q \times_X Q \\
{\scriptstyle \sigma_X}\downarrow & & & & \downarrow{\scriptstyle \sigma_Y} \\
Q \times X & \xleftarrow[1_Q \times f]{} & Q \times Q & \xrightarrow[1_Q \times g]{} & Q \times Y
\end{array}$$

using the notation of the previous section. The limit U of this diagram (which is also the homotopy limit) is the space of 4-tuples $(q_{ij})_{i,j=0,1} \in Q^4$ satisfying

$$g(q_{00}) = g(q_{10}), \quad f(q_{10}) = f(q_{11}), \quad g(q_{11}) = g(q_{01}), \quad f(q_{01}) = f(q_{00}).$$

We can illustrate such a point in U using the picture

$$\begin{array}{ccc}
y_0 & -q_{00}- & x_0 \\
| & & | \\
q_{10} & & q_{01} \\
| & & | \\
x_1 & -q_{11}- & y_1
\end{array}$$

Let J' be the subspace of U consisting of tuples such that *either* (i) $q_{00} = q_{01}$ and $q_{10} = q_{11}$ or (ii) $q_{00} = q_{10}$ and $q_{01} = q_{11}$. That is, J' consists of points of U which are of the form

$$\begin{array}{ccc} y & \overset{q_{0-}}{—} & x_0 \\ q_{1-}\big| & & \big|q_{0-} \\ x_1 & \underset{q_{1-}}{—} & y \end{array} \qquad \text{or} \qquad \begin{array}{ccc} y_0 & \overset{q_{-0}}{—} & x \\ q_{-0}\big| & & \big|q_{-1} \\ x & \underset{q_{-1}}{—} & y_1 \end{array}$$

There is a tautological map $J \to J'$ from the homotopy pushout of $Q \times_Y Q \leftarrow Q \to Q \times_X Q$ to J'.

More generally, we can represent objects and maps in the diagram (4.1) using the pictures

$$\begin{array}{ccccc} \left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ q_{10}\big| & & \\ x_1 & & \end{array}\right) & & & & \left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ & & \big|q_{01} \\ & & y_1 \end{array}\right) \\ \Downarrow & & & & \Downarrow \\ \left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ & & \\ x_1 & & \end{array}\right) & \Leftarrow & \left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ & & \\ x_1 & \underset{q_{11}}{—} & y_1 \end{array}\right) & \Rightarrow & \left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ & & \\ & & y_1 \end{array}\right) \end{array}$$

The maps p_X, p_Y, and τ from J in (3.3) are the evident ones determined by the inclusion $J' \subseteq U$ into the limit of (4.1).

A key step in the argument is the observation that the induced maps d' and d from J into each of the two pullbacks implicit in (4.1) is $(m+n)$-connected. These maps are induced by maps from J' having the form

$$\left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ q_{10}\big| & & \\ x_1 & \underset{q_{11}}{—} & y_1 \end{array}\right) \overset{d'}{\Leftarrow} \left(\begin{array}{ccc} y & \overset{q_{0-}}{—} & x_0 \\ q_{1-}\big| & & \big|q_{0-} \\ x_1 & \underset{q_{1-}}{—} & y \end{array} \quad \text{or} \quad \begin{array}{ccc} y_0 & \overset{q_{-0}}{—} & x \\ q_{-0}\big| & & \big|q_{-1} \\ x & \underset{q_{-1}}{—} & y_1 \end{array}\right) \overset{d}{\Rightarrow} \left(\begin{array}{ccc} y_0 & \overset{q_{00}}{—} & x_0 \\ & & \big|q_{01} \\ x_1 & \underset{q_{11}}{—} & y_1 \end{array}\right)$$

References

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