PROOF OF THE BLAKERS-MASSEY THEOREM

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ABSTRACT. An exposition of some proofs of the Freudenthal suspension theorem and the Blakers-Massey theorem. These are meant to be reverse engineered versions of proofs in homotopy type theory due to Lumsdaine, Finster, and Licata. The proof of Blakers-Massey given here is based on a formalization given by Favonia.

1. INTRODUCTION

This proof is a reverse engineered version of the homotopy type theoretic proof given by Lumsdaine, Finster, and Licata (stated in [TUFPI3, Theorem 8.10.1]), as formalized by Favonia (at http://github.com/HoTT/HoTT-Agda/tree/1.0).

I’ve written everything in (the homotopy theory of) spaces, but I expect that things go through in an arbitrary $\infty$-topos. “Pushout/pullback” really means “homotopy pushout/pullback”, etc.

The first section gives some preliminaries on $n$-truncated and $n$-connected maps. The second section gives a reverse engineered version of the proof of the Freudenthal suspension theorem given in [TUFPI3, Theorem 8.6.4]. The third section gives the reverse engineered proof of Blakers-Massey. The fourth section presents some allegedly helpful pictures.

1.1. Truncated and connected. Recall that a space $X$ is $n$-truncated if $X \to \text{Map}(S^{n+1}, X)$ is a weak equivalence. We say that a map $f: X \to Y$ is $n$-truncated if its homotopy fibers are $n$-truncated, or equivalently if $X \to \text{Map}(S^{n+1}, Y) \times^{h}_{\text{Map}(S^{n+1}, Y)} Y$ is a weak equivalence.

We say that a space $A$ is $n$-connected if $\text{Map}(A, X)$ is contractible for any $n$-truncated space $X$; equivalently, $A$ is $n$-connected if $n$-truncated map $f: X \to Y$ the map $\text{Map}(A, f): \text{Map}(A, Y) \to \text{Map}(A, X)$ is a weak equivalence.

More generally, we say that a map $j: A \to B$ is $n$-connected if for any $n$-truncated map $f: X \to Y$ the map

$$\text{Map}(B, X) \to \text{Map}(A, X) \times^{h}_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a weak equivalence. That is, $j$ is $n$-connected if and only if the (derived) space of solutions to the lifting problem

$$A \xrightarrow{j} X \xrightarrow{f} Y$$

is contractible for every $n$-connected $f$.

The following properties of $n$-connected maps are immediate.

1.2. Lemma. The composite of two $n$-connected maps is $n$-connected.

1.3. Lemma. The class of $n$-connected maps is closed under homotopy colimits. That is, given any diagram $\{f_i: A_i \to B_i\}$ such that each $f_i$ is $n$-connected, then $\text{colim} f_i: \text{colim} A_i \to \text{colim} B_i$ is $n$-connected.

We have the following characterization of $n$-connected maps of spaces.

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1.4. **Lemma.** A map $j: X \to Y$ of spaces is $n$-connected if and only if its homotopy fibers are $n$-connected.

This is a special case of the following.

1.5. **Lemma.** Consider a homotopy pullback square

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

If $f$ is $n$-connected, then $f'$ is $n$-connected. If $f'$ is $n$-connected and $g$ is $(-1)$-connected, then $f$ is $n$-connected.

**Proof.** To see that $n$-connected maps are closed under pullback, note that “pullback along $g$” admits (homotopically) a right adjoint, which necessarily preserves $n$-truncated maps.

If $g$ is $(-1)$-connected then $Y$ is equivalent to the realization of the Cech complex of $g$. The pullback $g'$ is also $(-1)$-connected, so $X$ is equivalent to the realization of the Cech complex of $g'$. That $f$ is $n$-connected follows easily using [1.3].

1.6. **Remark.** Our use of an “$n$-connected map” differs from the usual convention in topology by an offset of 1. Under this definition, $X \to *$ is $n$-connected iff $X$ is an $n$-connected space in the usual sense. A space is $(-1)$-connected if and only if it is non-empty, and thus a map is $(-1)$-connected if and only if it is surjective on path components.

We also have the following.

1.7. **Lemma.** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is such that $g$ and $gf$ are $n$-connected, then $f$ is $(n - 1)$-connected.

**Proof.** By looking at homotopy fibers, it suffices to show that if $X$ and $Y$ are $n$-connected spaces, then any map $f: X \to Y$ is $(n - 1)$-connected.

As a consequence of the definition of truncation, we have that if $X \to *$ is $n$-truncated, then $X \to \text{Map}(S^k, X)$ is $(n - k - 1)$-truncated. More generally, if $X \to Y$ is $n$-truncated, then $X \to \text{Map}(S^k, X) \times_{\text{Map}(S^k, Y)} Y$ is $(n - k - 1)$-truncated. More generally, we have the following.

1.8. **Proposition.** If $j: A \to B$ is $k$-connected and $f: X \to Y$ is $n$-truncated, then $\text{Map}(B, X) \to \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$ is $(n - k - 2)$-truncated.

Using the lifting criterion and cartesian closedness, we obtain the following.

1.9. **Lemma** (Join connectivity). If $A \to X$ is $m$-connected and $B \to Y$ is $n$-connected, then $(A \times Y) \cup_{A \times B} (X \times B) \to X \times Y$ is $(m + 2 + n)$-connected.

1.10. **Corollary.** If $X$ and $Y$ are pointed, and $X$ is $m$-connected and $Y$ is $n$-connected, then $X \vee Y \to X \times Y$ is $(m + n)$-connected.

**Proof.** If $X \to *$ is $m$-connected, then $* \to X$ is $(m - 1)$-connected by [1.7].

1.11. **Truncation and fiberwise truncation.** For every $X$, there exists an $n$-truncation $i: X \to |X|_n$, where $|X|_n$ is $n$-truncated and $i$ is $n$-connected. A space $X$ is $n$-truncated if and only if $|X|_n \approx *$.

More generally, given $f: X \to Y$, there exists a fiberwise $n$-truncation, which is a factorization

$$
X \xrightarrow{i} |f|_n \xrightarrow{j} Y
$$
where $i$ is $n$-connected and $j$ is $n$-truncated. This map is characterized by the fact that for each $y \in Y$, the induced map on homotopy fibers over $y$,

$$X_y \xrightarrow{i_y} (|f|_n)_y$$

presents $(|f|_n)_y$ as an $n$-truncation of $X_y$. It is essentially unique, characterized by being a factorization of $f$ into an $n$-connected map followed by an $n$-truncated map.

1.12. $n$-equifibered squares. Say that a commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow g & & \downarrow g' \\
Y & \xrightarrow{h} & Y'
\end{array}
$$

is $n$-equifibered if the induced square

$$
\begin{array}{ccc}
|f|_n & \xrightarrow{\sim} & |f'|_n \\
\downarrow j & & \downarrow j' \\
Y & \xrightarrow{g} & Y'
\end{array}
$$

involving fiberwise $n$-truncations is a homotopy pullback. Equivalently, it is $n$-equifibered if for each $y \in Y$ the maps $X_y \to X'_y$ between homotopy fibers become equivalences after applying $n$-truncation.

Note: “$n$-equifibered” is similar to, but distinct from, “$n$-cartesian”, which asserts that each of the maps $X_y \to X'_y$ between homotopy fibers are $(n \pm \epsilon)$-connected. (I don’t want to bother figuring out what $\epsilon$ is under the conventions I am using.)

The property “$n$-equifibered” (unlike $n$-cartesian) is not diagonally symmetric; it is really a condition on a map between maps $f \Rightarrow f'$.

We have the following result for recognizing $n$-equifibered squares.

1.14. Lemma. A commutative square as in (1.13) is $n$-equifibered if there exists a factorization of it into two commutative squares

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
\downarrow h & & \downarrow h' \\
Y & \xrightarrow{\sim} & Y'
\end{array}
$$

such that $g$ and $g'$ are $n$-connected and $h \Rightarrow h'$ is $n$-equifibered.

Proof. If $Z \xrightarrow{j} |h|_n \xrightarrow{i} Y$ is fiberwise $n$-truncation of $h$, then $X \xrightarrow{ig} |h|_n \xrightarrow{i} Y$ presents a fiberwise $n$-truncation of $f$, because $g$ is $n$-connected. □

The following observation, though trivial, is crucial to our arguments.

1.15. Lemma. If $f \Rightarrow f'$ is $n$-equifibered and $f'$ is $n$-connected, then $f$ is also $n$-connected.

For instance, in the proof of the Freudenthal suspension theorem given below, we will show that $\sigma : X \to \Omega \Sigma X$ is $(2n)$-connected (for $X$ $n$-connected) by constructing a $(2n)$-equifibered square

$$
\sigma \Rightarrow \rho.
$$
where \( \rho: Y \to P \) is actually a map between contractible spaces. A similar technique is used in the proof of homotopy excision which follows.

To use this idea, we will use the following “patching” result for \( n \)-equifibered squares.

1.16. Lemma. Consider a diagram of \( n \)-equifibered squares

\[
\begin{array}{ccc}
  f_1 & \cong & f_2 \\
  g_1 & \Rightarrow & g_2 \\
  f_0 & \Rightarrow & f_1 \\
  g_0 & \Rightarrow & g_2 \\
\end{array}
\]

and let \( f_{12} \) be the pushout of \( g_1 \) along \( g_2 \). Then the evident squares \( f_1 \Rightarrow f_{12} \Leftarrow f_2 \), as well as the composite square \( f_0 \Rightarrow f_{12} \) are \( n \)-equifibered.

Proof. Apply fiberwise \( n \)-truncations to each \( f_i \) for \( i = 0, 1, 2 \). The usual descent property implies that the squares relating these fiberwise \( n \)-truncations to their pushout are pullback squares. Now use (1.3) to identify the pushout of fiberwise \( n \)-truncations as the fiberwise \( n \)-truncation of \( f_{12} \). \( \square \)

Finally, we use the following result on composition of \( n \)-equifibered squares.

1.17. Lemma. Consider squares

\[
\begin{array}{ccc}
  X_1 & \longrightarrow & X_2 \longrightarrow X_3 \\
  f_1 & \downarrow & f_2 \downarrow & f_3 \\
  Y_1 & \stackrel{g}{\longrightarrow} & Y_2 \longrightarrow Y_3 \\
\end{array}
\]

(1) Suppose \( f_2 \Rightarrow f_3 \) is \( n \)-equifibered. Then \( f_1 \Rightarrow f_2 \) is \( n \)-equifibered iff \( f_1 \Rightarrow f_3 \) is \( n \)-equifibered.

(2) Suppose \( g \) is \((-1)\)-connected, and \( f_1 \Rightarrow f_2 \) and \( f_1 \Rightarrow f_3 \) are \( n \)-equifibered. Then \( f_2 \Rightarrow f_3 \) is \( n \)-equifibered.

Proof. Proved just as for pullback squares. \( \square \)

2. The Freudenthal suspension theorem

2.1. Theorem (Freudenthal). Suppose \( X \) is \( n \)-connected and pointed. Then \( \sigma: X \to \Omega \Sigma X \) is \((2n)\)-connected.

I give the proof below. First note that if \( n < 0 \), then the statement is vacuous (every map is \((-2)\)-connected), so we may assume \( n \geq 0 \).

Write \( G = \Omega \Sigma X \). I am going to pretend that \( G \) is actually a monoid.

Consider the commutative diagram

\[
\begin{array}{ccc}
  X & \overset{(id,\ast)}{\longrightarrow} & X \lor X \overset{(id,id)}{\longrightarrow} X \\
  \sigma & \downarrow & \tau \downarrow & \sigma \\
  G & \leftarrow \pi & G \times X & \stackrel{\mu}{\longrightarrow} & G \\
\end{array}
\]

(2.2)

Here \( \sigma \) is the standard unit map, \( \pi \) is projection, and \( \mu \) is “multiplication”, i.e., \( \mu(\gamma, x) = \gamma \cdot \sigma(x) \).

The map \( \tau \) is given on the left summand by \( (\sigma, \ast): X \to G \times X \), and on the right summand by \( (\ast, id): X \to G \times X \).

Take homotopy colimits in each row to get a map \( \rho: Y \to P \). We observe that

(1) \( Y \) is obviously contractible.

(2) \( P \) is contractible, since it presents a decomposition of the path space fibration \( P \to \Sigma X \), where \( \Sigma X \) is assembled as the colimit of \( \ast \leftarrow X \to \ast \).
We now claim that both squares in (2.2), i.e., the maps \( \sigma \Leftarrow \tau \Rightarrow \sigma \), are \((2n)\)-equifibered. As a consequence, it will follow from (1.16) that (either) square
\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\sigma & \downarrow & \rho \\
G & \longrightarrow & P
\end{array}
\]
is \((2n)\)-equifibered. Then (1.15) implies that \( \sigma \) is \((2n)\)-connected, because \( \rho \) is an equivalence and thus \((2n)\)-connected.

**The left-hand square of (2.2).** Consider the commutative diagram
\[
\begin{array}{ccc}
X & \overset{i_1}{\longrightarrow} & X \lor X & \overset{(\text{id},*)}{\longrightarrow} & X \\
\sigma & \downarrow & \sigma \times \text{id} & \downarrow & \sigma \\
G & \overset{(\text{id},*)}{\longrightarrow} & G \times X & \overset{\pi}{\longrightarrow} & G
\end{array}
\]
where \( f \) is the wedge inclusion. The two lower squares of (2.3) are manifestly homotopy pullbacks. The map \( f \) is \((2n)\)-connected by (1.10). It follows that the two tall rectangles in (2.3) are \((2n)\)-equifibered by (1.14). The right-hand tall rectangle of (2.3) is precisely the left-hand square of (2.2).

**The right-hand square of (2.2).** Consider
\[
\begin{array}{ccc}
X & \overset{i_1}{\longrightarrow} & X \lor X & \overset{(\text{id},\text{id})}{\longrightarrow} & X \\
\sigma & \downarrow & \sigma \times \text{id} & \downarrow & \sigma \\
G & \overset{(\text{id},*)}{\longrightarrow} & G \times X & \overset{\mu}{\longrightarrow} & G
\end{array}
\]
The left-hand square in (2.4) is the tall left-hand rectangle in (2.3), and so is \((2n)\)-equifibered. The composite rectangle of (2.4) is a pullback (the composite horizontal maps are identities), so is certainly \((2n)\)-equifibered. Because \( X \) is 0-connected, \((\text{id},*)\): \( G \to G \times X \) is \((-1)\)-connected, so statement (2) of (1.17) applies to show that the right-hand square in (2.4) is \((2n)\)-equifibered, and this square is precisely the right-hand square of (2.2).

We are done.

3. **The homotopy excision theorem**

3.1. **Theorem** (Blakers-Massey). Consider a homotopy pushout square
\[
\begin{array}{ccc}
Q & \overset{g}{\longrightarrow} & Y \\
f & \downarrow & \downarrow \\
X & \longrightarrow & P
\end{array}
\]
Let \( R := X \times^h P Y \) denote the homotopy pullback. If \( f \) is \( m \)-connected and \( g \) is \( n \)-connected, and \( m, n \geq -1 \), then the tautological map \( Q \to R \) is \((m+n)\)-connected.

Because connectivity (1.5), pushouts, and pullbacks are preserved under (homotopy) pullbacks, to prove the conclusion of the theorem it suffices to prove it after (homotopy) pullback along any map \( * \to P \). That is, we can immediately reduce to the following special case.
3.2. **Proposition.** Let $X \xleftarrow{f} Q \xrightarrow{g} Y$ be maps such that $f$ is $m$-connected, $g$ is $n$-connected, $m, n \geq -1$, and the homotopy pushout of $f$ along $g$ is contractible. Then $(f, g) : Q \to X \times Y$ is $(m + n)$-connected.

Without loss of generality, we can assume that $(f, g) : Q \to X \times Y$ is a fibration. I will sometimes use the following notation: if $(x, y) \in X \times Y$, then $Q(x, y)$ denotes the fiber of $(f, g)$ over $(x, y)$. I will also write $Q(X, y)$ for the pullback of $(f, g) : Q \to X \times Y$ along $X \times \{y\} \to X \times Y$, and similarly $Q(x, Y)$ for the pullback of $(f, g)$ along $\{x\} \times Y \to X \times Y$.

Note that $Q(X, y)$ is precisely the fiber of $g$ over $y$, and thus the hypotheses of the theorem assert that $Q(X, y)$ is $n$-connected. Likewise, $Q(x, Y)$ is precisely the fiber of $f$ over $x$, so the hypotheses of the theorem assert that $Q(x, Y)$ is $m$-connected.

Consider the pullback square

$$
\begin{array}{ccc}
Q \times Y & \xrightarrow{(q_{00}, q_{01}) \to q_{01}} & Q \\
\sigma_y := (q_{00}, q_{01}) \to (q_{00}, g(q_{01})) & \downarrow \sigma_y & \downarrow (f, g) \\
Q \times X & \xrightarrow{f \times 1_Y} & X \times Y
\end{array}
$$

Because $f$ is $(-1)$-connected, so is $f \times 1_Y$, and thus to prove the result it suffices to show that $\sigma_y$ is $(m + n)$-connected (1.5).

Let $J$ be the homotopy pushout

$$
Q \times Y \xleftarrow{\sigma_y} Q \xrightarrow{\sigma_y} Q \times X Q
$$

along diagonal inclusions. Let $j_X : Q \times Y Q \to J$ and $j_Y : Q \times X Q \to J$ denote the tautological maps.

Now we consider the following commutative diagram.

(3.3)

$$
\begin{array}{ccc}
Q \times Y & \xrightarrow{p_X} & J \\
\sigma_X & \downarrow \tau & \sigma_Y \\
Q \times X & \xleftarrow{1_Q \times f} & Q \times Y
\end{array}
$$

where the maps are defined as follows.

$$
\sigma_X(q_{00}, q_{10}) = (q_{00}, f(q_{10})), \\
\sigma_Y(q_{00}, q_{01}) = (q_{00}, g(q_{01})).
$$

The map $\sigma_Y$ is precisely the one we need to show is $(m + n)$-connected.

$$
\begin{align*}
p_X j_X(q_{00}, q_{10}) &= (q_{00}, q_{10}), \\
p_Y j_Y(q_{00}, q_{10}) &= (q_{00}, q_{10}), \\
\tau j_X(q_{00}, q_{10}) &= (q_{00}, q_{10}), \\
\tau j_Y(q_{00}, q_{10}) &= (q_{00}, q_{10}).
\end{align*}
$$

3.4. **Lemma.** The map $\mu$ induced by taking homotopy colimits along rows in (3.3) is equivalent to the identity map of $Q$.

**Proof.** For the top row, consider the commutative square

$$
\begin{array}{ccc}
J & \xrightarrow{p_Y} & Q \times X Q \\
p_X & & \downarrow (q_{00}, q_{10}) \to q_{00} \\
Q \times Y Q & \xrightarrow{(q_{00}, q_{01}) \to q_{00}} & Q
\end{array}
$$
To show this is a homotopy pushout, it suffices to show that the square of fibers over any point \( \{q_{00}\} \in Q \) is a homotopy pushout. The square of the fibers has the form

\[
\begin{array}{ccc}
Q(X, y_0) \cup \{q_{00}\} & \xrightarrow{\ast, \text{id}} & Q(x_0, Y) \\
(id, \ast) & \downarrow & \downarrow \\
Q(X, y_0) & \xrightarrow{\text{id}, \ast} & Q(x_0, Y)
\end{array}
\]

which is clearly a homotopy pushout.

The rest of the proof is straightforward, using the fact that the assumption that the pushout of

\[
X \xleftarrow{f} \xrightarrow{g} Y
\]

is contractible to show that the homotopy pushout of the bottom row is also equivalent to \( Q \). □

We define maps \( d, d' : J \rightarrow Q \times X \times Y \) as follows.

\[
dj (q_{0..}, q_{1..}) = (q_{0..}, q_{0..}, q_{1..}), \quad dj_Y (q_{0., q_{1..}}) = (q_{0..}, q_{1..}, q_{1..}),
\]

\[
d'j_X (q_{0..}, q_{1..}) = (q_{1..}, q_{1..}, q_{0..}), \quad d'j_Y (q_{0., q_{1..}}) = (q_{1..}, q_{0..}, q_{0..}).
\]

3.5. Lemma. The maps \( d \) and \( d' \) are \((m + n)\)-connected.

Proof. We actually have that \( d' = di \), where \( i : J \rightarrow J \) is the involution of \( J \) defined by

\[
ij_X (q_{0..}, q_{1..}) = (q_{1..}, q_{0..}), \quad ij_Y (q_{0..}, q_{1..}) = (q_{1..}, q_{0..}).
\]

Thus, it suffices to show that \( d \) is \((m + n)\)-connected. The map \( d \) is induced by the commutative square

\[
\begin{array}{ccc}
Q & \xrightarrow{(q_{0..}, q_{1..}) \mapsto (q_{0..}, q_{0..}, q_{1..})} & Q \times X \times Y \\
Q \times X & \xrightarrow{(q_{0..}, q_{1..}) \mapsto (q_{0..}, q_{0..}, q_{1..})} & Q \times X \times Y \\
Q \times Y & \xrightarrow{(q_{0..}, q_{0..}, q_{1..})} & Q \times X \times Y
\end{array}
\]

To show that \( d \) is \((m + n)\)-connected it suffices to show (by (1.5)) that for any point \( q_{01} \in Q(x_0, y_1) \subseteq Q \), the pullback of \( d \) along the inclusion

\[
k : Q(x_0, Y) \times Q(X, y_1) = Q \times X \{q_{01}\} \times Y Q \xrightarrow{\ast \times Y} Q \times X \times Y Q,
\]

is \((m + n)\)-connected. The pullback of (3.6) along \( k \) is the square

\[
\begin{array}{ccc}
\{q_{01}\} & \xrightarrow{\text{id}} & Q(X, y_1) \\
\downarrow & & \downarrow \\
Q(x_0, Y) & \xrightarrow{\text{id}} & Q(x_0, Y) \times Q(X, y_1)
\end{array}
\]

The pullback of \( d \) along \( k \) is the map

\[
Q(x_0, Y) \times \{q_{01}\} \cup \{q_{01}\} \times Q(X, y_1) \rightarrow Q(x_0, Y) \times Q(X, y_1),
\]

which is \((m + n)\)-connected by (1.10). □

3.7. Lemma. Each of the two squares in (3.3) is \((m + n)\)-equifibered.
Proof. For the right-hand square, consider the commutative square

\[
\begin{array}{ccc}
J & \xrightarrow{p_Y} & Q \times_X Q \\
\downarrow d & & \downarrow \sigma_Y \\
Q \times X Q & \xrightarrow{(q_{00},q_{01},q_{11}) \mapsto (q_{00},q_{01})} & Q \times X Q \\
\downarrow & & \downarrow \\
Q \times Q & \xrightarrow{1_{Q \times g}} & Q \times Y
\end{array}
\]

in which the composite of the left-hand column is \(\tau\), the lower square is a pullback, and and \(d\) is \((m + n)\)-connected by (3.5). The result follows by (1.14).

For he left-hand square, consider the commutative square

\[
\begin{array}{ccc}
Q \times Y Q & \xleftarrow{p_X} & J \\
\downarrow & & \downarrow d' \\
Q \times Y Q & \xrightarrow{(q_{00},q_{10}) \mapsto (q_{11},q_{10},q_{00})} & Q \times X Q \\
\downarrow \sigma_X & & \downarrow \sigma_Y \\
Q \times X Q & \xleftarrow{1_{Q \times f}} & Q \times Y
\end{array}
\]

in which the composite of the right-hand column is \(\tau\), the lower square is a pullback, and \(d'\) is \((m + n)\)-connected by (3.5). The result follows by (1.14).

Now we can finish the proof of (3.2), and thus of the homotopy excision theorem. As noted earlier, it suffices to show that \(\sigma_Y\) is \((m + n)\)-connected. By (3.7), each of the two squares in (3.3) is \((m + n)\)-equifibered. By (1.16) it follows that the square \(\sigma_Y \Rightarrow \mu\) relating \(\sigma_Y\) to the homotopy pushout of the rows of (3.3) is \((m + n)\)-equifibered. By (3.4) the map \(\mu\) is equivalent to the identity map of \(Q\), and thus in particular is \((m + n)\)-connected. It follows that \(\sigma_Y\) is \((m + n)\)-connected by (1.15).

4. Graphic remarks on the proof of homotopy excision

Fix a fibration \((f,g)\): \(Q \to X \times Y\), and consider the diagram

\[
\begin{array}{ccc}
Q \times Y Q & \xrightarrow{\sigma_X} & Q \times X Q \\
\downarrow & & \downarrow \sigma_Y \\
Q \times X Q & \xleftarrow{1_{Q \times f}} & Q \times Y
\end{array}
\]

using the notation of the previous section. The limit \(U\) of this diagram (which is also the homotopy limit) is the space of 4-tuples \((q_{ij})_{i,j=0,1} \in Q^4\) satisfying

\[
g(q_{00}) = g(q_{10}), \quad f(q_{10}) = f(q_{11}), \quad g(q_{11}) = g(q_{01}), \quad f(q_{01}) = f(q_{00}).
\]

We can illustrate such a point in \(U\) using the picture

\[
\begin{array}{c}
y_0 \xrightarrow{q_{00}} x_0 \\
q_{10} \quad q_{11} \\
x_1 \xrightarrow{q_{11}} y_1
\end{array}
\]
Let $J'$ be the subspace of $U$ consisting of tuples such that \( \text{either (i) } q_{00} = q_{01} \text{ and } q_{10} = q_{11} \text{ or (ii) } q_{00} = q_{10} \text{ and } q_{01} = q_{11} \). That is, $J'$ consists of points of $U$ which are of the form

\[
\begin{array}{c}
y \rightarrow q_{00} \rightarrow x_0 \\
q_0 \quad q_1 \\
x_1 \rightarrow q_{11} \rightarrow y
\end{array}
\quad \text{or} \quad
\begin{array}{c}
y \rightarrow q_{00} \rightarrow x \\
q_0 \quad q_1 \\
x \rightarrow q_{11} \rightarrow y_1
\end{array}
\]

There is a tautological map $J \rightarrow J'$ from the homotopy pushout of $Q \times_Y Q \leftarrow Q \rightarrow Q \times_X Q$ to $J'$.

More generally, we can represent objects and maps in the diagram (4.1) using the pictures

\[
\begin{array}{c}
\begin{pmatrix}
y_0 \rightarrow q_{00} \rightarrow x_0 \\
q_0 \quad q_1 \\
x_1
\end{pmatrix} \\
\downarrow
\end{array}
\leftarrow
\begin{array}{c}
\begin{pmatrix}
y \rightarrow q_{00} \rightarrow x_0 \\
q_0 \\
x_1 \rightarrow q_{11} \rightarrow y_1
\end{pmatrix}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{pmatrix}
y_0 \rightarrow q_{00} \rightarrow x_0 \\
q_1 \quad q_0 \\
y_1
\end{pmatrix}
\end{array}
\]

The maps $p_X$, $p_Y$, and $\tau$ from $J$ in (3.3) are the evident ones determined by the inclusion $J' \subseteq U$ into the limit of (4.1).

A key step in the argument is the observation that the induced maps $d'$ and $d$ from $J$ into each of the two pullbacks implicit in (4.1) is $(m + n)$-connected. These maps are induced by maps from $J'$ having the form

\[
\begin{array}{c}
\begin{pmatrix}
y_0 \rightarrow q_{00} \rightarrow x_0 \\
q_0 \quad q_1 \\
x_1 \rightarrow q_{11} \rightarrow y_1
\end{pmatrix} \\
\end{array}
\]