Koszul Resolutions of Power Operation Algebras

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"Plethory"
[Tall-Wraith,
Borger-Wieland,
Stacey-Whitehouse]

$$\begin{tikzcd} \mathcal{C} \arrow[d, "U"] \\ \mathrm{Alg}_R^* \arrow[u, bend left=60, "F"] \arrow[u, bend right=60, "G"'] \end{tikzcd}$$

— monadic & comonadic

U preserves limits & colimits

— (graded) commutative R-algebras

<u>Ex:</u> $\mathcal{C} \simeq$ Unstable algebras over Steenrod alg

$\downarrow$

$\mathrm{Alg}_R \simeq$ graded comm $\mathbb{F}_p$-alg

Cotangent Algebra :

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{U}}}}} Alg^*_R ,$$

$R \in \mathcal{C}$ initial object!

$A \in \mathcal{C}_{/R} \rightsquigarrow J_A/(J_A)^2$ "cotangent space at augmentation"

Prop : $J_A/(J_A)^2$ is naturally a module over assoc. ring $\Delta_{\mathcal{C}}$; $\supseteq R$ (not nec central).
(or Ab-cat.)

Construction : Let $P = F(R[x]) \in \mathcal{C}$
(free obj on one gen.)

$\rightsquigarrow \Delta \simeq J_P/(J_P)^2$ (augmentation $P \to R$, $x \mapsto 0$)

Examples of cotangent algebras

Ring w/ derivation:

- $(R, \partial_R$ satisfying Leibniz rule$)$
- $\mathcal{C} \triangleq$ R-algebras w/ derivation (compat w/ ∂_R)

$$\rightsquigarrow \quad P := F(R[x]) \simeq R[x, \partial(x), \partial^2(x), \dots]$$

$$\rightsquigarrow \quad \Delta_{\mathcal{C}} \simeq R\langle \partial \rangle,$$

subj to

$$\partial \cdot r = r \cdot \partial + \partial_R(r)$$

Unstable alg / Steenrod alg:

- Δ is a category (obj $\leftrightarrow \mathbb{N}$, grading)

$$\Delta(p,q) \xrightarrow{\sim} \mathrm{Hom}(H^p, H^q)$$

- Δ-modules $\simeq$ Unstable modules / Steenrod alg s.t. $Sq^n(x) = 0$ if $|x| \leq n$.

Power operations (vague)

$E :=$ strictly commutative ring spectrum

(or, equivariant ...,

or, ultracommutative (ie, globally equivariant), ...)

$$\begin{tikzcd} \mathrm{Alg}_E \arrow[d, "\mathrm{forget}"] \\ \mathrm{Mod}_E \arrow[u, bend left, "\mathbb{P}"] \end{tikzcd}$$

$\boxed{\text{Question}}$: ~~What~~

Find $\mathcal{C}$ st $\exists$

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "\mathrm{forget}"] \\ \mathrm{Alg}_E \arrow[r, "\pi_*"'] \arrow[ur, dotted] & \mathrm{Alg}^*_{E_*} \end{tikzcd}$$

- Ideally, $\pi_*(\mathbb{P}(E))$ = free obj on one gen in $\mathcal{C}$

- Ideally, $\mathcal{C}$ is a plethory over E_*

$\Longrightarrow$ What is $\Delta_{\mathcal{C}}$?

Examples of Power Operation Plethories

1) $E = H\mathbb{Q}$ $\rightsquigarrow$ $\mathcal{C} \xrightarrow{\simeq} \mathrm{Alg}^*_{\mathbb{Q}}$

$\boxed{\Delta_{\mathcal{C}} \simeq \mathbb{Q}}$

"trivial" plethory over gr. comm $\mathbb{Q}$-alg

2) $E = H\mathbb{F}_p$ $\rightsquigarrow$ $\mathcal{C} \longrightarrow \mathrm{Alg}^*_{\mathbb{F}_p}$

alg over mod p Dyer-Lashof algebra ["H_∞-book", BMMS]

$\Delta_{\mathcal{C}}$-modules $\simeq$ $\mathbb{Z}$-gr. modules over mod p D.L.-alg st $Q^n(x) = 0$ if $|x| \leq n$ $(p=2)$

3) $E = K^\wedge_p$, p-complete comm $K^\wedge_p$-alg $\rightsquigarrow$ $\mathcal{C} \longrightarrow \mathrm{Alg}^*_{\mathbb{Z}_p}$

(graded) θ^p-rings [McClure, Bousfield, Hopkins]

θ^p-ring := (comm. ring R, $\theta^p : R \to R$)

st
- $\theta^p(x+y) = \theta^p(x) + \theta^p(y) - \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} x^k y^{p-k}$
- $\theta^p(xy) = x^p\theta^p(y) + y^p\theta^p(x) + p\theta^p(x)\theta^p(y)$
- $\theta^p(1) = 0$ $\Longrightarrow$ $\psi^p(x) := x^p + p\cdot\theta^p(x)$

$\Delta_{\mathcal{C}} \simeq \mathbb{Z}_p[\theta^p]$

Examples (ct'd)

4) $E = E_{\mathbb{G}/k}$ Morava E-theory, $(\mathbb{G}/k = \text{ht } n)$ (K(n)-local algebras)

$\mathcal{C} \longrightarrow \mathrm{Alg}^*_{E_*}$ ← plethory, can be described.

→ $\Delta_e =$ a certain assoc ring, containing E_0, determined by the formal group assoc to E [Strickland].

e.g., for $\mathbb{G}/\mathbb{F}_2$ assoc to supersingular curve;

$E_0 = \mathbb{Z}_2[\![a]\!]$

$\Delta = E_0\langle Q_0, Q_1, Q_2 \rangle \Big/ \left(\begin{array}{l} Q_0 a = a^2 Q_0 - 2a Q_1 + 6 Q_2 \\ Q_1 a = 3 Q_0 + a Q_2 \\ Q_2 a = -a Q_0 + 3 Q_1 \\ \hline Q_1 Q_0 = 2 Q_2 Q_1 - 2 Q_0 Q_2 \\ Q_2 Q_0 = Q_0 Q_1 + a Q_0 Q_2 - 2 Q_1 Q_2 \end{array} \right)$

Equivariant "Examples" (imprecise in some ~~cases~~ details).

5) $E = KU_{gl}$:= global equivariant $\mathbb{C}$-top K-theory $\rightsquigarrow$ "ultracommutative"

- $\mathcal{C} \longrightarrow Alg_{\mathbb{Z}} \Rightarrow$ λ-rings $(R, \{\lambda^n : R \to R\}, \text{st} \ldots)$
- $\Delta_e \simeq \mathbb{Z}[\bar{\theta}^p,\ p \text{ prime}]$

6) $E = KTate_{gl}$:= equivariant Tate K-theory [Ganter]

- $\mathcal{C} \longrightarrow Alg_{\mathbb{Z}((q))} \Rightarrow$ "elliptic λ-rings" $\left(R,\ \lambda^n : R \to R\ n \geq 1,\ \mu^{n,i} : R \to R,\ 0 \leq i < n\right)$

$\Delta_e \simeq \mathbb{Z}((q))\langle \bar{\theta}_p,\ p \text{ prime};\ \bar{\mu}^{p,i},\ p \text{ prime},\ 0 \leq i < p \rangle / \sim$

7) Conj: E = Certain kind of globally equiv. elliptic coh. over $\mathcal{O}_S$

$\mathcal{C} \longrightarrow Alg_{\mathcal{O}_S} \Rightarrow$ (analogous to story for Morava E-theory)

Remarks about examples:

- In each case, $P := F(R[x])$ (free object in $\mathcal{C}$ on one generator) is (as a comm R-alg) a polynomial algebra

  (For Morava E-theory, non-trivial [Strickland, Kashiwabara])

- In each case, Δ_e is Koszul.

$\rightarrow$ Prove this for "classical" cases 1)-4) (non-equivariant).

Case of E = Morava-E-theory

- $\mathbb{D} : \mathrm{Mod}_E^{K(n)\text{-local}} \circlearrowleft$,

$$\left.\begin{aligned} \mathbb{D}(X) &\simeq \bigvee_{m \geq 0} \mathbb{D}_m(X) \\ &\simeq \bigvee_{m \geq 0} \left(X^{\wedge_E m}\right)_{h\Sigma_m} \\ &\simeq E \vee \widetilde{\mathbb{D}}(X) \end{aligned}\right\} \text{K(n)-localize everything}$$

Then

- $$P := \pi_* \mathbb{D}(E) \simeq \widehat{\bigoplus_{m \geq 0}} \underbrace{E^\wedge_* B\Sigma_m}_{\text{f.g free } E_0\text{-module [Strickland]}}$$

Multiplication in ~~[illegible]~~ $\mathbb{D}(X)$ given by

$$\mathbb{D}(X) \wedge_E \mathbb{D}(X) \simeq \mathbb{D}(X \vee X) \xrightarrow{\mathbb{D}(\mathrm{fold})} \mathbb{D}(X)$$

Case of Morava E-theory (2)

Cotangent algebra $\Delta := J_p/(J_p)^2$, $J_p = \mathrm{Ker}[\,P \to E_0\,] = \pi_0 \widetilde{\mathbb{D}}(E)$

$$\rightsquigarrow \quad \Delta = \bigoplus_{m \geq 1} \Delta_m,$$

"linearization"

$$\Delta_m \simeq \mathrm{Cok}\left[\pi_* \mathbb{D}_m(E \vee E) \xrightarrow{\pi_*[\mathbb{D}_m(\mathrm{fold}) - \mathbb{D}_m(\pi_1) - \mathbb{D}_m(\pi_2)]} \pi_* \mathbb{D}_m(E)\right]$$

$$\simeq \mathrm{Cok}\left[\bigoplus_{0<i<m} E_0^\wedge B(\Sigma_i \times \Sigma_{m-i}) \to E_0^\wedge B\Sigma_m\right]$$

Easy Fact: $\Delta_m = 0$ unless $m = p^k$

$$\Rightarrow \quad \Delta \simeq \bigoplus_{k \geq 0} \Delta_{p^k} =: \bigoplus_{k \geq 0} \Delta_{[k]}$$

Hard Fact: [Strickland] Each $\Delta_{[k]}$ is f.g. free E_0-module

Koszul Rings: [Priddy]

$$\Delta = \bigoplus \Delta_{[k]} \quad \text{augmented graded ring}/_{E_0}.$$

$$
\begin{array}{ccccccccc}
s=0 & & s=1 & & s=2 & & s=3 & & \\
\Delta_{[0]} = E_0 & \leftarrow & & \leftarrow & & \leftarrow & & & k=0 \\
 & \leftarrow & \Delta_{[1]} & & & \leftarrow & & & k=1 \\
 & \leftarrow & \Delta_{[2]} & \longleftarrow & \Delta_{[1]} \otimes \Delta_{[1]} & \leftarrow & & & k=2 \\
 & \leftarrow & \Delta_{[3]} & \longleftarrow & \Delta_{[2]} \otimes \Delta_{[1]} + \Delta_{[1]} \otimes \Delta_{[2]} & \longleftarrow & \Delta_{[1]} \otimes \Delta_{[1]} \otimes \Delta_{[1]} & & k=3 \\
 & & & & \vdots & & & &
\end{array}
$$

Normalized Bar complex

$$\bar{B}(E_0, \Delta, E_0) \simeq \bigoplus_{k \geq 0} \bar{B}(E_0, \Delta, E_0)_{[k]}$$

Δ is <u>Koszul</u> if $H_* \bar{B}(E_0, \Delta, E_0)_{[k]} = H_{\underline{\underline{k}}}$.

Koszul Rings (2):

If Δ is Koszul $\rightsquigarrow$ functorial "Koszul" resolution for Δ-modules

$$\to \Delta \otimes_{E_0} C(k) \otimes_{E_0} M \longrightarrow \Delta \otimes_{E_0} C(k-1) \otimes_{E_0} M \longrightarrow \cdots\cdots$$

where $C(k) = H_k(\widehat{B}(E_0, \Delta, E_0)_{[k]})$

Thm: E = Morava E-theory $\rightsquigarrow$ Δ is Koszul.
wrt the evident grading

Idea: Bar complex of Δ
= linearization of monadic Bar complex of $\mathbb{T}$

+ Bredon homology of partition complex

Linearization :

$$F : \mathcal{A} \longrightarrow \mathcal{B} \quad \Longrightarrow \quad \mathcal{L}(F) : \mathcal{A} \longrightarrow \mathcal{B}$$

additive functor, abelian

or $\mathcal{L}_F$ ← additive functor

$$\mathcal{L}_F(X) := \mathrm{Cok}\left[F(X \oplus X) \underset{F(\pi_1)+F(\pi_2)}{\overset{F(\pi_1+\pi_2)}{\rightrightarrows}} F(X) \right]$$

[Johnson - McCarthy] Derived Linearization $D_1(F) : \mathcal{A} \longrightarrow Ch(\mathcal{B})$

s.t. $H_0 D_1(F) = \mathcal{L}_F$

Chain-Rule : $D_1(F) \circ D_1(G) \underset{q\text{-iso}}{\simeq} D_1(F \circ G)$

- with additional hyp on $G(X)$, can show this implies [eg. $G(X)$, $\mathcal{L}_G(X)$ projective]

$$\mathcal{L}_{F \circ G}(X) \simeq \mathcal{L}_F \circ \mathcal{L}_G(X)$$

Linearization (2)

- Apply to $\pi_* \widetilde{\mathbb{D}} : \mathrm{Mod}_E^{K(n)\text{-local}} \longrightarrow \mathrm{Mod}_{E_*}$

$$\Rightarrow \quad \mathcal{L}_{\pi_* \widetilde{\mathbb{D}}}(X) \simeq \Delta \otimes_{E_*} \pi_* X \quad (\pi_* X \text{ projective})$$

- Apply to $\pi_* \underbrace{\widehat{\mathbb{D}} \circ \cdots \circ \widetilde{\mathbb{D}}}_{k\text{-times}} : \mathrm{Mod}_E^{K(n)\text{-local}} \longrightarrow \mathrm{Mod}_{E_*}$

$$\Rightarrow \quad \mathcal{L}_{\pi_* \widehat{\mathbb{D}} \circ \cdots \circ \widetilde{\mathbb{D}}}(X) \simeq \Delta \otimes_{E_*} \cdots \otimes_{E_*} \Delta \otimes \pi_* X \quad (\text{chain rule})$$

Thus:

$$\mathcal{L}_{\bar{B}(\pi_* \widehat{\mathbb{D}}, \widehat{\mathbb{D}}, \widehat{\mathbb{D}}(-))}(X) \simeq \bar{B}(\Delta, \Delta, \Delta \otimes_{E_*} \pi_* X)$$

We care about a $B(E_0, \Delta, E_0)$
(a <u>quotient</u> of this)

Partition Complex:

$\mathcal{P}_m$ = "Big" Partition complex
= nerve of poset of equiv rel. on $\{1, \ldots, m\}$
(incl. two trivial cases).

$\mathcal{O}$ = (non-unital) $\mathcal{E}_\infty$-operad.

$$\Rightarrow \quad (\underbrace{\mathcal{O} \circ \cdots \circ \mathcal{O}}_{q+2})\langle m \rangle \simeq (\mathcal{P}_m)_q \quad \circlearrowleft \Sigma_m$$

$\nearrow$ q-simplices.

If X = space.

$$B(\pi_* \mathbb{D}, \widetilde{\mathbb{D}}, \widetilde{\mathbb{D}}(\Sigma^\infty_+ X))_q \simeq \widehat{\bigoplus_{m \geq 1}} E_*^\vee((\mathcal{P}_m)_q \times_{h\Sigma_m} X^m)$$

want to linearize at $X = *$
i.e $\Sigma^\infty_+ X = S^0$

"Transitive Homology":

Fix K = finite Σ_m-set

$\rightsquigarrow$ functor $\quad$ Spectra $\longrightarrow$ E_*-mod.

$$X \longmapsto E^\vee_*(K_+ \wedge_{h\Sigma_m} X^{\wedge m})$$

Define: $\quad Q_{\Sigma_m}(K) := \mathcal{L}_{E^\vee_*(K_+ \wedge_{h\Sigma_m} (-)^{\wedge m})}(S^0)$

$\in E_*$-mod

$\Longrightarrow$ Q_{Σ_m}: Fin Σ_m-Set $\longrightarrow \mathrm{Mod}_{E_*}$

~~etc~~ extends to a Mackey-Functor

$Q_{\Sigma_m}: A(\Sigma_m) \longrightarrow \mathrm{Mod}_{E_*}$

$\uparrow$ Burnside category

Rem: If $K = \Sigma_m/G$ s.t. $G \curvearrowright \{1,\dots,m\}$ does not act transitively, then $Q_{\Sigma_m}(K) \simeq 0$

Key Observation:

$$B(\Delta, \Delta, \Delta \otimes_{E_*} E_*)_q \simeq \mathcal{L}_{\widehat{B}(\Pi_*\widehat{\mathbb{D}}, \widehat{\mathbb{D}}, \widehat{\mathbb{D}}(-))_q}(E)$$

$$\simeq \bigoplus_m \mathcal{L}_{E_*^\vee\left((P_m)_q \times_{h\Sigma_m} (\ )^{\wedge m}\right)}(S^0)$$

$$\simeq \bigoplus_m Q_{\Sigma_m}\left((P_m)_q\right)$$

Quotient: $\overline{P_m} := P_m / P_m^{\Diamond}$, $\quad P_m^{\Diamond} = P_m^{\wedge} \cup P_m^{\vee}$ sub posets of P_m

$$\Rightarrow \quad B(E_*, \Delta, E_*)_q \simeq \bigoplus_m \bar{Q}_{\Sigma_m}\left((\overline{P_m})_q\right)$$

Conclusion

$$H_*(B(E_*,\Delta,E_*)_\cdot) \simeq \bigoplus_m \underbrace{H_*(\bar{Q}_{\Sigma_m}((\bar{P}_m)_\cdot))}_{\cong \; \tilde{H}_*^{Br}(\bar{P}_m;\, Q_{\Sigma_m})}$$

Δ is Koszul if

$\tilde{H}_q^{Br}(\bar{P}_m; Q_{\Sigma_m})$ is concentrated in dim $q=k$ when $m=p^k$

(Note: $H_*^{Br}(\bar{P}_m; Q_{\Sigma_m}) \equiv 0$ if $m \neq p^k$)

Finish the Proof

- Had a fairly complex argument for vanishing of
$$H_*^{Br}(\overline{\mathcal{P}}_{p^k}; Q_{\Sigma_{p^k}}) \quad \text{if } * \neq k.$$

- Now can use Arone-Dwyer-Lesh: ("Bredon homology of partition complexes").

Thm: If Q is a "p-constrained" Σ_{p^k}-Mackey functor satisfying a certain condition $(*)$, then
$$H_*^{Br}(\overline{\mathcal{P}}_{p^k}; Q)$$
is concentrated in degree $* = k$.

p-constrained: composite $Q(X) \xrightarrow{f^*} Q(Y) \xrightarrow{f_*} Q(X)$ is iso for $f: Y \to X$ map of Σ_{p^k}-sets w/ fibers of size prime to p.

[True for us, because $Q(-)$ is a quotient of $E_*(\ )_{h\Sigma_{p^k}}$]

$(*)$ = a condition on $Q(\Sigma_{p^k}/G)$ for some "non-transitive" G, trivial in our case.

Remarks:

- I stated this for Morava E-theory, but it all works the same for $H\mathbb{F}_p$.

- ADL gives a formula

$$C(k) := H_k^{Br}(\overline{P}_{p^k}; \mathbb{Q}_{\Sigma_{p^k}})$$

$$\simeq St_k \otimes_{\Sigma_{p^k}} \mathbb{Q}(\Sigma_{p^k}/\Delta_k), \qquad \Delta_k \simeq (\mathbb{Z}/p)^k$$

$$\simeq St_k \otimes_{\Sigma_{p^k}} \left(E_* B\Delta_k / E_*\left(\begin{smallmatrix}\text{proper}\\ \text{subgps}\end{smallmatrix}\right)\right)$$

- Proves a result about formal groups, but uses topology in essential way (alg pf exists for $ht \leq 2$).

- Is there an integral analogue which applies to λ-rings, elliptic coh, etc?