A STREAMLINED PROOF OF GOODWILLIE'S n-EXCISIVE APPROXIMATION

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Abstract. We give a shorter proof of Lemma 1.9 from Goodwillie, "Calculus III", which is the key step in proving that the construction P_nF gives an n-excisive functor.

1. Introduction

For a homotopy functor F from spaces to spaces, Goodwillie has defined the notion of an "n-excisive approximation", which is a homotopy functor P_nF together with a natural transformation $p_nF\colon F \to P_nF$. In [Goo03, Thm. 1.8] it is shown that the functor P_nF is in fact an n-excisive functor, and therefore that p_nF it is the universal example of a map from F to an n-excisive functor. The notable feature of this proof is that no hypotheses involving connectivity are needed. Goodwillie's proof relies on a clever lemma [Goo03, Lemma. 1.9], which is, as he notes, "a little opaque".

The purpose of this note is to give a streamlined proof of Goodwillie's lemma. Our proof uses his ideas, but is simpler, and we believe less opaque. We will assume that the reader is familiar with [Goo03], and we assume the context and notation of §1 of that paper.

2. Lemma 1.9 of Calculus III

Let $\mathcal{P}(n)$ denote the poset of subsets of $\{1, \cdots, n\}$, and let $\mathcal{P}_0(n) \subset \mathcal{P}(n)$ be the poset of non-empty subsets.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a homotopy functor, Goodwillie define a functor $T_nF\colon \mathcal{C} \to \mathcal{D}$ and natural map $t_nF\colon F \to T_{n-1}F$ by

$$F(X) \xrightarrow{t_nF} \operatorname{holim}_{U\in\mathcal{P}_0(n+1)} F(X * U).$$

2.1. **Lemma.** *Let $\mathcal{X}$ be any strongly cocartesian n-cube in $\mathcal{U}$, and let F be any homotopy functor. The map of cubes $(t_nF)(\mathcal{X})\colon F(\mathcal{X}) \to (T_nF)(\mathcal{X})$ factors through some cartesian cube.*

Proof. We write n instead of $n+1$. Given any cube $\mathcal{X}$ and a set $U \in \mathcal{P}(n)$, define a cube $\mathcal{X}_U$ by

$$\mathcal{X}_U(T) = \operatorname{hocolim}\left(\mathcal{X}(T) \leftarrow \coprod_{s\in U} \mathcal{X}(T) \to \coprod_{s\in U} \mathcal{X}(T\cup\{s\})\right).$$

We have $\mathcal{X}_\varnothing(T) \approx \mathcal{X}(T)$, and there is an evident map $\alpha\colon \mathcal{X}_U(T) \to \mathcal{X}(T) * U$, which is natural in both T and U.

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The map $(t_{n-1}F)(\mathcal{X})$ factors as follows:

$$F(\mathcal{X}(T)) \to \operatorname{holim}_{U\in\mathcal{P}_0(n)} F(\mathcal{X}_U(T)) \to \operatorname{holim}_{U\in\mathcal{P}_0(n)} F(\mathcal{X}(T) * U) \approx (T_{n-1}F)(\mathcal{X}(T)).$$

Now suppose that $\mathcal{X}$ is strongly cocartesian. Then there are natural weak equivalences $\mathcal{X}_U(T) \approx \mathcal{X}(T \cup U)$. The maps $\mathcal{X}(T\cup U) \to \mathcal{X}(T \cup \{s\} \cup U)$ are isomorphisms for $s \in U$, and thus if U is non-empty the cube $T \mapsto F(\mathcal{X}_U(T))$ is cartesian. Therefore $\operatorname{holim}_{U\in\mathcal{P}_0(n)} F(\mathcal{X}_U(T))$ is a homotopy limit of cartesian cubes, and thus is cartesian. $\square$

Note that this shows that if T is non-empty, then $U \mapsto F(\mathcal{X}_U(T))$ is cartesian, so that $F(\mathcal{X}(T)) \to \operatorname{holim}_{U\in\mathcal{P}_0(n)} F(\mathcal{X}_U(T))$ is a weak equivalence for $T \neq \varnothing$. For $T = \varnothing$, we see that $\operatorname{holim}_{U\in\mathcal{P}_0(n)} F(\mathcal{X}_U(\varnothing)) \approx \operatorname{holim}_{U\in\mathcal{P}_0(n)} F(\mathcal{X}(U))$.

References

[Goo03] Thomas G. Goodwillie, *Calculus. III. Taylor series*, Geom. Topol. **7** (2003), 645–711 (electronic). MR MR2026544

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