abstract

GENERALIZATIONS OF WITT VECTORS IN ALGEBRAIC TOPOLOGY

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ABSTRACT. Witt vector constructions (both integral and p-adic) are the underlying functors of the comonads which define λ-rings and p-derivation rings, which arise algebraic structures on certain K-theory rings. In algebraic topology, this is a special case of structure that exist for a number of generalized cohomology theories. In this talk I will describe the analogue of this structure for "Morava E-theories", which are cohomology theories associated to universal deformations of 1-dimensional formal groups of finite height.

1. Introduction

Talk for the workshop on "Witt vectors, Deformations, and Anabelian Geometry" at U. of Vermont, July 16–21, 2018.

The goal of this talk is to describe some generalizations of "Witt vectors" and "λ-rings" which show up in algebraic topology, in the guise of algebras of "power operations" for certain cohomology theories. I won't dwell much on the algebraic topology aspect: the set-up I describe arises directly from the arithemetic algebraic geometry of formal groups and isogenies.

comonad	algebra	group object	cohomology theory
$\mathbb{W}$ (big Witt)	λ-rings	$\mathbb{G}_m$	(equivariant) K-theory
$\mathbb{W}_p$ (p-typical Witt)	p-derivation ring	$\widehat{\mathbb{G}}_m$	K-theory w/ p-adic coeff
??	$\mathbb{T}$-algebra	universal def. of fg	Morava E-theory
??	??	certain p-divisible group	$K(h)$-localization of Morava E-theory
??	??	elliptic curves	(equivariant) elliptic cohomology
??	??	Tate curve	(equivariant) Tate K-tehory.

There are other examples, corresponding to certain p-divisible groups (local), or elliptic curves (global).

The case of Morava E-theory was observed by Ando, Hopkins, and Strickland, together with some contributions by me.

2. Λ-rings

λ-rings were invented by Grothendieck[1] to formalize certain operations on vector bundles, or representations. Thus,

$$\lambda^n V = n\text{th exterior power of } V,$$

satisfying various conditions. For instance, for direct sum:

$$\lambda^n(V+W) = \sum \lambda^i V \otimes \lambda^{n-i} W.$$

There are also formulas for exterior power of tensor product and for composition of exterior powers:

$$\lambda^n(V \otimes W) = \text{polynomial in } \lambda^i V,\ \lambda^j W,$$

$$\lambda^m \lambda^n V = \text{polynomial in } \lambda^i V.$$

These polynomials involve possibly negative integer coefficients, so we must regard these as acting on a Grothendieck ring, e.g., $K^0 X$ or RG.

The forgetful functor

$$(\lambda\text{-rings}) \to (\text{commutative rings})$$

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[1]By which I mean what he called "special λ-rings"

has both a left and right adjoint, and in fact is both monadic and comonadic. The functor of the comonad is the **big Witt vector** construction. λ-rings are precisely "coalgebras" for this comonad.

3. Adams operations and Frobenius lifts

Other operations on λ-ring R are symmetric powers σ^n, which can be expressed (on a Grothendieck group) in terms of exterior powers. We also have **Adams operations** $\psi^m(V)$.

$$\sum_{k\geq 0} \sigma_k(V)\, t^k = \Big(\sum_{k\geq 0} \lambda_k(V)\,(-t)^k\Big)^{-1} = \exp\Big[\sum_{m\geq 1} \frac{1}{m}\psi^m(V)\, t^m\Big].$$

Taking a log-derivative gives $\psi^m(V) =$ polynomial in $\lambda^i(V)$ with $\mathbb{Z}$-coefficients. Adams observed:

(1) ψ^m are ring homomorphisms,
(2) $\psi^m\psi^n = \psi^{mn}$,
(3) $\psi^p(x) \equiv x^p \mod pR$ if p prime.

That is, ψ^p is a lift of Frobenius for each prime, and lifts for different primes commute.

Note that $\lambda^k(x) =$ polynomial in $\psi^m(x)$ with $\mathbb{Q}$-coefficients, so a λ-ring structure is determined by Adams operations *rationally*.

There is a partial converse.

3.1. **Theorem** (Wilkerson). *Let R be a commutative ring equipped with functions $\psi^m\colon R\to R$ satisfying the above conditions (1)–(3). If R is torsion free, then there exists a unique λ-ring structure on R with the given ψ^ms as the Adams operations.*

It turns out that a free λ-ring on a set of generators is torsion free as an abelian group (in fact, it's a polynomial ring). This means that the entire theory of λ-rings can be recovered from the theory of ψ^ps.

4. Multiplicative group

Consider the representation ring of the circle:

$$R(U(1)) = \mathbb{Z}[T, T^{-1}].$$

Multiplication $\mu\colon U(1)\times U(1)\to U(1)$ is a homomorphism, so we get a coproduct

$$\mu^*\colon R(U(1)) \to R(U(1)\times U(1)) \approx R(U(1))\otimes R(U(1)).$$

This Hopf algebra is the ring of functions on $\mathbb{G}_m$, i.e.,

$$\mathbb{G}_m = \operatorname{Spec} R(U(1)).$$

The Adams operations ψ^n, $n\geq 1$ thus give rise to endomorphisms of the group scheme $\mathbb{G}_m$. A straightforward calculation ($\psi^n(T) = T^n$) shows that the endomorphism is $[n]\colon \mathbb{G}_m\to\mathbb{G}_m$.

Note that on $(\mathbb{G}_m)_{\mathbb{F}_p}$, the endomorphism $[p]$ is also the Frobenius.

(Remark: in complex K-theory, there is also a ψ^{-1}, corresponding to complex conjugation. This induces $[-1]$ on $\mathbb{G}_m$.)

5. Power operations

To see how this generalizes, we look at a different construction of ψ^m. First note the "total mth power" operation:

$$P_m(V) := V^{\otimes m} = \sum_\pi \rho_\pi\otimes\sigma_\pi(V), \qquad P_m\colon K^0(X)\to R\Sigma_m\otimes K^0(X),$$

sum over irreps π of Σ_m.

- $\sigma_{\text{triv}} = \sigma^m$, and $\sigma_{\text{sign}} = \lambda^m$.
- $\sum_\pi(\dim\rho_\pi)\sigma_\pi(V) = V^{\otimes m}$.

It turns out that every $\sigma_\pi \in \mathbb{Z}[\lambda^k,\ k \geq 1]$. See Atiyah, "Power operations in K-theory".

The power construction is multiplicative: $(V \otimes W)^{\otimes m} \approx V^{\otimes m} \otimes W^{\otimes m}$. It is not additive; instead:

$$(V + W)^{\otimes m} = \sum_{i+j=m} V^{\otimes i} \otimes W^{\otimes j} \uparrow_{\Sigma_i \times \Sigma_j}^{\Sigma_m}.$$

The failure of additivity comes from induced representations from proper subgroups $\Sigma_i \times \Sigma_{m-i} \subsetneq \Sigma_m$. We can get a ring homomorphism by quotienting by the "transfer ideal":

$$K^0(X) \xrightarrow{P_m} R\Sigma_m \otimes K^0(X) \to \big(R\Sigma_m / I_{\mathrm{tr}}\big) \otimes K^0(X) \approx \mathbb{Z} \otimes K^0(X).$$

This gives the Adams operation ψ^m, which is a ring homomorphism.

6. Power operations for commutative ring spectra

The above analysis is specific to K-theory, but there is a class of cohomology theories which allow us to do something similar, namely those arising from **commutative ring spectra**, also called **$\mathbb{E}_\infty$-ring spectra**. For such E we have

$$E^0(X) \xrightarrow{P_m} E^0(B\Sigma_m \times X) \xleftarrow{\sim ?} E^0(B\Sigma_m) \otimes_{E^0(\mathrm{pt})} E^0(X) \to E^0(B\Sigma_m)/I_{\mathrm{tr}} \otimes_{E^0(\mathrm{pt})} E^0(X).$$

The function P_m is a refinement of the ordinary mth power map $x \mapsto x^m$.

In certain cases, the second map is an isomorphism, in which case we get a ring homomorphism

$$E^0(X) \to E^0(B\Sigma_m)/I_{\mathrm{tr}} \otimes_{E^0(\mathrm{pt})} E^0(X).$$

Example. If $E = H\mathbb{F}_2$ is ordinary mod 2 cohomology, and $m = 2$, then $I_{\mathrm{tr}} = 0$, and we get the classical Steenrod operations.

Example. Let $E = K_p$, the p-completion of the complex K-theory spectrum. Then

$$K_p^0(B\Sigma_{p^k})/I_{\mathrm{tr}} \approx \begin{cases} \mathbb{Z}_p & \text{if } m = p^k \\ 0 & \text{else.} \end{cases}$$

We get natural ring maps

$$\psi^{p^k} \colon K_p^0(X) \to K_p^0(X) \otimes K_p^0(B\Sigma_{p^k})/I = K_p^0(X).$$

One has that $K_p^0(B\Sigma_p) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, so that the total operation is

$$P_p(x) = (\psi^p(x),\ \delta(x)),$$

where $\psi^p(x) = x^p + p\,\delta(x)$. One can show that the entire structure the P_ms produce in this case are exactly a "ring with p-derivation".

The object $K_p^0(BU(1)) \approx \mathbb{Z}_p[\![t]\!] \approx \mathcal{O}_{\widehat{\mathbb{G}}_m}$ is functions on the **formal multiplicative group** $\widehat{\mathbb{G}}_m$. On this ψ^{p^k} realize $[p^k] \colon \widehat{\mathbb{G}}_m \to \widehat{\mathbb{G}}_m$.

7. The deformation category

I now describe how this works out for a family of theories called **Morava E-theory**.

Fix

- κ perfect field char p,
- Γ one-dimensional formal group of finite height n over κ.

Deformations. Given a complete local ring R, a deformation of Γ/k to R is data

$$\big(G/R, \quad i \colon \kappa \to R/\mathfrak{m}, \quad \alpha \colon i^*\Gamma \xrightarrow{\sim} G_{R/\mathfrak{m}}\big).$$

G is a formal group over R, and we have

$$\begin{array}{ccccccc} G & \longleftarrow & G_0 & \xleftarrow[\sim]{\alpha} & i^*\Gamma & \longrightarrow & \Gamma \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spf} R & \longleftarrow & \operatorname{Spec} R/\mathfrak{m} & = & \operatorname{Spec} R/\mathfrak{m} & \xrightarrow[i]{} & \operatorname{Spec} \kappa \end{array}$$

$G_0 =$ special fiber of $G \to \operatorname{Spf} R$. (If $\kappa = \mathbb{F}_p$ we can ignore i.)

Isogenies. Consider isogenies of formal groups over R.

$$\begin{array}{ccccc} \operatorname{Ker} f & \rightarrowtail & G & \overset{f}{\twoheadrightarrow} & G' \\ & & & \searrow & \uparrow \wr \\ & & & & G/\operatorname{Ker} f \end{array}$$

$\operatorname{Ker} f$ is a finite subgroup scheme of G. Up to canonical isomorphism, the data of f is the same as the data of the pair $(G, \operatorname{Ker} f)$.

Deformation structures can be pushed forward along isogenies using relative Frobenius:

$$G \xrightarrow{f} G'$$

$$(i, \alpha) \longmapsto (i', \alpha')$$

with $\deg f = p^k$, where $i' = i \circ \sigma^k$ and

$$\begin{array}{ccc} G_0 & \xrightarrow{f_0} & G_0' \\ \alpha \uparrow \wr & & \wr \uparrow \exists!\, \alpha' \\ i^*\Gamma & \xrightarrow[\mathrm{Fr}^k]{} & (i \circ \sigma^k)^*\Gamma \end{array}$$

where $\sigma(x) = x^p$ is absolute Frobenius.

Category of deformations and isogenies. For each R we get a category

$$\operatorname{Def}_\Gamma(R) = \begin{cases} \textbf{obj:} & \text{deformations } (G, i, \alpha), \\ \textbf{mor:} & f\colon G \to G' \text{ st. } (i, \alpha) \mapsto (i', \alpha'). \end{cases}$$

Fact. $\operatorname{Def}_\Gamma$ is representable:

$$\operatorname{Def}_\Gamma = \begin{cases} \operatorname{Spf} A, \\ \coprod_{k \geq 0} \operatorname{Spf} A_k. \end{cases}$$

We have $A = A_0$. The category structure is encoded by ring maps:

$$A \overset{s}{\underset{t}{\rightrightarrows}} A_1, \qquad A_{k+\ell} \xrightarrow{\mu} A_k{}^s\otimes_A{}^t A_\ell.$$

$\operatorname{Spf} A$ is the **Lubin-Tate deformation space** of Γ/k, so that

$$(\text{iso classes of } (G, i, \alpha)) \approx (\text{local homomorphisms } A \to R).$$

There is a universal deformation $(G_{\text{univ}}, i, \alpha)$ to A, and a non-canonical isomorphism

$$A \approx \mathbb{W}_p\kappa[\![a_1, \ldots, a_{n-1}]\!], \qquad n = \operatorname{ht} \Gamma.$$

If two deformations are isomorphic, they are so by a **unique** isomorphism.

$\operatorname{Spf} A_k$ carries the **universal example of subgroup** of rank p^k of a deformation:

$$(\text{iso classes of } (G, i, \alpha, H \leq G),\ \operatorname{rk} H = p^k) \approx (\text{local homomorphisms } A_k \to R).$$

This was shown to exist by Strickland. "Finite subgroups of formal groups." Note that $s\colon A \to A_k$ is finite and flat.

Frobenius. If $\mathbb{F}_p \subseteq R$, then for a deformation (G, i, α) there is a canonical subgroup $\operatorname{Ker}\operatorname{Fr}^k \leq G$ of rank p^k, where $\operatorname{Fr}^k\colon G \to (\sigma^k)^* G$ is p^kth power relative Frobenius. This is represented by a map

$$\tau\colon A_k \to A/p.$$

Morava E-theory. For each Γ/κ, there is a generalized cohomology theory E called **Morava E-theory** with

- $E^0(\mathrm{pt}) = A$.
- $E^*(\mathrm{pt}) = A[\mu^{\pm}]$, $|\mu| = 2$. ("Even periodic".)
- $E^0(BU(1)) = \mathcal{O}_{G_{\mathrm{univ}}}$, the ring of functions on the universal deformation of Γ/κ.
- E is a commutative ring spectrum (the "Goerss-Hopkins-Miller" theorem).
- $E^0(B\Sigma_{p^k})/I_{\mathrm{tr}} = A_k$ (Strickland, "Morava E-theory of symmetric groups".)

8. Λ_Γ-rings

From the above category we produce an analogue of Witt vectors and λ-rings, which I'll call **Γ-Witt vectors** and **Λ_Γ-rings**.

We define categories

$$\mathcal{Q} = \mathrm{QCoh}_{\mathrm{Ring}}(\mathrm{Def}_\Gamma) \supseteq \mathcal{Q}^{\mathrm{Fr}}$$

as follows. A **quasicoherent sheaf of commutative rings on** Def_Γ is data

$$(R, \{\psi_k,\ k \geq 0\})$$

consisting an A-alegbra $A \to R$, maps

$$\psi_k\colon R \to {}^t{A_k}^s \otimes_A R$$

of A-algebras (where the target gets its A-algebra structure from $t\colon A \to A_k$), and such that

$$\begin{array}{ccc}
R & \xrightarrow{\psi_k} & {A_k}^s \otimes_A R \\
{\scriptstyle \psi_{k+\ell}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\psi_\ell} \\
{A_{k+\ell}}^s \otimes_A R & \xrightarrow[\mu\otimes\mathrm{id}]{} & {A_k}^s \otimes_A {}^t{A_\ell}^s \otimes_A R
\end{array}$$

Such an object satisfies the **Frobenius congruence** if

$$\begin{array}{ccc}
R & \xrightarrow{\psi_k} & {A_k}^s \otimes_A R \\
 & & \downarrow{\scriptstyle \tau\otimes\mathrm{id}} \\
\downarrow & & A/p \otimes_A R \\
 & & \| \\
R/p & \xrightarrow[x \mapsto x^{p^k}]{} & R/p
\end{array}$$

commutes, where $\tau\colon A_k \to A/p$ is the map classifying $\operatorname{Ker}\operatorname{Fr}^k$.

Remark. Thus, ψ_k is a "Frobenius lift", but not in general an endomorphism of R. In our setting, formal groups of height ≥ 2 are not "ordinary", i.e., do not have a canonical lift of $\ker\operatorname{Fr}$.

8.1. **Theorem.** *There exists a category called (Λ_Γ-rings), together with a diagram of functors*

$$\begin{array}{ccc}
(\Lambda_\Gamma\text{-rings}) & \xrightarrow{\text{forget}} & \mathcal{Q}^{\mathrm{Fr}} \\
{\scriptstyle \text{underlying}}\searrow & & \swarrow{\scriptstyle \text{underlying}} \\
 & (\text{comm. } A\text{-alg.}) &
\end{array}$$

such that (i) $(\Lambda_\Gamma\textit{-rings}) \to (\textit{comm. } A\textit{-alg.})$ *is monadic and comonadic, and (ii) "*forget*" induces an equivalence on the full-subcategories of objects whose underlying ring is p-torsion free.*

The functor of the comonad is $W = W_\Gamma$.

Key example. The ring $\mathcal{O}_{G_{\text{univ}}}$ is tautologically a Λ_Γ-ring. The maps ψ_k extend to ring homomorphisms

$$A_k{}^t\otimes_A \mathcal{O}_{G_{\text{univ}}} \to A_k{}^s\otimes_A \mathcal{O}_{G_{\text{univ}}}$$

which describe $G_{\text{univ}} \times^s_{\operatorname{Spf} A} \operatorname{Spf} A_k \to G_{\text{univ}} \times^t_{\operatorname{Spf} A} \operatorname{Spf} A_k$ the universal example of a p^k isogeny on G_{univ}.

Motivating exmaple. For every topological space X, $E^0(X)$ is naturally a Λ_Γ-ring.

In fact, for every $K(n)$-local commutative E-algebra B, $\pi_0 B$ is a Λ_Γ-ring.

Λ_Γ-rings at height 1. Let

$$\kappa = \mathbb{F}_p, \qquad \Gamma = \widehat{\mathbb{G}}_m.$$

Then

$$A = \mathbb{Z}_p, \qquad G_{\text{univ}} = \widehat{\mathbb{G}}_m, \qquad A_k = \mathbb{Z}_p.$$

An object of $\mathcal{Q}^{\text{Fr}}$ is a $\mathbb{Z}_p$-algebra with endomorphisms $\psi_k\colon R \to R$ such that $\psi_k = \psi_1 \circ \cdots \circ \psi_1$ and $\psi_k(x) \equiv x^{p^k} \mod pR$. A Λ_Γ-ring is precisely a $\mathbb{Z}_p$-algebra with p-derivation.

The Morava E-theory in this case is precisely p-adic K-theory K_p.

9. Simplifying results

To even get our hands on W_Γ, we need an infinite collecion of rings A_k together with s, t, μ, τ. In fact things are much better than this, by the following "quadraticity" theorem.

9.1. **Theorem** (R.). *The data $(R, \{\psi_k\})$ of an object of $\mathcal{Q}$ are uniquely determined by the following subset:*

1. *an A-algebra map $\psi_1\colon R \to {}^tA_1{}^s\otimes_A R$, such that*
2. *there exists a unique dotted arrow making*

$$\begin{array}{ccc} R & \xrightarrow{\psi_1} & A_1{}^s\otimes_A R \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\psi_1} \\ A_2{}^s\otimes_A R & \xrightarrow[\mu\otimes\mathrm{id}]{} & A_1{}^s\otimes_A {}^tA_1{}^s\otimes_A R \end{array}$$

commute.

The Frobenius condition for $\{\psi_k\}$ is equivalent to the Frobenius condition for ψ_1.

So we just need A_1, A_2 and the relevant s, t, μ, τ.

Height 2. When $n = \operatorname{ht}\Gamma = 2$ we can do a little better, by giving a formula for A_2.

Given an isogeny $f\colon G \to G'$ of degree p, there exists a unique **dual isogeny** $f^\vee\colon G' \to G$, defined by:

$$\begin{array}{ccccc} \ker f & \rightarrowtail & G & \xrightarrow{f} & G' \\ \downarrow & & \| & & \downarrow{\scriptstyle f^\vee} \\ G[p] & \rightarrowtail & G & \xrightarrow[{[p]}]{} & G \end{array}$$

The operation $(G, i, \alpha, \ker f) \mapsto (G', i', \alpha', \ker f^\vee)$ is represented by a ring map

$$w\colon A_1 \to A_1.$$

The identity $f^\vee f = [p]$ corresponds to a commutative square:

$$\begin{array}{ccc} A_2 & \xrightarrow{\mu} & A_1 {}^s\!\otimes_A {}^t A_1 \\ \gamma \big\downarrow & & \big\downarrow (w,\mathrm{id}) \\ A & \xrightarrow[s]{} & A_1 \end{array}$$

where γ represents the operation $G \mapsto (G[p] \le G)$.

Fact. The above square is a pullback, and remains a pullback after tensoring with ${}^s\!\otimes_A R$.

Thus, the "associativity condition" on ψ_1 becomes the existence of a dotted arrow in:

$$\begin{array}{ccc} R & \xrightarrow{\psi_1} & A_1 {}^s\!\otimes_A R \\ \vdots & & \big\downarrow \mathrm{id}\otimes\psi_1 \\ \vdots & & A_1 {}^s\!\otimes_A {}^t A_1 {}^s\!\otimes_A R \\ \vdots & & \big\downarrow (w,\mathrm{id})\otimes\mathrm{id} \\ R & \xrightarrow[s]{} & A_1 {}^s\!\otimes_A R \end{array}$$

So to describe objects of $\mathcal{Q}^{\mathrm{Fr}}$, and thus the category of Λ_Γ-rings in the height 2 case, we only need

$$\begin{array}{ccccc} A & \underset{t}{\overset{s}{\rightrightarrows}} & A_1 & \xrightarrow{w} & A_1 \\ & & \big\downarrow \tau & & \\ & & A/p & & \end{array}$$

Explicit example. Let

$$\kappa = \mathbb{F}_2, \qquad \Gamma = \widehat{C}_{\mathrm{ss}}, \qquad \text{where } C_{\mathrm{ss}} = \left(Y^2Z + YZ^2 = X^3\right),$$

the formal completion of a supersingular curve. Then

$$A = \mathbb{Z}_2[\![a]\!], \qquad G_{\mathrm{univ}} = \widehat{\widetilde{C}}, \qquad \text{where } \widetilde{C} = \left(Y^2Z + aXYZ + YZ^2 = X^3\right).$$

Furthermore

$$A_1 = \mathbb{Z}_2[\![a]\!][d]/(d^3 - ad - 2).$$

Note: $P = (d : 1 : -d^3)$ is 2-torsion point, and the Cartier divisor $[O] + [P]$, $O = (0 : 1 : 0)$, is the divisor of a subgroup.

The maps are given by

$$\begin{aligned} s &\colon a \mapsto a, \\ t &\colon a \mapsto a' = a^2 + 3d - ad^2, \\ w &\colon a \mapsto a', \\ &\colon d \mapsto a - d^2, \\ \tau &\colon a \mapsto a, \\ &\colon d \mapsto 0. \end{aligned}$$

From this we can (in principle) completely recover W_Γ.

Remark. Yifei Zhu has given similar explicit formulas at $p = 3$ and $p = 5$, and has described the procedure producing formulas at any prime using appropriate elliptic curve data.

10. Describing W_Γ

It is in fact difficult to describe W_Γ-explicitly.

For instance, $\kappa = A/\mathfrak{m}$ is an A-algebra, so it is naturally to ask about $W_\Gamma(\kappa)$. I would like to say $A \xrightarrow{\sim} W_\Gamma(\kappa)$ but I don't know.

We have an inverse limit $W(R) = \lim_n W_n(R)$, and I can describe W_1.

Let B_1 be the pullback of A-algebras:

$$\begin{array}{ccc} B_1 & \longrightarrow & A_1 \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ A & \twoheadrightarrow & A/p \end{array}$$

Note: in topology, $B_1 = E^0 B\Sigma_p$.

We have

$$\begin{array}{ccccc} W_1(R) & \longrightarrow & B_1 \otimes_A R & \longrightarrow & A_1{}^s\otimes_A R \\ \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow[x\mapsto x^p]{} & R & \longrightarrow & R/p \end{array}$$

The left-hand square is a pullback of sets (but not generally of rings, since pth power isn't a ring map).

If R is p-torsion free, then the right-hand square is a pullback of rings, and therefore the big rectangle is a pullback of rings.

11. The Koszul theorem

Let me indicate where the quadraticity theorem comes from.

We have a sequence of cochain complexes of A-modules:

$$\mathcal{C}^0\colon \quad A \longrightarrow 0$$

$$\mathcal{C}^1\colon \quad 0 \longrightarrow A_1 \longrightarrow 0$$

$$\mathcal{C}^2\colon \quad 0 \longrightarrow A_2 \longrightarrow A_1{}^s\otimes_A{}^t A_1 \longrightarrow 0$$

$$\mathcal{C}^3\colon \quad 0 \longrightarrow A_3 \longrightarrow \begin{array}{c} A_2{}^s\otimes_A{}^t A_1 \\ \times \\ A_1{}^s\otimes_A{}^t A_2 \end{array} \longrightarrow A_1{}^s\otimes_A{}^t A_1{}^s\otimes_A{}^t A_1 \longrightarrow 0$$

The components of the boundary maps are produced using μ, with signs to make it a chain complex.

11.1. **Theorem** (R.). *We have*

$$H_q\mathcal{C}^k \approx 0 \qquad \text{if } q \neq k,$$

while

$$H_k\mathcal{C}^k \approx \text{ free } A\text{-module of rank } \left[{n \atop k}\right]_p.$$

(Number of k-dimensional subspaces in $\mathbb{F}_p^n$.)

That $H_1\mathcal{C}^k = 0$ for $k > 1$ says that the family of maps $\mu\colon A_k \to A_i{}^s\otimes_A{}^t A_{k-i}$ for fixed k are injective. This implies that in $\mathcal{Q}$, the higher ψ_k must be uniquely determined by ψ_1.

That $H_2\mathcal{C}^k = 0$ for $k > 2$ then implies that the "associativity" relation on the ψs are forced by the "first" one.

Idea of proof. When height =1 this is trivial. When height =2 there is a proof using information about the moduli of subgroups of elliptic curves found in Katz-Mazur.

For higher height the only proof I know is topological. Namely, let Part_m denote the poset of partitions (=equivalence relations) on $\{1, \ldots, m\}$; we exclude the top and bottom elements.

The nerve $N\mathrm{Part}_m$ is a simplical complex. Write $(N\mathrm{Part}_m)_q$ for the set of q-simplices.

Let $m = p^k$ and take E-Borel cohomology of each set of simplices. Then one can show there is an isomorphism of chain complexes:

$$E^0\left[(N\mathrm{Part}_{p^k})_\bullet\right]_{h\Sigma_{p^k}}/I_{\mathrm{tr}} \approx \mathcal{C}^k.$$

Then you can prove the result for the left-hand side. The key observation is that the functor $X \mapsto E^0 X_{h\Sigma_m}/I_{\mathrm{tr}}$ on Σ_m-sets is a **Mackey functor**. The vanishing result we need is in a paper by Arone, Dwyer, Lesh. □

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