abstract

GENERALIZATIONS OF WITT VECTORS IN ALGEBRAIC TOPOLOGY

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Abstract. Witt vector constructions (both integral and p-adic) are the underlying functors of the comonads which define λ-rings and p-derivation rings, which arise algebraic structures on certain K-theory rings. In algebraic topology, this is a special case of structure that exist for a number of generalized cohomology theories. In this talk I will describe the analogue of this structure for “Morava E-theories”, which are cohomology theories associated to universal deformations of 1-dimensional formal groups of finite height.

1. Introduction


The goal of this talk is to describe some generalizations of “Witt vectors” and “λ-rings” which show up in algebraic topology, in the guise of algebras of “power operations” for certain cohomology theories. I won’t dwell much on the algebraic topology aspect: the set-up I describe arises directly from the arithmetic algebraic geometry of formal groups and isogenies.

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There are other examples, corresponding to certain p-divisible groups (local), or elliptic curves (global).

The case of Morava E-theory was observed by Ando, Hopkins, and Strickland, together with some contributions by me.

2. λ-rings

λ-rings were invented by Grothendieck\(^1\) to formalize certain operations on vector bundles, or representations. Thus,

\[ \lambda^n V = \text{nth exterior power of } V, \]

satisfying various conditions. For instance, for direct sum:

\[ \lambda^n (V + W) = \sum \lambda^i V \otimes \lambda^{n-i} W. \]

There are also formulas for exterior power of tensor product and for composition of exterior powers:

\[ \lambda^n (V \otimes W) = \text{polynomial in } \lambda^i V, \lambda^i W, \]

\[ \lambda^m \lambda^n V = \text{polynomial in } \lambda^i V. \]

These polynomials involve possibly negative integer coefficients, so we must regard these as acting on a Grothendieck ring, e.g., \( K^0 X \) or \( RG \).

The forgetful functor

\[ (\lambda\text{-rings}) \rightarrow (\text{commutative rings}) \]

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\(^1\)By which I mean what he called “special λ-rings”
has both a left and right adjoint, and in fact is both monadic and comonadic. The functor of the comonad is the \textbf{big Witt vector} construction. \(\lambda\)-rings are precisely “coalgebras” for this comonad.

3. Adams operations and Frobenius lifts

Other operations on \(\lambda\)-ring \(R\) are symmetric powers \(\sigma^n\), which can be expressed (on a Grothendieck group) in terms of exterior powers. We also have \textbf{Adams operations} \(\psi^m(V)\).

\[
\sum_{k \geq 0} \sigma_k(V) t^k = \left( \sum_{k \geq 0} \lambda_k(V) (-t)^k \right)^{-1} = \exp \left[ \sum_{m \geq 1} \frac{1}{m} \psi^m(V) t^m \right].
\]

Taking a log-derivative gives \(\psi^m(V) = \text{polynomial in } \lambda(V) \text{ with } \mathbb{Z}\text{-coefficients.} \)

Adams observed:

1. \(\psi^m\) are ring homomorphisms,
2. \(\psi^m \psi^n = \psi^{mn}\),
3. \(\psi^p(x) \equiv x^p \mod pR\) if \(p\) prime.

That is, \(\psi^p\) is a lift of Frobenius for each prime, and lifts for different primes commute.

Note that \(\lambda^k(x) = \text{polynomial in } \psi^m(x) \text{ with } \mathbb{Q}\text{-coefficients, so a } \lambda\text{-ring structure is determined rationally.} \)

There is a partial converse.

3.1. \textbf{Theorem} (Wilkerson). \textit{Let }\(R\) \textit{be a commutative ring equipped with functions }\(\psi^m : R \to R\) \textit{satisfying the above conditions (1)–(3). If }\(R\) \textit{is torsion free, then there exists a unique }\(\lambda\)-\textit{ring structure on }\(R\) \textit{with the given }\(\psi^m\)'s \textit{as the Adams operations.} \)

It turns out that a free \(\lambda\)-ring on a set of generators is torsion free as an abelian group (in fact, it’s a polynomial ring). This means that the entire theory of \(\lambda\)-rings can be recovered from the theory of \(\psi^p\)'s.

4. Multiplicative group

Consider the representation ring of the circle:

\(R(U(1)) = \mathbb{Z}[T, T^{-1}].\)

Multiplication \(\mu : U(1) \times U(1) \to U(1)\) is a homomorphism, so we get a coproduct

\[
\mu^* : R(U(1)) \to R(U(1) \times U(1)) \approx R(U(1)) \otimes R(U(1)).
\]

This Hopf algebra is the ring of functions on \(\mathbb{G}_m\), i.e.,

\[
\mathbb{G}_m = \text{Spec } R(U(1)).
\]

The Adams operations \(\psi^n, n \geq 1\) thus give rise to endomorphisms of the group scheme \(\mathbb{G}_m\). A straightforward calculation \((\psi^n(T) = T^n)\) shows that the endomorphism is \([n] : \mathbb{G}_m \to \mathbb{G}_m.\)

Note that on \((\mathbb{G}_m)_{\mathbb{F}_p}\), the endomorphism \([p]\) is also the Frobenius.

(Remark: in complex K-theory, there is also a \(\psi^{-1}\), corresponding to complex conjugation. This induces \([-1]\) on \(\mathbb{G}_m\).)

5. Power operations

To see how this generalizes, we look at a different construction of \(\psi^m\). First note the “total \(m\)th power” operation:

\[
P_m(V) := V^{\otimes m} = \sum_{\pi} \rho_{\pi} \otimes \sigma_{\pi}(V), \quad P_m : K^0(X) \to R\Sigma_m \otimes K^0(X),
\]

\[\sum_{\pi} (\dim \rho_{\pi}) \sigma_{\pi}(V) = V^{\otimes m}.\]
It turns out that every $\sigma_p \in \mathbb{Z}[\lambda^k, \ k \geq 1]$. See Atiyah, "Power operations in K-theory".

The power construction is multiplicative: $(V \otimes W)^{\otimes m} \approx V^{\otimes m} \otimes W^{\otimes m}$. It is not additive; instead:

$$(V + W)^{\otimes m} = \sum_{i+j=m} V^{\otimes i} \otimes W^{\otimes j} \uparrow_{\Sigma_i \times \Sigma_j}.$$ 

The failure of additivity comes from induced representations from proper subgroups $\Sigma_i \times \Sigma_{m-i} \subsetneq \Sigma_m$. We can get a ring homomorphism by quotienting by the "transfer ideal":

$$\psi$$

This gives the Adams operation $\psi^m$, which is a ring homomorphism.

6. Power operations for commutative ring spectra

The above analysis is specific to $K$-theory, but there is a class of cohomology theories which allow us to do something similar, namely those arising from **commutative ring spectra**, also called $\mathbb{E}_\infty$-ring spectra. For such $E$ we have

$$E^0(X) \xrightarrow{P_m} E^0(B\Sigma_m \times X) \xleftarrow{\sim} E^0(B\Sigma_m) \otimes_{E^0(pt)} E^0(X) \rightarrow E^0(B\Sigma_m)/I_{tr} \otimes_{E^0(pt)} E^0(X).$$

The function $P_m$ is a refinement of the ordinary $m$th power map $x \mapsto x^m$.

In certain cases, the second map is an isomorphism, in which case we get a ring homomorphism

$$E^0(X) \rightarrow E^0(B\Sigma_m)/I_{tr} \otimes_{E^0(pt)} E^0(X).$$

**Example.** If $E = H\mathbb{F}_2$ is ordinary mod 2 cohomology, and $m = 2$, then $I_{tr} = 0$, and we get the classical Steenrod operations.

**Example.** Let $E = K_p$, the $p$-completion of the complex $K$-theory spectrum. Then

$$K^0_p(B\Sigma_p)/I_{tr} \approx \begin{cases} \mathbb{Z}_p & \text{if } m = p^k \\ 0 & \text{else.} \end{cases}$$

We get natural ring maps

$$\psi^p : K^0_p(X) \rightarrow K^0_p(X) \otimes K^0_p(B\Sigma_p)/I = K^0_p(X).$$

One has that $K^0_p(B\Sigma_p) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, so that the total operation is

$$P_p(x) = (\psi^p(x), \delta(x)),$$

where $\psi^p(x) = x^p + p \delta(x)$. One can show that the entire structure the $P_m$s produce in this case are exactly a "ring with $p$-derivation".

The object $K^0_p(BU(1)) \approx \mathbb{Z}_p[t] \approx O_{\hat{\mathbb{G}_m}}$ is functions on the **formal multiplicative group** $\hat{\mathbb{G}}_m$.

On this $\psi^p$ realize $[p^k] : \hat{\mathbb{G}}_m \rightarrow \hat{\mathbb{G}}_m$.

7. The deformation category

I now describe how this works out for a family of theories called **Morava E-theory**.

Fix

- $\kappa$ perfect field char $p$,
- $\Gamma$ one-dimensional formal group of finite height $n$ over $\kappa$.

**Deformations.** Given a complete local ring $R$, a deformation of $\Gamma/k$ to $R$ is data

$$\left( G/R, \ i : \kappa \rightarrow R/m, \ \alpha : i^*\Gamma \sim \rightarrow G_{R/m} \right).$$
$G$ is a formal group over $R$, and we have

$$
\begin{array}{c}
G & \xleftarrow{\alpha} & i^*\Gamma & \xrightarrow{\sim} & \Gamma \\
\downarrow & & \downarrow & & \downarrow \\
Spf\ R & \xleftarrow{\sim} & \Spec\ R/\mathfrak{m} & \xrightarrow{i} & \Spec\ \kappa
\end{array}
$$

$G_0 = \text{special fiber of } G \to Spf\ R$. (If $\kappa = \mathbb{F}_p$ we can ignore $i$.)

**Isogenies.** Consider isogenies of formal groups over $R$.

$$
\begin{array}{c}
\Ker f \xrightarrow{\sim} G \xrightarrow{f} G' \\
\downarrow & & \uparrow \\
G/\Ker f
\end{array}
$$

$\Ker f$ is a finite subgroup scheme of $G$. Up to canonical isomorphism, the data of $f$ is the same as the data of the pair $(G, \Ker f)$.

Deformation structures can be pushed forward along isogenies using relative Frobenius:

$$
\begin{array}{c}
G \xrightarrow{f} G' \\
(i, \alpha) \mapsto (i', \alpha')
\end{array}
$$

with $\deg f = p^k$, where $i' = i \circ \sigma^k$ and

$$
\begin{array}{c}
G_0 \xrightarrow{f_0} G_0' \\
\alpha \xrightarrow{\sim} \alpha' \\
i^*\Gamma \xrightarrow{\sim} (i \circ \sigma^k)^*\Gamma
\end{array}
$$

where $\sigma(x) = x^p$ is absolute Frobenius.

**Category of deformations and isogenies.** For each $R$ we get a category

$$
\text{Def}_\Gamma(R) = \begin{cases}
\text{obj:} & \text{deformations } (G, i, \alpha), \\
\text{mor:} & f: G \to G' \text{ st. } (i, \alpha) \mapsto (i', \alpha').
\end{cases}
$$

**Fact.** $\text{Def}_\Gamma$ is representable:

$$
\text{Def}_\Gamma = \begin{cases}
\text{Spf } A, \\
\coprod_{k \geq 0} \text{Spf } A_k.
\end{cases}
$$

We have $A = A_0$. The category structure is encoded by ring maps:

$$
\begin{array}{c}
A \xrightarrow{s} A_1, \\
A_{k+\ell} \xrightarrow{\mu} A_k^s \otimes_A A_{\ell}
\end{array}
$$

$\text{Spf } A$ is the **Lubin-Tate deformation space** of $\Gamma/k$, so that

$$
(\text{iso classes of } (G, i, \alpha)) \approx (\text{local homomorphisms } A \to R).
$$

There is a universal deformation $(\mathcal{G}_{\text{univ}}, i, \alpha)$ to $A$, and a non-canonical isomorphism

$$
A \approx \mathcal{W}_p\kappa[a_1, \ldots, a_{n-1}], \quad n = \text{ht } \Gamma.
$$

If two deformations are isomorphic, they are so by a **unique** isomorphism.

$\text{Spf } A_k$ carries the **universal example of subgroup** of rank $p^k$ of a deformation:

$$
(\text{iso classes of } (G, i, \alpha, H \leq G), \text{rk } H = p^k) \approx (\text{local homomorphisms } A_k \to R).$$
This was shown to exist by Strickland. “Finite subgroups of formal groups.” Note that $s: A \to A_k$ is finite and flat.

**Frobenius.** If $\mathbb{F}_p \subseteq R$, then for a deformation $(G, i, \alpha)$ there is a canonical subgroup $\text{Ker Fr}_k \leq G$ of rank $p^k$, where $\text{Fr}_k: G \to (\sigma^k)^*G$ is $p^k$th power relative Frobenius. This is represented by a map $\tau: A_k \to A/p$.

**Morava $E$-theory.** For each $\Gamma / \kappa$, there is a generalized cohomology theory $E$ called Morava $E$-theory with

- $E^0(\text{pt}) = A$.
- $E^*(\text{pt}) = A[\mu^\pm], |\mu| = 2$. (“Even periodic”.)
- $E^0(BU(1)) = \mathcal{O}_{G_{\text{univ}}}$, the ring of functions on the universal deformation of $\Gamma / \kappa$.
- $E$ is a commutative ring spectrum (the “Goerss-Hopkins-Miller” theorem).
- $E^0(B\Sigma_p)/I_{tr} = A_k$ (Strickland, “Morava $E$-theory of symmetric groups”).

8. $\Lambda^\Gamma$-rings

From the above category we produce an analogue of Witt vectors and $\lambda$-rings, which I’ll call $\Gamma$-Witt vectors and $\Lambda^\Gamma$-rings.

We define categories $Q = \text{Qcoh}_{\text{Ring}}(\text{Def}_\Gamma) \supseteq Q^{\text{Fr}}$ as follows. A quasicoherent sheaf of commutative rings on $\text{Def}_\Gamma$ is data $(R, \{\psi_k, k \geq 0\})$

consisting an $A$-alegbra $A \to R$, maps

\[ \psi_k: R \to {}^tA_k^s \otimes_A R \]

of $A$-algebras (where the target gets its $A$-algebra structure from $t: A \to A_k$), and such that

\[ \begin{CD}
R @>{\psi_k}>> A_k^s \otimes_A R \\
@V{\psi_{k+\ell}}VV @VV{id \otimes \psi_\ell}V \\
A_{k+\ell}^s \otimes_A R @>{\mu \otimes \text{id}}>> A_k^s \otimes_A {}^tA_\ell^s \otimes_A R
\end{CD} \]

Such an object satisfies the Frobenius congruence if

\[ \begin{CD}
R @>{\psi_k}>> A_k^s \otimes_A R \\
@VV{\tau \otimes \text{id}}V @V{\tau/p \otimes A}VV \\
A/p \otimes_A R @. R/p
\end{CD} \]

commutes, where $\tau: A_k \to A/p$ is the map classifying $\text{Ker Fr}_k$.

**Remark.** Thus, $\psi_k$ is a “Frobenius lift”, but not in general an endomorphism of $R$. In our setting, formal groups of height $\geq 2$ are not “ordinary”, i.e., do not have a canonical lift of $\text{ker Fr}$.

8.1. **Theorem.** There exists a category called $(\Lambda^\Gamma$-rings), together with a diagram of functors

\[ (\Lambda^\Gamma$-rings) \xrightarrow{\text{forget}} Q^{\text{Fr}} \xleftarrow{\text{underlying}} \xrightarrow{\text{underlying}} (\text{comm. } A\text{-alg.}) \]
such that (i) \((\Lambda_{\Gamma}-\text{rings}) \to (\text{comm. } A-\text{alg.})\) is monadic and comonadic, and (ii) “forget” induces an equivalence on the full-subcategories of objects whose underlying ring is \(p\)-torsion free.

The functor of the comonad is \(W = W_{\Gamma}\).

**Key example.** The ring \(O_{\text{univ}}\) is tautologically a \(\Lambda_{\Gamma}\)-ring. The maps \(\psi_k\) extend to ring homomorphisms

\[
A_k^t \otimes_A O_{\text{univ}} \to A_k^s \otimes_A O_{\text{univ}}
\]

which describe \(G_{\text{univ}} \times^{\text{Spf } A} \text{Spf } A_k \to G_{\text{univ}} \times^{\text{Spf } A} \text{Spf } A_k\) the universal example of a \(p^k\) isogeny on \(G_{\text{univ}}\).

**Motivating example.** For every topological space \(X\), \(E^0(X)\) is naturally a \(\Lambda_{\Gamma}\)-ring.

In fact, for every \(K(n)\)-local commutative \(E\)-algebra \(B\), \(\pi_0 B\) is a \(\Lambda_{\Gamma}\)-ring.

\(\Lambda_{\Gamma}\)-rings at height 1.

Let \(\kappa = \mathbb{F}_p\), \(\Gamma = \hat{\mathbb{G}}_m\).

Then

\[
A = \mathbb{Z}_p, \quad G_{\text{univ}} = \hat{\mathbb{G}}_m, \quad A_k = \mathbb{Z}_p.
\]

An object of \(Q^{\text{Fr}}\) is a \(\mathbb{Z}_p\)-algebra with endomorphisms \(\psi_k: R \to R\) such that \(\psi_k = \psi_1 \circ \cdots \circ \psi_1\) and \(\psi_k(x) \equiv x^{p^k} \mod pR\). A \(\Lambda_{\Gamma}\)-ring is precisely a \(\mathbb{Z}_p\)-algebra with \(p\)-derivation.

The Morava \(E\)-theory in this case is precisely \(p\)-adic \(K\)-theory \(K_p\).

9. **Simplifying results**

To even get our hands on \(W_{\Gamma}\), we need an infinite collection of rings \(A_k\) together with \(s, t, \mu, \tau\). In fact things are much better than this, by the following “quadraticity” theorem.

9.1. **Theorem** \((R.)\). The data \((R, \{\psi_k\})\) of an object of \(Q\) are uniquely determined by the following subset:

1. an \(A\)-algebra map \(\psi_1: R \to {^tA_1}^s \otimes_A R\), such that
2. there exists a unique dotted arrow making

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_1} & {^tA_1}^s \otimes_A R \\
\downarrow & & \downarrow \text{id} \otimes \psi_1 \\
A_2^s \otimes_A R & \xrightarrow{\mu \otimes \text{id}} & {^tA_1}^s \otimes_A {^tA_1}^s \otimes_A R \\
\end{array}
\]

commute.

The Frobenius condition for \(\{\psi_k\}\) is equivalent to the Frobenius condition for \(\psi_1\).

So we just need \(A_1, A_2\) and the relevant \(s, t, \mu, \tau\).

**Height 2.** When \(n = \text{ht } \Gamma = 2\) we can do a little better, by giving a formula for \(A_2\).

Given an isogeny \(f: G \to G'\) of degree \(p\), there exists a unique dual isogeny \(f^\vee: G' \to G\), defined by:

\[
\begin{array}{ccc}
\ker f & \hookrightarrow & G \\
\downarrow & & \downarrow f \\
G[p] & \xrightarrow{i} & G \\
\end{array}
\quad
\begin{array}{ccc}
G[p] & \xrightarrow{[p]} & G \\
\downarrow f^\vee & & \downarrow f \\
G[p] & \xrightarrow{i'} & G \\
\end{array}
\]

The operation \((G, i, \alpha, \ker f) \mapsto (G', i', \alpha', \ker f^\vee)\) is represented by a ring map

\[w: A_1^s \to A_1^t.\]
The identity \( f^\vee f = [p] \) corresponds to a commutative square:

\[
\begin{array}{ccc}
A_2 & \xrightarrow{\mu} & A_1^s \otimes_A tA_1 \\
\downarrow & & \downarrow (w, \text{id}) \\
A & \xrightarrow{s} & A_1
\end{array}
\]

where \( \gamma \) represents the operation \( G \mapsto (G[p] \leq G) \).

**Fact.** The above square is a pullback, and remains a pullback after tensoring with \( ^s \otimes_A R \).

Thus, the “associativity condition” on \( \psi_1 \) becomes the existence of a dotted arrow in:

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_1} & A_1^s \otimes_A R \\
\downarrow & & \downarrow \text{id} \otimes \psi_1 \\
A_1^s \otimes_A tA_1 & \xrightarrow{\text{id} \otimes \psi_1} & A_1^s \otimes_A R \\
\downarrow & & \downarrow (w, \text{id}) \otimes \text{id} \\
R & \xrightarrow{s} & A_1^s \otimes_A R
\end{array}
\]

So to describe objects of \( Q^{Fr} \), and thus the category of \( A_\Gamma \)-rings in the height 2 case, we only need

\[
\begin{array}{ccc}
A & \xrightarrow{s} & A_1 \\
\downarrow & & \downarrow w \\
\tau & & A/p
\end{array}
\]

**Explicit example.** Let

\( \kappa = \mathbb{F}_2, \Gamma = \widehat{C}_{\text{ss}}, \) where \( C_{\text{ss}} = (Y^2Z + YZ^2 = X^3) \), the formal completion of a supersingular curve. Then

\[
A = \mathbb{Z}_2[[a]], \quad G_{\text{univ}} = \widehat{C}, \quad \text{where} \ \widehat{C} = (Y^2Z + aXYZ + YZ^2 = X^3).
\]

Furthermore

\[
A_1 = \mathbb{Z}_2[a][d]/(d^3 - ad - 2).
\]

Note: \( P = (d : 1 : -d^3) \) is 2-torsion point, and the Cartier divisor \([O] + [P], O = (0 : 1 : 0)\), is the divisor of a subgroup.

The maps are given by

\[
\begin{align*}
s &: a \mapsto a, \\
t &: a \mapsto a' = a^2 + 3d - ad^2, \\
w &: a \mapsto a', \\
: d &\mapsto a - d^2, \\
\tau &: a \mapsto a, \\
: d &\mapsto 0.
\end{align*}
\]

From this we can (in principle) completely recover \( W_\Gamma \).

**Remark.** Yifei Zhu has given similar explicit formulas at \( p = 3 \) and \( p = 5 \), and has described the procedure producing formulas at any prime using appropriate elliptic curve data.
10. Describing $W_\Gamma$

It is in fact difficult to describe $W_\Gamma$-explicitly.
For instance, $\kappa = A/m$ is an $A$-algebra, so it is naturally to ask about $W_\Gamma(\kappa)$. I would like to say $A \xrightarrow{\sim} W_\Gamma(\kappa)$ but I don't know.

We have an inverse limit $W(R) = \lim_n W_n(R)$, and I can describe $W_1$.

Let $B_1$ be the pullback of $A$-algebras:

$$
\begin{array}{c}
B_1 \\
\downarrow \\
A
\end{array} \longrightarrow 
\begin{array}{c}
A_1 \\
\downarrow \tau
\end{array} \\
\begin{array}{c}
A \\
\longrightarrow
\end{array} \longrightarrow 
\begin{array}{c}
A/p
\end{array}
$$

Note: in topology, $B_1 = E^0 \Sigma_p$.

We have

$$
\begin{array}{c}
W_1(R) \\
\downarrow
\end{array} \longrightarrow 
\begin{array}{c}
B_1 \otimes_A R \\
\downarrow
\end{array} \longrightarrow 
\begin{array}{c}
A_1^s \otimes_A R \\
\downarrow
\end{array} \\
\begin{array}{c}
R \\
\longrightarrow
\end{array} \longrightarrow 
\begin{array}{c}
R/p
\end{array}
$$

The left-hand square is a pullback of sets (but not generally of rings, since $p$th power isn’t a ring map).

If $R$ is $p$-torsion free, then the right-hand square is a pullback of rings, and therefore the big rectangle is a pullback of rings.

11. The Koszul theorem

Let me indicate where the quadraticity theorem comes from.
We have a sequence of cochain complexes of $A$-modules:

$$
\begin{array}{c}
\mathcal{C}^0: \\
A \longrightarrow 0
\end{array}
$$

$$
\begin{array}{c}
\mathcal{C}^1: \\
0 \longrightarrow A_1 \longrightarrow 0
\end{array}
$$

$$
\begin{array}{c}
\mathcal{C}^2: \\
0 \longrightarrow A_2 \longrightarrow A_1^s \otimes A^t A_1 \longrightarrow 0
\end{array}
$$

$$
\begin{array}{c}
\mathcal{C}^3: \\
0 \longrightarrow A_3 \longrightarrow A_2^s \otimes A^t A_1 \times A_1^s \otimes A^t A_2 \longrightarrow A_1^s \otimes A^t A_1 \longrightarrow 0
\end{array}
$$

The components of the boundary maps are produced using $\mu$, with signs to make it a chain complex.

11.1. Theorem (R.). We have

$$
H_q \mathcal{C}^k \approx 0 \quad \text{if } q \neq k,
$$

while

$$
H_k \mathcal{C}^k \approx \text{free } A\text{-module of rank } \binom{n}{k}_p.
$$

(Number of $k$-dimensional subspaces in $F^n_p$.)

That $H_1 \mathcal{C}^k = 0$ for $k > 1$ says that the family of maps $\mu: A_k \rightarrow A_i^s \otimes A^t A_{k-i}$ for fixed $k$ are injective. This implies that in $Q$, the higher $\psi_k$ must be uniquely determined by $\psi_1$.

That $H_2 \mathcal{C}^k = 0$ for $k > 2$ then implies that the “associativity” relation on the $\psi$s are forced by the “first” one.
Idea of proof. When height $=1$ this is trivial. When height $=2$ there is a proof using information about the moduli of subgroups of elliptic curves found in Katz-Mazur.

For higher height the only proof I know is topological. Namely, let $\text{Part}_m$ denote the poset of partitions (=equivalence relations) on $\{1, \ldots, m\}$; we exclude the top and bottom elements.

The nerve $N\text{Part}_m$ is a simplicial complex. Write $(N\text{Part}_m)_q$ for the set of $q$-simplices.

Let $m = p^k$ and take $E$-Borel cohomology of each set of simplices. Then one can show there is an isomorphism of chain complexes:

$$E^0[(N\text{Part}_{p^k})_\bullet]_{h\Sigma_{p^k}}/I_{tt} \approx C_k.$$

Then you can prove the result for the left-hand side. The key observation is that the functor $X \mapsto E^0X_{h\Sigma_m}/I_{tt}$ on $\Sigma_m$-sets is a Mackey functor. The vanishing result we need is in a paper by Arone, Dwyer, Lesh. □

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