ANALYTIC COMPLETION

CHARLES REZK

Abstract. This is an expository treatment of what we call "analytic completion" of R-modules, which is a kind of completion defined in terms of quotients of power series modules. It is closely related to the left derived functors of adic completion, in which guise it has been studied by several authors.

1. Introduction

The purpose of this note is describe something which I'll call the "analytic completion" of an R-module with respect to a finite set of elements in R. This is not a new notion. The case of $\{p\} \subset \mathbb{Z}$ appears often occurs in the homotopy theory literature under the name "Ext-p completion", and is well-known to be closely linked to computing the homotopy groups of p-completions of spaces and spectra; for instance, see [BK72, Ch. VI].

Analytic completion also coincides with the 0th left derived functor of adic completion. Derived functors of adic completion (in the case of a complete local ring R) are studied in [Mat74] and [GM92], and topological applications of this theory are given in [HS99, App. A]; see also [Hov08]. This note is largely a reworking of some of the results of these papers, and not much is really new. The particular way of describing analytic completion given here seems new. I have not seen the extension of the theory to simplicial algebras elsewhere. The treatment of the derived category of analytically complete modules overlaps with [Val].

I've tried to make things as self-contained and as elementary as possible. The main purpose of this note is simply to assert that analytic completion is a Good Thing.

I'd like to thank Martin Frankland for many discussions which helped clarify my thinking about these ideas.

1.1. **Topological and analytic p-completion of integers.** Let $p \in \mathbb{Z}$ be a prime. The completion of the integers at p, called the ring $\mathbb{Z}_p$ of **p-adic integers**, can be constructed in two different ways.

Topological p-completion of $\mathbb{Z}$. The most familiar construction of the p-adics is as the inverse limit of quotients of powers of the ideal $(p) \subset \mathbb{Z}$. Thus

$$\mathbb{Z}_p \approx \lim \mathbb{Z}/(p^n).$$

To say it slightly differently, elements of $\mathbb{Z}_p$ can be identified as equivalence classes of sequences (a_n) of integers which are Cauchy with respect to the p-adic filtration; that is, $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the p-adic topology. Thus, we can think of this construction of the p-adics as a "topological" completion.

Analytic p-completion of $\mathbb{Z}$. There is another description of the p-adic integers which can be found in any commutative algebra textbook. Let $\mathbb{Z}[\![x]\!]$ denote the ring of formal power series on one variable, with coefficients in $\mathbb{Z}$. There is an isomorphism

$$\mathbb{Z}[\![x]\!]/(x-p)\mathbb{Z}[\![x]\!] \to \mathbb{Z}_p$$

where the left-hand side is the quotient ring by the principal ideal generated by $x-p$, and the right-hand side is the topological p-completion. (A proof that this map is an isomorphism

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will be given below.) If we identify $\mathbb{Z}_p$ with the limit $\lim \mathbb{Z}/p^n$, then the above isomorphism is described by the maps $\mathbb{Z}[\![x]\!] \to \mathbb{Z}/p^n$ which send $f(x) \mapsto f(p)$. An element of $\mathbb{Z}[\![x]\!]/(x-p)\mathbb{Z}[\![x]\!]$ can be thought of as a "formal power series in p" of the form

$$\sum_{k\geq 0} a_k\, p^k = a_0 + a_1\, p + a_2\, p^2 + \cdots ,$$

where the $a_k \in \mathbb{Z}$. Finite sums and products are defined as usual for power series, and we are also allowed to make the identification

$$\sum_{k\geq 0} a_k\, p^{k+1} = p \sum_{k\geq 0} a_k\, p^k$$

between formal series. Because of this power series formulation, we will think of this construction of the p-adics as an "analytic" completion.

Now observe that in either of these two completion constructions, we can replace $\mathbb{Z}$ with an arbitrary abelian group M, thus obtaining "topological" and "analytic" p-completions $M^\wedge_p = \lim M/p^k M$ and $\mathcal{A}_p(M) = M[\![x]\!]/(x-p)M[\![x]\!]$ of M. Say that M is **topologically (resp. analytically) p-complete** if the evident map $M \to M^\wedge_p$ (resp. $M \to \mathcal{A}_p(M)$) is an isomorphism. It turns out that the two notions of completion do not coincide in general, and that though every topologically complete group is analytically complete, the converse is not true.

A major advantage of analytic completion is that the full subcategory of analytically complete abelian groups is an abelian subcategory of the category of abelian groups, with kernels and cokernels computed exactly as they are in abelian groups. This is not true for topological completion; the cokernel of a map between topologically p-complete groups need not be topologically p-complete (though it must be analytically p-complete).

It is analytic completion which seems to occur more naturally in homotopy theory. For instance, a spectrum X is p-complete if and only if its homotopy groups $\pi_* X$ are *analytically* p-complete.

1.2. **Organization.** In §2, we define the notion of a *p-analytic module* over a commutative ring R with a chosen element p. We show that the full subcategory of p-analytic modules is well-behaved with respect to the usual operations in abelian categories. Then in §3 we extend our definitions to analytic completion with respect to sequences $\underline{p} = (p_1, \dots, p_r)$ of elements in R.

In §4 we describe *analytic completion funtor* and investigate its properties. In §5 we prove an analogue of Nakayama's lemma for analytic modules, and in §6 we investigate how analytic completion interacts with tensors and homs. In §7 we show that analytic completion with respect to a finite sequence $\underline{p}$ in fact only depends on the radical of the ideal generated by $\underline{p}$.

In §8 we define the notion of *tameness*; a module M is tame if its higher "koszul homology" vanishes. In §9 we define the notion of a *tame sequence* in R, and show that the higher derived functors of analytic completion with respect to a tame sequence coincide with koszul homology. We go on to analyze some basic features of homological algebra in categories of analytic modules.

In §10 we study the derived category $\widehat{\mathcal{D}}_R$ of $\underline{p}$-analytic modules. We show that when $\underline{p}$ is a tame sequence, then $\widehat{\mathcal{D}}_R$ is equivalent to an easily identified full subcategory of the usual derived category $\mathcal{D}_R$ of R-modules. In §11 we begin the study of the homotopy theory of algebraic objects which are analytically complete, and we consider the specific example of analytically complete simplicial commutative non-unital R-algebras.

1.3. **Notation.** In the following, R will be a commutative ring. We write $\mathrm{Mod} = \mathrm{Mod}_R$ for the category of R-modules. Given $M \in \mathrm{Mod}_R$, let $M[\![x]\!]$ denote the $R[\![x]\!]$-module of power series on one indeterminate x with coefficients in M; the construction $M \mapsto M[\![x]\!]$ defines a functor $\mathrm{Mod}_R \to \mathrm{Mod}_{R[\![x]\!]}$. We note that this functor is exact, and commutes with arbitrary products.

We write $i_M \colon M \to M[\![x]\!]$ for the evident inclusion of M as the set of constant power series in $M[\![x]\!]$; it will often be convenient to abuse notation and identify M with its image $i_M(M) \subseteq M[\![x]\!]$.

More generally, we may consider power series in several variables: $M[\![x_1, \dots, x_n]\!] \approx M[\![x_1, \dots, x_{n-1}]\!][\![x_n]\!]$. Given elements $f_k \in (x_1, \dots, x_n)R[\![x_1, \dots, x_n]\!]$, $k = 1, \dots, m$, we have a "coordinate change" map

$$M[\![y_1, \dots, y_m]\!] \xrightarrow{\phi} M[\![x_1, \dots, x_n]\!]$$
$$\sum a_{k_1 \dots k_m} y_1^{k_1} \cdots y_m^{k_m} \mapsto \sum a_{k_1 \dots k_m} f_1^{k_1} \cdots f_m^{k_m}.$$

We typically write $g(f_1, \dots, f_m)$ for the image of $g \in M[\![y_1, \dots, y_m]\!]$ under ϕ. Note that if $m = n$ and the matrix $((\partial f_i / \partial x_j)(0))$ is invertible, then ϕ is an isomorphism of rings.

2. p-analytic modules

2.1. **Definition.** Let $p \in R$. We say that $M \in \mathrm{Mod}$ is **analytically p-complete**, or simply **p-analytic**, if for every $f \in M[\![x]\!]$, there exists a unique $c \in M$ such that

$$f - c \in (x - p)M[\![x]\!].$$

In other words, there exists a unique $c \in M \subseteq M[\![x]\!]$ such that an equation of the form

$$f = c + (x - p)g$$

holds for some $g \in M[\![x]\!]$. Note that we do not *require* that there be a unique g for which this equation holds. However, it does happen to be unique, assuming M is p-analytic.

2.2. **Proposition.** *Let $p \in R$, and let M be a p-analytic module. Given $f \in M[\![x]\!]$, let $c \in M$ be the unique element such that $f - c \in (x - p)M[\![x]\!]$. Then there exists a* unique *element $g \in M[\![x]\!]$ such that*

$$f = c + (x - p)g.$$

The proposition is immediate from the following lemma.

2.3. **Lemma.** *Let $p \in R$, and let M be a p-analytic module. Then multiplication by $(x - p)$ is injective on $M[\![x]\!]$.*

Proof. Given $g \in M[\![x]\!]$, write $g = x^n(a + xh)$, with $a \in M$ and $h \in M[\![x]\!]$. We will show that $(x - p)g = 0$ implies that $a = 0$; from this it is straightforward to derive the lemma by induction on n. Since multiplication by x is injective on $M[\![x]\!]$, the identity $(x - p)g = 0$ implies $(x - p)(a + xh) = 0$, and therefore the equality

$$pa = x(a + (x - p)h)$$

holds in $M[\![x]\!]$. Since $pa \in M$, we must have $a + (x - p)h = 0$. Since M is p-analytic, a is the unique element of M such that $a \in (x - p)M[\![x]\!]$, whence $a = 0$ as desired. □

2.4. *Remark.* Let $\mathbb{Z}[t] \to R$ be the unique ring homomorphism sending $t \mapsto p$. Then an R-module is p-analytic if and only if it is t-analytic as a $\mathbb{Z}[t]$-module, since $M[\![x]\!]/(x-p)M[\![x]\!] = M[\![x]\!]/(x - t)M[\![x]\!]$. Thus, p-analyticity of M is a property purely of the abelian group endomorphism of M defined by multiplication by p; it does not depend on any other part of an R-module structure.

2.5. **p-evaluation of power series.** If M is p-analytic, and $f \in M[\![x]\!]$, we write $f(p)$ (or $f|_{x=p}$) for the unique element $c \in M$ such that $f - c \in (x-p)M[\![x]\!]$. We call the resulting function

$$f \mapsto f(p) \colon M[\![x]\!] \to M$$

the **p-evaluation function**.

2.6. **Proposition.** *Let M be p-analytic.*

(1) *p-evaluation coincides with the usual notion of evaluation on polynomials $M[x] \subseteq M[\![x]\!]$. In particular, p-evaluation restricts to the identity map on the image of $i_M \colon M \to M[\![x]\!]$.*
(2) *p-evaluation factors through an isomorphism $M[\![x]\!]/(x-p)M[\![x]\!] \xrightarrow{\sim} M$.*
(3) *If $g \in R[\![x]\!]$ and $d \in R$ are any elements such that $g - d \in (x-p)R[\![x]\!]$, then*

$$(gf)(p) = df(p) \qquad \text{for any } f \in M[\![x]\!].$$

(4) *p-evaluation is a map of $R[x]$-modules, in the sense that*

$$(gf)(p) = g(p)f(p) \qquad \text{for any } g \in R[x] \text{ and } f \in M[\![x]\!].$$

In particular, p-evaluation is a map of R-modules.
(5) *If R is itself p-analytic, then*

$$(gf)(p) = g(p)f(p) \qquad \text{for any } g \in R[\![x]\!] \text{ and } f \in M[\![x]\!].$$

In particular, the p-evaluation map $R[\![x]\!] \to R$ is an R-algebra homomorphism.

Proof. Properties (1) and (2) are immediate from the definitions. For (3), note that if $f - c \in (x-p)M[\![x]\!]$ and $g - d \in (x-p)R[\![x]\!]$, then $gf - dc = (g-d)f + d(f-c) \in (x-p)M[\![x]\!]$. Given this, (4) and (5) are immediate. □

2.7. **Taylor's formula.** If M is p-analytic, we obtain a well-defined "Taylor expansion" of an element $f = f_0 \in M[\![x]\!]$ at p, by successively "solving" equations

$$f_n = c_n + (x-p)f_{n+1}$$

for $c_n \in M$ and $f_{n+1} \in M[\![x]\!]$. We have the following.

2.8. **Proposition.** *If M is p-analytic, then there exists a sequence $c_0, c_1, \dots$ of elements of M such that for each $n \geq 0$, the sequence $c_0, \dots, c_{n-1}$ is the unique one such that*

$$f \equiv \sum_{k=0}^{n-1} c_k (x-p)^k \mod (x-p)^n M[\![x]\!].$$

Proof. We have already noted the existence of such elements. To prove uniqueness, it is enough to show that $\sum_{k=0}^{n-1} c_k(x-p)^k \in (x-p)^n M[\![x]\!]$ implies that $c_0 = \cdots = c_{n-1} = 0$. By induction on n, it is enough to consider the case when $c_0 = \cdots = c_{n-2} = 0$, i.e., to show that $c(x-p)^{n-1} \in (x-p)^n M[\![x]\!]$ implies $c = 0$. By (2.3), this can be reduced to showing $c \in (x-p)M[\![x]\!]$ implies $c = 0$, which follows because M is p-analytic. □

We refer to c_n as the nth **Taylor coefficient** of f at p. We can actually recover the classical formula for the Taylor expansion. Given $f = \sum a_k x^k \in M[\![x]\!]$, let

$$\partial_x^{(n)} f = \sum_{k=0}^{\infty} \binom{k+n}{n} a_{k+n} x^k \in M[\![x]\!].$$

Formally, $\partial_x^{(n)} =$ "$(1/n!) d^n/dx^n$". The function $M[\![x]\!] \to M$ defined by $f \mapsto (\partial_x^{(n)} f)(p)$, is a homomorphism of R-modules.

2.9. **Proposition.** *If M is p-analytic, and $f \in M[\![x]\!]$, then for all $n \geq 0$ we have*

$$f \equiv \sum_{k=0}^{n-1} (\partial_x^{(k)} f)(p)\,(x-p)^k \mod (x-p)^n M[\![x]\!].$$

Proof. Standard algebraic manipulation shows that

$$f \equiv \sum_{k=0}^{n-1} (\partial_x^{(k)} f)|_{x=t}\,(x-t)^k \mod (x-t)^n M[\![t,x]\!]$$

for any $f \in M[\![x]\!] \subseteq M[\![t,x]\!]$. The result is obtained by passing to the quotient $M[\![t,x]\!]/(t-p) \approx M[\![x]\!]$. □

2.10. **The full subcategory of p-analytic modules.** Let $\widehat{\mathrm{Mod}}_p = \widehat{\mathrm{Mod}}_{R,p}$ denote the full subcategory of p-analytic modules in Mod_R.

2.11. **Proposition.** *The full subcategory $\widehat{\mathrm{Mod}}_p$ of Mod is closed under arbitrary limits in Mod_R, and under cokernels of maps between p-analytic modules.*

Proof. Suppose that $\phi\colon M \to N$ is a homomorphism between p-analytic modules, and let $K = \operatorname{Ker}\phi \subseteq M$. To show that K is p-analytic, observe that because M is p-analytic, for $f \in K[\![x]\!]$, there exist unique $c \in M$ and $g \in M[\![x]\!]$ (using (2.2)) such that $f = c + (x-p)g$. It thus suffices to show that $c \in K$ and $g \in K[\![x]\!]$. In fact, we see that in $N[\![x]\!]$ we have

$$0 = \phi(f) = \phi(c) + (x-p)\phi(g),$$

and since N is also p-analytic, we must have $\phi(c) = 0$ and $\phi(g) = 0$. Here we write $\phi\colon M[\![x]\!] \to N[\![x]\!]$ for the map induced by $\phi\colon M \to N$ on coefficients; note that the kernel of this map is precisely $K[\![x]\!] \subseteq M[\![x]\!]$. Thus $c \in K$ and $g \in K[\![x]\!]$ as desired.

Likewise, $\widehat{\mathrm{Mod}}_p$ is closed under arbitrary products of modules, by a straightforward argument using the fact that the construction $M \mapsto M[\![x]\!]$ preserves such products.

We have shown that $\widehat{\mathrm{Mod}}_p$ is closed under all products and kernels; thus it is closed under limits.

Suppose $\phi\colon M \to N$ is a homomorphism between p-analytic R-modules, let $C = \operatorname{Cok}\phi$, and let $\pi\colon N \to C$ denote the quotient map. Given $f \in C[\![x]\!]$, we must show there exists a unique $c \in C$ such that $f - c \in (x-p)C[\![x]\!]$. To prove existence, choose $f' \in N[\![x]\!]$ such that $\pi(f') = f$. Since N is p-analytic, we have $f' - c' \in (x-p)N[\![x]\!]$ for $c' = f'|_p \in N$; thus $f - \pi(c') \in (x-p)C[\![x]\!]$.

For uniqueness, suppose $c \in C$ such that $c = (x-p)g$ in $C[\![x]\!]$ for some $g \in C[\![x]\!]$; we want to show that $c = 0$. Choose $c' \in N$ and $g' \in N[\![x]\!]$ such that $\pi(c') = c$ and $\pi(g') = g$. Thus $f' = c' - (x-p)g' \in N[\![x]\!]$ is in the kernel of π, and thus $f' = \phi(f'')$ for some $f'' \in M[\![x]\!]$. Since M is p-analytic, we have that there is a $c'' \in M$ such that $f'' - c'' \in (x-p)M[\![x]\!]$, and therefore $\phi(f'' - c'') = f' - \phi(c'') \in (x-p)N[\![x]\!]$. Since N is p-analytic, we conclude that $c' = \phi(c'')$, and thus $c = 0$. □

2.12. **Corollary.** *The image of a map $\phi\colon M \to N$ between p-analytic modules is p-analytic.*

Proof. The image of ϕ is isomorphic to the cokernel of $\operatorname{Ker}\phi \to M$. □

2.13. **Proposition.** *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, and any two of M', M, M'' are p-analytic, so is the third.*

Proof. Given (2.11), it only remains to show that if M' and M'' are p-analytic, so is M. Given $f \in M[\![x]\!]$, consider its image $f'' \in M''[\![x]\!]$. Since M'' is p-analytic, we have $f'' = c'' + (x-p)g''$ for some $c'' \in M''$ and $g'' \in M''[\![x]\!]$. Lift c'' and g'' to elements $c_1 \in M$ and $g_1 \in M[\![x]\!]$, and write $f_1 = c_1 + (x-p)g_1$. Then $f' = f - f_1 \in M'[\![x]\!]$, and since M' is p-analytic we have $f' = c' + (x-p)g'$ for some $c' \in M'$ and $g' \in M'[\![x]\!]$.

Thus we have an identity

$$f = f_1 + f' = (c_1 + c') + (x-p)(g_1 + g')$$

in $M[\![x]\!]$, with $c = c_1 + c' \in M$. It is straightforward to show that c is unique such that $f - c \in (x-p)M[\![x]\!]$, using successively the uniqueness of the choices of $c'' \in M''$ and $c' \in M'$. $\square$

3. Analyticity for sequences

Let $\underline{p}$ be a set of elements in R. We say that $M \in \mathrm{Mod}$ is **$\underline{p}$-analytic** if it is p analytic for all $p \in \underline{p}$. We write $\widehat{\mathrm{Mod}}_{\underline{p}} = \bigcap_{p\in\underline{p}} \widehat{\mathrm{Mod}}_p$ for the full subcategory of $\underline{p}$-analytic modules in Mod. (For many statements below, the set $\underline{p}$ will be given as an ordered sequence $p_1, p_2, \dots$ of elements of R, so we will usually refer to $\underline{p}$ as sequence rather than as a set.)

3.1. **Proposition.** *For a set $\underline{p} \subseteq R$, the full subcategory $\widehat{\mathrm{Mod}}_{\underline{p}}$ of* Mod *is closed under small limits and cokernels in* Mod*. The image of a map between $\underline{p}$-analytic modules is also $\underline{p}$-analytic. For any short exact sequence of R-modules in which two terms are $\underline{p}$-analytic, the third term is also $\underline{p}$-analytic.*

Proof. Immediate from (2.11) and (2.13). $\square$

We will show in §7 that the condition of $\underline{p}$-analyticity only depends on the radical ideal generated by $\underline{p}$ in R.

3.2. **$\underline{p}$-evaluation for sequences.** We can evaluate power series $f \in M[\![x_1, \dots, x_r]\!]$ in several variables at sequences $\underline{p} = (p_1, \dots, p_r)$ such that M is $\underline{p}$-analytic.

3.3. **Lemma.** *Suppose $\underline{p} = (p_1, \dots, p_r)$ is a finite sequence in R, and let M be $\underline{p}$-analytic. Then for any $f \in M[\![x_1, \dots, x_r]\!]$ there exists a unique $c \in M$ such that*

$$f - c \in \sum_{k=1}^{r} (x_k - p_k) M[\![x_1, \dots, x_r]\!].$$

Proof. Write $M_k := M[\![x_1, \dots, x_k]\!] \subseteq M[\![x_1, \dots, x_r]\!]$. Since M is $\underline{p}$-analytic, so is each M_k, since products of analytic modules are analytic (2.11).

We work by induction on r. The base case $r = 1$ holds by definition, so let $r \geq 2$.

First we show existence of c. Since M_{r-1} is $\underline{p}$-analytic and hence p_r-analytic, and $M_r = M_{r-1}[\![x_r]\!]$, there exists $g \in M_{r-1}$ such that

$$f - g \in (x_r - p_r) M_{r-1}[\![x_r]\!] = (x_r - p_r) M_r.$$

By induction on r, there exists $c \in M$ such that $g - c \in \sum_{k=1}^{r-1} (x_k - p_k) M_{r-1}$, whence $f - c = (f-g) + (g-c) \in \sum_{k=1}^{r-1} (x_k - p_k) M_{r-1} + (x_r - p_r) M_r \subseteq \sum_{k=1}^{r} (x_k - p_k) M_r$ as desired.

To show uniqueness of c, we may assume $f = 0$, and thus show that

$$c \in M \quad \text{and} \quad c \in \sum_{k=1}^{r} (x_k - p_k) M_r \quad \text{implies} \quad c = 0.$$

Choose $g_k \in M_r$ so that $c = \sum_{k=1}^{r} (x_k - p_k) g_k$. Since M_r is $\underline{p}$-analytic and hence p_r-analytic, we can evaluate this expression at $x_{r-1} = p_r$. The p_r-evaluation map $M_r = M_{r-1}[\![x_r]\!] \to M_{r-1}$ is a map of $R[\![x_1, \dots, x_{r-1}]\!][x_r]$-modules by (2.6), so we see that

$$c = \sum_{k=1}^{r-1} (x_k - p_k) \overline{g}_k \qquad \text{where} \qquad \overline{g}_k = g_k|_{x_r = p_r} \in M_{r-1}.$$

Thus $c \in \sum_{k=1}^{r-1}(x_k - p_k)M_{r-1}$, so by induction on r we conclude that $c = 0$. □

Thus we have a $\underline{p}$-evaluation function $M[\![x_1, \dots, x_r]\!] \to M$ for any $\underline{p}$-analytic M, so that $c = f(\underline{p}) \in M$ is the unique $c \in M$ such that $f - c \in \sum_{k=1}^{r}(x_k - p_k)M[\![x_1, \dots, x_r]\!]$. This function satsifies properties analogous to (2.6), whose statement we leave to the reader.

3.4. **Topologically complete modules are analytic.** Let $\underline{p} = (p_1, \dots, p_r)$. Say that M is **topologically $\underline{p}$-complete** if it is $\underline{p}$-complete in the usual adic sense, i.e., if $M \to \lim_n M/(p_1, \dots, p_r)^n M$ is an isomorphism.

3.5. **Proposition.** *If M is topologically $\underline{p}$-complete for a finite set $\underline{p}$, then it is $\underline{p}$-analytic.*

Proof. Since $\widehat{\mathrm{Mod}}_{\underline{p}}$ is closed under limits, to show that $\lim_n M/(p_1, \dots, p_r)^n M$ is $\underline{p}$-analytic it is enough to show that all $(p_1, \dots, p_r)^n$-torsion modules are $\underline{p}$-analytic. Since $\widehat{\mathrm{Mod}}_{\underline{p}}$ is closed under extensions, it suffices to consider the case when $(p_1, \dots, p_r)M = 0$. This is immediate, for in this case $(x - p_i)M[\![x]\!] = xM[\![x]\!]$ for all $i = 1, \dots, r$. □

3.6. *Example.* (This example is taken from [HS99, §A.1].) Not every analytic module is topologically complete. For instance, let $R = \mathbb{Z}$ and let p be a prime, let $F = \bigoplus_{n=0}^{\infty} \mathbb{Z}$ be a countably generated free module, and let $F_p^\wedge$ denote its (ordinary) p-adic completion. Elements of $F_p^\wedge$ are sequences $(a_n)_{n\geq 0}$ in $\mathbb{Z}_p$ such that for all $k \geq 0$, we have $a_n \in p^k\mathbb{Z}_p$ for all but finitely many n. Let M be the cokernel

$$F_p^\wedge \xrightarrow{f=(\bigoplus p^n)_p^\wedge} F_p^\wedge \to M \to 0$$

of the map f between topological p-completions of F. The module M is not topologically p-complete, as may be seen from the fact that the element $a = (1, p, p^2, p^3, \dots) \in F_p^\wedge$ is not contained in $\mathrm{Im}(f)$, but is contained in $\mathrm{Im}(f) + p^n F_p^\wedge$ for all n.

On the other hand, M is p-analytic, being the cokernel of a homomorphism between p-analytic modules.

3.7. *Remark.* The theory of Taylor expansions (2.7) shows that an R-module M is p-analytic if and only if the $R[\![x]\!]$-module $M[\![x]\!]$ is topologically $(x - p)$-complete.

3.8. **Convergence in analytic modules.** Let $p \in R$ be an element in a ring, and M an R-module. Then M is topologically p-complete if and only if, for every sequence $(a_k)_{k\geq 1}$ in M such that $a_{k+1} \equiv a_k \mod p^k M$ for all $k \geq 1$, there exists a unique element $a \in M$ such that $a \equiv a_k \mod p^k M$ for all $k \geq 1$. In other words, M is topologically p-complete if the limit $a = \lim_{k\to\infty} a_k$ in the p-adic topology exists and is unique.

Even though p-analytic modules can fail to be topologically p-complete, there is still a notion of "limit" in the analytic setting.

Thus, suppose M is p-analytic, and consider a sequence $(a_k)_{k\geq 1}$ in M such that $a_{k+1} \equiv a_k \mod p^k M$ for $k \geq 1$. For each $k \geq 0$ choose $b_k \in M$ such that

$$a_1 = b_0, \qquad \text{and} \quad a_{k+1} = a_k + p^k b_k \quad \text{for all } k \geq 1.$$

Define

$$f(x) := \sum_{k\geq 0} b_k x^k \in M[\![x]\!]$$

and set $a := f(p)$, the p-evaluation of f at p.

3.9. **Lemma.** *The element a constructed above does not depend on the choice of the sequence (b_k), and has the property that $a \equiv a_k \mod p^k M$ for all $k \geq 0$.*

Proof. Suppose (b_k) and (b'_k) are two sequences such that $a_{k+1} = a_k + p^k b_k = a_k + p^k b'_k$, whence $p^k(b'_k - b_k) = 0$. Let $f = \sum_{k\geq 0} b_k x^k$ and $f' = \sum_{k\geq 0} b'_k x^k$. Then

$$\begin{aligned} f' - f &= \sum_{k\geq 1} (b'_k - b_k)x^k \\ &= \sum_{k\geq 1} (b'_k - b_k)(x^k - p^k) \\ &= (x-p)\sum_{k\geq 1} (b'_k - b_k)(x^{k-1} + \cdots + p^{k-1}) \in (x-p)M[\![x]\!] \end{aligned}$$

It follows that $f'(p) = f(p)$ as desired. For the second claim, note that if we write $f(x) = g(x) + x^k h(x)$ with $g(x) \in M[x]$ a polynomial of degree $< k$, then $f(p) = g(p) + p^k h(p)$ by (3.5)(1), and $g(p) = \sum_{j=0}^{k-1} p^j b_j = a_k$. □

Thus, given a sequence $(a_k)_{k\geq 1}$ in M such that $a_{k+1} \equiv a_k \mod p^k M$, we define its limit to be

$$a = \lim_{k\to\infty} a_k := f(p)$$

where $f = \sum_{k\geq 0} b_k x^k \in M[\![x]\!]$ is *any* power series such that $\sum_{j=0}^{k-1} b_j p^j = a_k$ for all $k \geq 1$.

3.10. *Remark.* Note that we have only defined limits for sequences (a_k) in p-analytic modules with a very strong convergence property. It does not seem to be possible to define such limits for arbitrary Cauchy sequences. Given a strictly monotone function $\gamma\colon \mathbb{N} \to \mathbb{N}$, one can use a version of the above method to define limits for all sequences (a_k) such that $a_i \equiv a_j \mod p^r M$ for all $i, j \geq \gamma(r)$. However, there seems to be no guarantee that the result will not depend on the choice of γ.

We can extend this limit construction to modules which are analytically complete with respect to a finite sequence.

3.11. **Proposition.** *Let $\underline{p} = (p_1, \dots, p_r)$ be a finite sequence, and suppose that M is $\underline{p}$-analytic. Let $J = (\underline{p}) = (p_1, \dots, p_r) \subseteq R$, and suppose $(a_k)_{k\geq 1}$ is a sequence in M such that*

$$a_{k+1} \equiv a_k \mod J^k M \qquad \text{for all } k \geq 1.$$

1. *There exists $f \in M[\![x_1, \dots, x_r]\!]$ such that $f_k(\underline{p}) = a_k$ for all $k \geq 1$, where $f_k \in M[x_1, \dots, x_r]$ is the polynomial consisting of all terms of degree $< k$ in f.*
2. *For any two $f, f' \in M[\![x_1, \dots, x_r]\!]$ as in (1), we have that $f(\underline{p}) = f'(\underline{p})$.*

Therefore we may define

$$a = \lim_{k\to\infty} a_k := f(p) \qquad \text{for any } f \text{ as in (1).}$$

Proof. The proofs are straightforward. For (2) use the fact that for elements $u_1, \dots, u_r,\ v_1, \dots, v_r$ in any commutative ring S, we have that

$$(u_1 \cdots u_r) - (v_1 \cdots v_r) = \sum_{k=1}^{r} (u_k - v_k) w_k \qquad \text{for some } w_1, \dots, w_k \in S.$$

□

4. The analytic completion functor

In this section, we describe, for a *finite* sequence $\underline{p} = (p_1, \dots, p_r)$ of R, a left adjoint to the inclusion functor $\widehat{\mathrm{Mod}}_{\underline{p}} \subseteq \mathrm{Mod}_R$, which we call the **analytic $\underline{p}$-completion functor**. The key properties of this completion functor are that, viewed as an endofunctor of Mod_R, it is idempotent and right exact.

4.1. **Idempotent monads.** Let C be a category. An **idempotent monad** on C is monad (A, η, μ) on C, with the property that $\mu\colon AA \to A$ is a natural isomorphism; it follows (i) that $A\eta\colon A \to AA$ and $\eta A\colon A \to AA$ are natural isomorphisms, and (ii) that $A\eta = \eta A$. Conversely, given a pair (A, η) consisting of an endofunctor $A\colon C \to C$ and a natural transformation $\eta\colon I \to A$, there exists a (necessarily unique) $\mu\colon AA \to A$ such that (A, η, μ) is an idempotent monad, *if and only if* (i) and (ii) hold.

Given an idempotent monad (A, η, μ) on C, let C^A denote the full subcategory of C consisting of all objects M such that $\eta\colon M \to A(M)$ is an isomorphism. It is immmediate that the functor A takes its image in the subcategory C^A, and that the resulting functor $\overline{A}\colon C \to C^A$ is left adjoint to the inclusion functor $C^A \to C$. Conversely, the idempotent monad (A, η, μ) is determined, up to unique natural isomorphism, by the full subcategory C^A. Furthermore, if $f\colon X \to Y$ is a morphism in C, then Af is an isomorphism if and only if for all objects W in C^A, the induced function $\operatorname{Hom}_C(f, \mathrm{id}_W)\colon \operatorname{Hom}_C(X, W) \to \operatorname{Hom}_C(Y, W)$ is a bijection.

4.2. **The p-analytic completion functor.** Let $p \in R$. We define a monad $(\mathcal{A}_p, \eta_p, \mu_p)$ on Mod as follows. Let

$$\mathcal{A}_p(M) \overset{\text{def}}{=} M[\![x]\!]/(x-p)M[\![x]\!] = \operatorname{Cok}\bigl[M[\![x]\!] \xrightarrow{\cdot(x-p)} M[\![x]\!]\bigr].$$

Let $\eta_p\colon M \to \mathcal{A}_p(M)$ be the composite

$$M \xrightarrow{i_M} M[\![x]\!] \xrightarrow{\text{quotient}} \mathcal{A}_p(M).$$

Observe that since $M[\![x_1, x_2]\!] \approx (M[\![x_2]\!])[\![x_1]\!]$, and since $M \mapsto M[\![x]\!]$ is an exact functor, the module $\mathcal{A}_p(\mathcal{A}_p(M))$ is a quotient of a module of the form $M[\![x_1, x_2]\!]$, and in fact we have isomorphisms

$$\begin{aligned}
\mathcal{A}_p(\mathcal{A}_p(M)) &= \frac{\mathcal{A}_p(M)[\![x_1]\!]}{(x_1 - p)\mathcal{A}_p(M)[\![x_1]\!]} \\
&= \frac{(M[\![x_2]\!]/(x_2 - p)M[\![x_2]\!])[\![x_1]\!]}{(x_1 - p)(M[\![x_2]\!]/(x_2 - p)M[\![x_2]\!])[\![x_1]\!]} \\
&\approx M[\![x_1, x_2]\!]/(x_1 - p, x_2 - p)M[\![x_1, x_2]\!].
\end{aligned}$$

Let $\mu_p\colon \mathcal{A}_p\mathcal{A}_p(M) \to \mathcal{A}_p(M)$ be the map

$$M[\![x_1, x_2]\!]/(x_1 - p, x_2 - p)M[\![x_1, x_2]\!] \to M[\![x]\!]/(x-p)M[\![x]\!]$$

which sends $x_1 \mapsto x$ and $x_2 \mapsto x$. It is straightforward to check that $(\mathcal{A}_p, \eta_p, \mu_p)$ is a monad.

4.3. **Proposition.** *The monad $(\mathcal{A}_p, \eta_p, \mu_p)$ is an idempotent monad. An object M in* Mod *is p-analytic if and only if $\eta_p\colon M \to \mathcal{A}_p(M)$ is an isomorphism.*

Proof. To show that μ_p is an isomorphism, note that we may factor it as

$$\begin{tikzcd}
M[\![x_1, x_2]\!]/(x_1 - p, x_2 - p)M[\![x_1, x_2]\!] \arrow[d, "\substack{x_1 \mapsto x \\ x_2 \mapsto x + y}"'] \arrow[dr, "\mu_p"] & \\
M[\![x, y]\!]/(x - p, y)M[\![x, y]\!] \arrow[r, "\substack{x \mapsto x \\ y \mapsto 0}"'] & M[\![x]\!]/(x-p)M[\![x]\!]
\end{tikzcd}$$

The bottom horizontal map is clearly an isomorphism, and the left vertical map is also seen to be an isomorphism, using the inverse transformation $x \mapsto x_1$, $y \mapsto x_2 - x_1$.

Now let M be an R-module. Given $f \in M[\![x]\!]$, let $\bar{f}$ denote the image of f in the quotient $\mathcal{A}_p(M)$, and let $\eta_p^{-1}(\bar{f}) \subseteq M$ be the preimage of $\bar{f}$ in M. The set $\eta_p^{-1}(\bar{f})$ consists precisely of elements $c \in M$ such that $f - c \in (x-p)M[\![x]\!]$, and thus η_p is an isomorphism if and only if M is p-analytic. $\square$

As a consequence, the functor $\mathcal{A}_p$ takes its image in $\widehat{\mathrm{Mod}}_p$, so the resulting functor $\overline{\mathcal{A}}_p\colon \mathrm{Mod} \to \widehat{\mathrm{Mod}}_p$ is left adjoint to the inclusion functor $\widehat{\mathrm{Mod}}_p \to \mathrm{Mod}$.

Finally we note that analytic completion is R-linear.

4.4. **Proposition.** *The map* $\mathrm{Hom}_R(M,N) \to \mathrm{Mod}_R(\mathcal{A}_p(M), \mathcal{A}_p(N))$ *induced by the functor* $\mathcal{A}_p$ *is a map of* R*-modules.*

Proof. Straightforward from the definition of $\mathcal{A}_p$. $\square$

4.5. **Analytic completion with respect to sequences.** Given a *finite* sequence $\underline{p} = (p_1, \dots, p_r)$ of elements of R, we define a functor $\mathcal{A}_{\underline{p}}\colon \mathrm{Mod} \to \mathrm{Mod}$, and a natural map $\eta_{\underline{p}}\colon M \to \mathcal{A}_{\underline{p}}(M)$, as follows. We set

$$\mathcal{A}_{\underline{p}}(M) \overset{\text{def}}{=} M[\![x_1, \dots, x_r]\!]/(x_1 - p_1, \dots, x_r - p_r)M[\![x_1, \dots, x_r]\!],$$

and we let $\eta_{\underline{p}}$ be the composite

$$M \xrightarrow{i_M} M[\![x_1, \dots, x_r]\!] \xrightarrow{\text{quotient}} \mathcal{A}_{\underline{p}}(M).$$

Observe that if $\underline{p} = (p_1, \dots, p_r)$ and $\underline{q} = (q_1, \dots, q_s)$ are two sequences, then there is an evident natural isomorphism

$$\phi_{\underline{p},\underline{q}}\colon \mathcal{A}_{\underline{p},\underline{q}} \to \mathcal{A}_{\underline{p}}\mathcal{A}_{\underline{q}},$$

where $\underline{p},\underline{q}$ is the concatenated sequence $(p_1, \dots, p_r, q_1, \dots, q_s)$. This is just the evident isomorphism

$$M[\![x_i, y_j]\!]/(x_i - p_i, y_j - q_j)M[\![x_i, y_j]\!] \approx \frac{(M[\![y_j]\!]/(y_j - q_j)M[\![y_j]\!])[\![x_i]\!]}{(x_i - p_i)(M[\![y_j]\!]/(y_j - q_j)M[\![y_j]\!])[\![x_i]\!]}.$$

Furthermore, we have the following.

4.6. **Proposition.** *If* $\underline{p}$ *and* $\underline{q}$ *are finite sequences in* R*, then the diagram*

$$\begin{tikzcd}
M \arrow[rr, "\eta_{\underline{p}}"] \arrow[dr, "\eta_{\underline{pq}}"] \arrow[dd, "\eta_{\underline{q}}"'] & & \mathcal{A}_{\underline{p}}(M) \arrow[dd, "\mathcal{A}_{\underline{p}}(\eta_{\underline{q}})"] \\
& \mathcal{A}_{\underline{p},\underline{q}}(M) \arrow[dr, "\phi_{\underline{p},\underline{q}}"] & \\
\mathcal{A}_{\underline{q}}(M) \arrow[rr, "\eta_{\underline{p}}"'] & & \mathcal{A}_{\underline{p}}\mathcal{A}_{\underline{q}}(M)
\end{tikzcd}$$

commutes.

Proof. Straightforward. $\square$

Using the evident isomorphism

$$\mathcal{A}_{\underline{p}}\mathcal{A}_{\underline{p}}(M) \approx M[\![y_i, z_i]\!]/(y_i - p_i, z_i - p_i)M[\![y_i, z_i]\!]$$

we define $\mu_{\underline{p}}\colon \mathcal{A}_{\underline{p}}\mathcal{A}_{\underline{p}}(M) \to \mathcal{A}_{\underline{p}}$ by sending $y_i \mapsto x_i$ and $z_i \mapsto x_i$.

4.7. **Lemma.** *If* $\underline{p} = (p_1, \dots, p_r)$ *is a sequence in* R*, and* $1 \le k \le r$*, then* $\eta_{p_k}\colon \mathcal{A}_{\underline{p}}(M) \to \mathcal{A}_{p_k}\mathcal{A}_{\underline{p}}(M)$ *and* $\mathcal{A}_{\underline{p}}(\eta_{p_k})\colon \mathcal{A}_{\underline{p}}(M) \to \mathcal{A}_{\underline{p}}\mathcal{A}_{p_k}(M)$ *are isomorphisms for all* M*.*

Proof. Note that we can reorder the elements of the sequence $\underline{p}$ without changing the functor $\mathcal{A}_{\underline{p}}$ or its coaugmentation. Thus, the statement can be reduced to the case that $r = 1$, which is (4.3). □

4.8. **Proposition.** *Let $\underline{p}$ be a finite sequence in R. The data $(\mathcal{A}_{\underline{p}}, \eta_{\underline{p}}, \mu_{\underline{p}})$ is an idempotent monad on* Mod*. An object M of* Mod *is $\underline{p}$-analytic if and only if $\eta_{\underline{p}} \colon M \to \mathcal{A}_{\underline{p}}(M)$ is an isomorphism. For a map $f \colon M \to N$ of R-modules, $\mathcal{A}_{\underline{p}}(f)$ is an isomorphism if and only if* $\mathrm{Hom}_R(f, \mathrm{id}_C) \colon \mathrm{Hom}_R(N, C) \to \mathrm{Hom}_R(M, C)$ *is an isomorphism for all $\underline{p}$-analytic modules C.*

Proof. That this data determines a monad is proved just as for sequences of length 1. That it is idempotent is a consequence of the above lemma (4.7). □

As a consequence, the functor $\mathcal{A}_{\underline{p}}$ takes its image in $\widehat{\mathrm{Mod}}_{\underline{p}}$, and the resulting functor $\overline{\mathcal{A}}_{\underline{p}} \colon \mathrm{Mod} \to \widehat{\mathrm{Mod}}_{\underline{p}}$ is left adjoint to the inclusion functor $\widehat{\mathrm{Mod}}_{\underline{p}} \to \mathrm{Mod}$.

4.9. **Proposition.** *Let $\underline{p}$ be a finite sequence in R. The map* $\mathrm{Hom}_R(M, N) \to \mathrm{Mod}_R(\mathcal{A}_{\underline{p}}(M), \mathcal{A}_{\underline{p}}(N))$ *induced by the functor $\mathcal{A}_{\underline{p}}$ is a map of R-modules.*

4.10. **Exactness properties of analytic completion.**

4.11. **Proposition.** *For any finite sequence $\underline{p}$ of R, the functor $\mathcal{A}_{\underline{p}} \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is right exact.*

Proof. This is immediate from the construction of $\mathcal{A}_{\underline{p}}$. Alternately, use the facts that the completion functor $\mathrm{Mod}_R \to \widehat{\mathrm{Mod}}_{\underline{p}}$ is a left adjoint, and that the inclusion functor $\widehat{\mathrm{Mod}}_{\underline{p}} \to \mathrm{Mod}_R$ is an exact functor (3.1). □

We will see later that $\mathcal{A}_{\underline{p}}$ is not generally left exact.

4.12. **Proposition.** *For any finite sequence $\underline{p}$ of R, the functor $\mathcal{A}_{\underline{p}} \colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ commutes with arbitrary products.*

Proof. Immediate from the construction of $\mathcal{A}_{\underline{p}}$, and the fact that taking products in Mod_R is exact. □

4.13. **Comparison with topological completion.** Given $p \in R$, consider the map $\tilde{\gamma} \colon M[\![x]\!] \to M_p^\wedge = \lim_k M/p^n M$ induced by maps $M[\![x]\!] \to M/p^n M$ which send $\sum a_k x^k \mapsto \sum_{k=0}^{n-1} a_k p^k$. It is clear that $\tilde{\gamma}((x-p)M[\![x]\!]) = 0$, and thus we obtain a **comparison map**

$$\gamma_p \colon \mathcal{A}_p(M) \to M_p^\wedge$$

with the property that $\gamma_p \circ \eta_p \colon M \to M_p^\wedge$ coincides with the usual map of M to its p-adic completion.

4.14. **Proposition.** *For all M, the comparison map $\gamma_p \colon \mathcal{A}_p(M) \to M_p^\wedge$ is surjective.*

Proof. Given $m \in M_p^\wedge$ represented by a sequence of elements $m_n \in M$ for $n \geq 1$ such that $m_n \equiv m_{n+1} \mod p^n M$, we must produce $f = \sum a_k x^k \in M[\![x]\!]$ such that $\tilde{\gamma}(f) = m$. In fact, by hypothesis for every $n \geq 1$ we can choose $a_n \in R$ such that $p^n a_n = m_{n+1} - m_n$. Taking $a_0 = m_1$, we see that $\sum_{k=0}^{n-1} a_k p^k = m_n$ for all $n \geq 1$, and thus we may take $f = \sum a_k x^k$. □

We have already noted (3.6) that not every p-analytic module is topologically p-complete, and thus γ_p need not be an isomorphism.

For a finite sequence $\underline{p} = (p_1, \dots, p_r)$, we may likewise define a comparison map $\gamma_{\underline{p}} \colon \mathcal{A}_{\underline{p}}(M) \to M^\wedge_{\underline{p}}$ inductively, by

$$\mathcal{A}_{\underline{p}}(M) \approx \mathcal{A}_{p_r}(\mathcal{A}_{p_1,\dots,p_{r-1}}(M)) \xrightarrow{\mathcal{A}_{p_r}(\gamma_{p_1,\dots,p_{r-1}})} \mathcal{A}_{p_r}(M^\wedge_{p_1,\dots,p_{r-1}}) \xrightarrow{\gamma_{p_r}} (M^\wedge_{p_1,\dots,p_{r-1}})^\wedge_{p^r} \approx M^\wedge_{\underline{p}}.$$

Since $\mathcal{A}_{p_r}$ is right exact, we have the following.

4.15. **Proposition.** *For all M, the comparison map $\gamma_{\underline{p}} \colon \mathcal{A}_{\underline{p}}(M) \to M^\wedge_{\underline{p}}$ is surjective.*

5. Nakayama lemma

In this section, we prove a kind of Nakayama lemma, saying that $\mathcal{A}_{\underline{p}}(M) \approx 0$ if and only if $M \otimes_R R/(\underline{p}) \approx 0$.

5.1. **Lemma.** *Let M be an R-module, and $\underline{p} = (p_1, \dots, p_r)$ a sequence of elements of R. Then the map $\eta_{\underline{p}} \colon M \to \mathcal{A}_{\underline{p}}(M)$ descends to an isomorphism*

$$M/(p_1, \dots, p_r)M \xrightarrow{\sim} \mathcal{A}_{\underline{p}}(M)/(p_1, \dots, p_r)\mathcal{A}_{\underline{p}}(M).$$

In particular, it follows that $M/(p_1, \dots, p_r)M$ is $\underline{p}$-analytic.

Proof. Immediate from the fact that $\mathcal{A}_{\underline{p}} \colon \mathrm{Mod} \to \mathrm{Mod}$ is right exact (4.11) and R-linear (4.9). □

5.2. **Lemma.** *Let M be an R-module, and $p \in R$. Then $\mathcal{A}_p(M) \approx 0$ if and only if $M/pM \approx 0$.*

Proof. Applying the right-exact and R-linear functor $\mathcal{A}_p$ to the sequence $M \xrightarrow{\cdot p} M \to M/pM \to 0$ gives an exact sequence

$$\mathcal{A}_p(M) \xrightarrow{\cdot p} \mathcal{A}_p(M) \to \mathcal{A}_p(M/pM) \to 0,$$

so $\mathcal{A}_p(M/pM) \approx \mathcal{A}_p(M)/p\mathcal{A}_p(M)$. It is clear that $\eta_p \colon M/pM \to \mathcal{A}_p(M/pM) = (M/pM)[\![x]\!]/(x-p)(M/pM)[\![x]\!]$ is an isomorphism. Thus, $\mathcal{A}_p(M) \approx 0$ implies $M/pM \approx 0$.

Now suppose that $M/pM \approx 0$, so that $\cdot p \colon M \to M$ is surjective. Then $\cdot(x-p) \colon M[\![x]\!] \to M[\![x]\!]$ is surjective, since given $f = \sum a_k x^k \in M[\![x]\!]$ there exists a sequence $b_k \in M$ such that

$$pb_0 = -a_0 \quad \text{and} \quad pb_k = -a_k + b_{k-1} \quad \text{for} \quad k > 0.$$

Then $f = (x-p)\sum b_k x^k$. Thus $M/pM \approx 0$ implies $\mathcal{A}_p(M) \approx 0$. □

5.3. **Proposition.** *Let $\underline{p} = (p_1, \dots, p_r)$ be a sequence of elements in R, and let M be an R-module. Then $\mathcal{A}_{\underline{p}}(M) \approx 0$ if and only if $M/(p_1, \dots, p_r)M \approx 0$.*

Proof. For the only if direction, if $\mathcal{A}_{\underline{p}}(M) \approx 0$, then

$$M/(p_1, \dots, p_r)M \approx \mathcal{A}_{\underline{p}}(M)/(p_1, \dots, p_r)\mathcal{A}_{\underline{p}}(M) \approx 0.$$

For the if direction, we use induction on r, having proved the case $r = 1$ above. Suppose that $M/(p_1, \dots, p_r) \approx 0$. Thus, $N = M/p_1M$ is such that $N/(p_2, \dots, p_r)N \approx 0$, and therefore by the inductive hypothesis $\mathcal{A}_{p_2,\dots,p_r}(N) \approx 0$. Since $\mathcal{A}_{p_2,\dots,p_r}(N) \approx \mathcal{A}_{p_2,\dots,p_r}(M)/p_1\mathcal{A}_{p_2,\dots,p_r}(M)$, we conclude that

$$\mathcal{A}_{\underline{p}}(M) \approx \mathcal{A}_{p_1}\mathcal{A}_{p_2,\dots,p_r}(M) \approx 0$$

as desired. □

6. Hom and tensor

6.1. **Proposition.** *Let $\underline{p}$ be a set of elements of R. If M, N are R-modules and N is $\underline{p}$-analytic, then $\mathrm{Hom}_R(M,N)$ is $\underline{p}$-analytic.*

Proof. Choose a free presentation $\bigoplus_J R \to \bigoplus_I R \to M \to 0$ of M. Then there is an exact sequence of R-modules,

$$0 \to \mathrm{Hom}_R(M,N) \to \prod_I N \to \prod_J N$$

and since $\widehat{\mathrm{Mod}}_{\underline{p}}$ is closed under products and kernels in Mod, it follows that $\mathrm{Hom}_R(M,N)$ is $\underline{p}$-analytic. □

6.2. **Proposition.** *If M and N are R-modules, then for any finite sequence $\underline{p}$ in R, the maps*

$$\mathcal{A}_{\underline{p}}(M \otimes_R N) \xrightarrow{\mathcal{A}_{\underline{p}}(\eta_{\underline{p}} \otimes \mathrm{id})} \mathcal{A}_{\underline{p}}(\mathcal{A}_{\underline{p}}(M) \otimes_R N), \qquad \mathcal{A}_{\underline{p}}(M \otimes_R N) \xrightarrow{\mathcal{A}_{\underline{p}}(\mathrm{id} \otimes \eta_{\underline{p}})} \mathcal{A}_{\underline{p}}(M \otimes_R \mathcal{A}_{\underline{p}}(N))$$

are isomorphisms.

Proof. We do the first case. By (4.8), it suffices to show for every $\underline{p}$-analytic module C that

$$\mathrm{Hom}_R(\eta_{\underline{p}} \otimes N, C) \colon \mathrm{Hom}_R(\mathcal{A}_{\underline{p}}(M) \otimes_R N, C) \to \mathrm{Hom}_R(M \otimes_R, C)$$

is an isomorphism. This is clear, since the map is isomorphic to

$$\mathrm{Hom}_R(\eta_{\underline{p}}, \mathrm{Hom}_R(N,C)) \colon \mathrm{Hom}_R(\mathcal{A}_{\underline{p}}(M), \mathrm{Hom}_R(N,C)) \to \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,C))$$

and $\mathrm{Hom}_R(N,C)$ is $\underline{p}$-analytic by (6.1). □

Thus for a finite sequence $\underline{p}$ in R we can use the left adjoint $\overline{\mathcal{A}}_{\underline{p}} \colon \mathrm{Mod}_R \to \widehat{\mathrm{Mod}}_{\underline{p}}$ to inclusion to define a "$\underline{p}$-analytically completed tensor product"

$$M, N \mapsto M \widehat{\otimes} N := \overline{\mathcal{A}}_{\underline{p}}(M \otimes_R N) \colon \widehat{\mathrm{Mod}}_{\underline{p}} \times \widehat{\mathrm{Mod}}_{\underline{p}} \to \widehat{\mathrm{Mod}}_{\underline{p}}$$

on the category of analytically $\underline{p}$-complete modules. A straightforward argument using (6.2) shows that $\widehat{\otimes}$ is part of a *symmetric monoidal structure* on $\widehat{\mathrm{Mod}}_{\underline{p}}$, with unit object $\overline{\mathcal{A}}_{\underline{p}}(R)$. Furthermore, this symmetric monoidal structure is *closed*, since $-\widehat{\otimes} M \colon \widehat{\mathrm{Mod}}_{\underline{p}} \to \widehat{\mathrm{Mod}}_{\underline{p}}$ is left adjoint to $\mathrm{Hom}_R(M,-) \colon \widehat{\mathrm{Mod}}_{\underline{p}} \to \widehat{\mathrm{Mod}}_{\underline{p}}$ using (6.1) and (4.8). Finally, note that $\overline{\mathcal{A}}_{\underline{p}}$ is a strongly symmetric monoidal functor, essentially by construction.

7. The analyticity ideal

We have defined the notion of analytic completeness with respect to a sequence (or subset) $\underline{p}$ of elements of R. In this section we observe that this notion in fact depends only on the radical of the ideal generated by $\underline{p}$.

Suppose M is an R-module. Let I_M denote the set of $p \in R$ such that M is p-analytic. By the following proposition, I_M is an ideal of R, called the **analyticity ideal** of M.

7.1. **Proposition.** *Let M be an R-module.*

(1) *M is 0-analytic.*
(2) *If $a, p \in R$, and M is p-analytic, then M is ap-analytic.*
(3) *If $p, q \in R$, and M is both p-analytic and q-analytic, then M is $(p+q)$-analytic.*

Proof.

(1) Immediate, since $\eta_0 \colon M \to M[\![t]\!]/tM[\![t]\!]$ is always an isomorphism.

(2) Consider the diagram

$$\begin{tikzcd}[column sep=small] & M[\![z]\!]/(z-p)M[\![z]\!] \arrow[dl, "z\mapsto z"'] \arrow[dr, "z\mapsto z"] & \\ M[\![x,z]\!]/(x,z-p)M[\![x,z]\!] \arrow[rr, "{x\mapsto y-az},\ {z\mapsto z}"'] & & M[\![y,z]\!]/(y-ap,z-p)M[\![y,z]\!] \end{tikzcd}$$

of R-module maps. The left-hand vertical map and the bottom horizontal map are isomorphisms, and therefore the right-hand vertical map is an isomorphism. If M is p-analytic, then $\eta_p\colon M \to M[\![z]\!]/(z-p)M[\![z]\!]$ is an isomorphism, and using this the right-hand vertical map $\mathcal{A}_p(M) \to \mathcal{A}_{ap}(\mathcal{A}_p(M))$ can be identified with $\eta_{ap}\colon M \to \mathcal{A}_{ap}(M)$, whence M is ap-analytic since this map is iso.

(3) Consider the diagram

$$\begin{tikzcd}[column sep=small] & M[\![z]\!]/(z-q)M[\![z]\!] \arrow[dl, "z\mapsto z"'] \arrow[dr, "z\mapsto z"] & \\ M[\![x,z]\!]/(x-p,z-q)M[\![x,z]\!] \arrow[rr, "{x\mapsto y-z},\ {z\mapsto z}"'] & & M[\![y,z]\!]/(y-(p+q),z-q)M[\![y,z]\!] \end{tikzcd}$$

of R-module maps. The bottom horizontal map is an isomorphism. The left-hand vertical map may be identified with $\mathcal{A}_q(\eta_p)\colon \mathcal{A}_q(M) \to \mathcal{A}_q\mathcal{A}_p(M)$, and thus is an isomorphism since M is p-analytic. Therefore, the right-hand vertical map is an isomorphism. Since M is q-analytic, the map $\eta_q\colon M \to M[\![z]\!]/(z-q)M[\![z]\!]$ is an isomorphism, and so the right-hand vertical map $\mathcal{A}_q(M) \to \mathcal{A}_{p+q}(\mathcal{A}_q(M))$ can be identified with $\eta_{p+q}\colon M \to \mathcal{A}_{p+q}(M)$, whence M is $(p+q)$-analytic since this map is iso.

$\square$

7.2. **Proposition.** *For any $p \in R$ and $n \geq 1$, there is an isomorphism $\psi\colon \mathcal{A}_{p^n} \to \mathcal{A}_p$ of functors, such that $\psi \circ \eta_{p^n} = \eta_p$.*

7.3. **Corollary.** *If M is an R-module, $p \in R$, and $n \geq 1$, then M is p-analytic if and only if it is p^n-analytic. That is, the analyticity ideal I_M is radical.*

Proof. We define explicit natural homomorphisms of R-modules $\psi\colon \mathcal{A}_{p^n} \to \mathcal{A}_p$ and $\phi\colon \mathcal{A}_p \to \mathcal{A}_{p^n}$, and we show (i) that $\psi \circ \eta_{p^n} = \eta_p$, and (ii) that ψ and ϕ are inverse to each other.

We let $\psi\colon M[\![y]\!]/(y-p^n)M[\![y]\!] \to M[\![x]\!]/(x-p)M[\![x]\!]$ be the map induced after passing to quotients by the R-module homomorphism

$$\begin{aligned} M[\![y]\!] &\to M[\![x]\!] \\ g(y) &\mapsto g(x^n). \end{aligned}$$

To see that this is well-defined, it suffices to observe that any element which is of the form $(y-p^n)g(y) \in M[\![y]\!]$ is sent to $(x^n-p^n)g(x^n)$, which is contained in $(x-p)M[\![x]\!]$.

It is straightforward to verify (i).

We let $\phi\colon M[\![x]\!]/(x-p)M[\![x]\!] \to M[\![y]\!]/(y-p^n)M[\![y]\!]$ be the map induced after to passing to quotients by the R-module homomorphism

$$\begin{aligned} M[\![x]\!] &\to M[\![y]\!] \\ f = \sum_{k=0}^{n-1} x^k\, f_k(x^n) &\mapsto \sum_{k=0}^{n-1} p^k\, f_k(y). \end{aligned}$$

Here the f_k are formal series in one variable with coefficients in M. To see that this is well-defined, it suffices to observe that any element which is of the form $(x-p)f \in M[\![x]\!]$ is sent to an element of $(y-p^n)M[\![y]\!]$. In fact, if we write $f = \sum_{k=0}^{n-1} x^k f_k(x^n)$, then

$$\begin{aligned}(x-p)f &= (x-p)\sum_{k=0}^{n-1} x^k f_k(x^n)\\ &= [x^n f_{n-1}(x^n) - pf_0(x^n)] + \sum_{k=1}^{n-1} x^k[f_{k-1}(x^n) - pf_k(x^n)]\\ &\mapsto [yf_{n-1}(y) - pf_0(y)] + \sum_{k=1}^{n-1} p^k[f_{k-1}(y) - pf_k(y)]\\ &= (y-p^n)f_{n-1}(y).\end{aligned}$$

Finally, it is straightforward to verify (ii); i.e., that $\phi\circ\psi = \mathrm{id}$ and $\psi\circ\phi = \mathrm{id}$. □

7.4. **Corollary.** *Let $\underline{p} = (p_1,\dots,p_r)$ and $\underline{q} = (q_1,\dots,q_s)$ be two finite sequences in R. If the ideals generated by $\underline{p}$ and $\underline{q}$ have the same radical, then $\widehat{\mathrm{Mod}}_{\underline{p}} = \widehat{\mathrm{Mod}}_{\underline{q}}$.*

8. Tame modules and the Koszul complex

8.1. **Koszul complex.** Let $p \in R$. For an R-module M, write $\mathcal{K}_\bullet(M,p)$ for the chain complex given by

$$\mathcal{K}_0(M,p) = \mathcal{K}_1(M,p) = M[\![x]\!], \qquad d(f) = (x-p)f \quad \text{for } f \in \mathcal{K}_1(M,p).$$

It is immediate that $H_0\mathcal{K}_\bullet(M,p) \approx \mathcal{A}_p(M)$.

More generally, for a finite sequence $\underline{p} = (p_1,\dots,p_r)$ in R, we define a chain complex $\mathcal{K}_\bullet(M,\underline{p})$ by

$$\mathcal{K}_m(M,\underline{p}) = \bigoplus_{\substack{I\subseteq\{1,\dots,n\}\\ \#I=m}} M[\![x_1,\dots,x_r]\!]\cdot e_I,$$

where e_I is a formal symbol to keep track of summands, and

$$d(f\cdot e_{i_1,\dots,i_m}) = \sum_{j=1}^{m}(-1)^{j-1}(x_{i_j} - p_{i_j})f\cdot e_{i_1\dots\widehat{i_j}\dots i_m}.$$

We call $\mathcal{K}_\bullet(M,\underline{p})$ the **Koszul complex** of M with respect to $\underline{p}$.[1] Note that though the definition of the Koszul complex depends on the order of the sequence $\underline{p}$, reordering the sequence $\underline{p}$ gives an isomorphic complex.

We write

$$K_m(M,\underline{p}) \overset{\text{def}}{=} H_m(\mathcal{K}_\bullet(M,\underline{p}))$$

for the homology of $\mathcal{K}_\bullet(M,\underline{p})$, the **koszul homology**. It is immediate that $K_0(M,\underline{p}) \approx \mathcal{A}_{\underline{p}}(M)$. It is straightforward to show that there is an isomorphism of complexes

$$\mathcal{K}_\bullet(M,\underline{p}) \approx \operatorname{Tot}\mathcal{K}_\bullet\big(\mathcal{K}_\bullet(M,(p_2,\dots,p_r)),p_1\big).$$

Note that for each m the functor $M \mapsto \mathcal{K}_m(M,\underline{p})$ is exact (since $M \mapsto M[\![x_1,\dots,x_r]\!]$ is exact). As a consequence, given a short exact sequence $0\to M'\to M\to M''\to 0$ of R-modules, we obtain a long exact sequence

$$\cdots \to K_{q+1}(M'',\underline{p}) \to K_q(M',\underline{p}) \to K_q(M,\underline{p}) \to K_q(M'',\underline{p}) \to \cdots. \tag{8.2}$$

[1] This is not really a very good name. Although the complex we are using is vaguely similar to a Koszul complex, it is not at all the same thing.

More generally, given a chain complex $C_\bullet$ of R-modules, we obtain two spectral sequences associated to the double complex $\mathcal{K}_\bullet(C_\bullet, \underline{p})$, taking the forms

$$ {}^{\mathrm{I}}E^1_{ij} = K_i(C_j, \underline{p}) \Longrightarrow H_{i+j}(\operatorname{Tot}\mathcal{K}_\bullet(C_\bullet, \underline{p})) \tag{8.3}$$

$$ {}^{\mathrm{II}}E^2_{ij} = K_j(H_iC_\bullet, \underline{p}) \Longrightarrow H_{i+j}(\operatorname{Tot}\mathcal{K}_\bullet(C_\bullet, \underline{p})). \tag{8.4}$$

In particular, the second spectral sequence applied to the double complex $\mathcal{K}_\bullet(\mathcal{K}_\bullet(M, (p_2, \dots, p_r)), p_1)$ gives a **composite koszul homology spectral sequence** of the form

$$ {}^{\mathrm{II}}E^2_{ij} = K_j\big(K_i(M, (p_2, \dots, p_r)), p_1\big) \Longrightarrow K_{i+j}(M, \underline{p}). \tag{8.5}$$

8.6. **Tameness.** Fix a finite sequence $\underline{p} = (p_1, \dots, p_r)$ in R. We say that a module M is **tame** with respect to $\underline{p}$ (or $\underline{p}$**-tame**) if $K_q(M, \underline{p}) = 0$ for $q \neq 0$. Note that tameness does not depend on the order of $\underline{p}$, since reordering the sequence gives isomorphic koszul homology.

For sequences of length one, tameness is easy to characterize.

8.7. **Lemma.** *Let $p \in R$. Then an R-module M is p-tame if and only if $\operatorname{Hom}_R(R/p^\infty, M) \approx 0$, where $R/p^\infty = \operatorname{colim}_k R/p^kR$ is the direct limit in* Mod *along the sequence $R/p^k \xrightarrow{p\cdot} R/p^{k+1}$. In particular, if multiplication by p is injective on M, then M is p-tame.*

Proof. Suppose $f = \sum a_k x^k \in M[\![x]\!]$ is such that $(x-p)f = 0$. Then $pa_0 = 0$ and $pa_{k+1} = a_k$ for $k \geq 0$. The sequence of elements (a_k) fit together to produce a map $R/p^\infty \to M$, and in fact this construction defines a bijection

$$K_1(M, p) = \operatorname{Ker}\big[(x-p)\cdot\colon M[\![x]\!] \to M[\![x]\!]\big] \xrightarrow{\sim} \operatorname{Hom}(R/p^\infty, M).$$

$\square$

8.8. **Corollary.** *If multiplication by p is an isomorphism on M, then $K_j(M, p) \approx 0$ for all j.*

Proof. By (5.2), $M/pM \approx 0$ implies $K_0(M, p) = \mathcal{A}_p(M) \approx 0$, while (8.7) gives $K_1(M, p) \approx 0$. $\square$

It is more difficult to characterize tame modules for sequences with more than one element. The following is useful for identifying tame modules.

Recall that a module M is **regular** with respect to $\underline{p}$ if each sequence

$$0 \to M/(p_1, \dots, p_{k-1})M \xrightarrow{\cdot p_k} M/(p_1, \dots, p_{k-1})M \to M/(p_1, \dots, p_k)$$

is exact for $k = 1, \dots, r$.

8.9. **Proposition.** *Let $\underline{p} = (p_1, \dots, p_r)$. If M is p_1-regular and if M/p_1M is $\underline{q} = (p_2, \dots, p_r)$ tame, then M is $\underline{p}$-tame.*

Proof. If we apply $K_\bullet(-, \underline{q})$ to the exact sequence $0 \to M \xrightarrow{p_1} M \to M/p_1M \to 0$, the long exact sequence (8.2) and the hypothesis that $K_i(M/p_1M, \underline{q}) \approx 0$ for $i \geq 1$ implies that

$$K_i(M, \underline{q}) \xrightarrow{p_1} K_i(M, \underline{q})$$

is an isomorphism for all $i \geq 1$, and is injective for $i = 0$, so that $K_0(M, \underline{q}) \approx \mathcal{A}_{\underline{q}}(M)$ is p_1-regular. Now feed this into the spectral sequence (8.5)

$${}^{\mathrm{II}}E^2_{ij} = K_j(K_i(M, \underline{q}), p_1) \Longrightarrow K_{i+j}(M, \underline{p}).$$

Using (8.8), we see that ${}^{\mathrm{II}}E^2_{ij} \approx 0$ for $(i, j) \neq (0, 0)$, whence $K_m(M, \underline{p}) \approx 0$ for $m \geq 1$. $\square$

8.10. **Proposition.** *If M is regular with respect to a sequence $\underline{p}$, then it is also tame with respect to $\underline{p}$.*

Proof. The case of a sequence of length one is (8.7). For sequences of length $r \geq 2$, the result follows using induction on r, since if M is $\underline{p}$-regular, then M/p_1M is $\underline{q} = (p_2, \dots, p_r)$-regular, and hence $\underline{q}$-tame by induction, and we can therefore apply (8.9). □

Tameness is a significantly weaker property than regularity. For instance, any analytic module is tame.

8.11. **Proposition.** *If M is $\underline{p}$-analytic, then it is $\underline{p}$-tame.*

Proof. For sequences of length one, this is (2.2). For sequences of length $r \geq 2$, let $\underline{q} = (p_2, \dots, p_r)$. If M is $\underline{p}$-analytic, then it is necessarily $\underline{q}$-analytic, and thus by induction it is $\underline{q}$-tame. Thus

$$K_i(M, \underline{q}) \approx 0 \quad \text{if } i \geq 0, \qquad K_0(M, \underline{q}) \approx \mathcal{A}_{\underline{q}} M \approx M.$$

The spectral sequence (8.5) ${}^{\mathrm{II}}E^2_{ij} = K_j(K_i(M, \underline{q}), p_1) \implies K_{i+j}(M, \underline{p})$ thus collapses to $K_m(M, \underline{p}) \approx K_m(M, p_1)$, which vanish for $m \geq 1$ because M is p_1-analytic and therefore p_1-tame by (2.2). □

9. Tame sequences and derived functors of analytic completion

9.1. **Tame sequences.** We say that a sequence $\underline{p} = (p_1, \dots, p_r)$ in R is **tame** if every free R-module is tame. By (8.10), every *regular* finite sequence in R is tame, since such a sequence will be regular on any free R-module.

9.2. *Warning.* As we have defined it, "$\underline{p}$ is tame in the ring R" does not mean the same thing as "R is a tame module with respect to $\underline{p}$".

There are many tame sequences which are not regular. For instance, any sequence $\underline{p} = (p_1, \dots, p_r)$ of nilpotent elements in R is tame, because every module is $\underline{p}$-analytic by (7.3).

9.3. *Remark.* If the sequence $\underline{p} = (p)$ has length one, then it is not hard to see that the sequence is tame if and only if R, the free module on one generator, is tame, i.e., if and only if $\mathrm{Hom}_R(R/p^\infty, R) = 0$ by (8.7). To prove this, note that (8.7) implies that any submodule of a p-tame module is p-tame, and that a free module is a submodule of a *product* of copies of R, which will necessarily be tame if R is. This argument does not seem generalize to longer sequences.

9.4. **Derived functors of analytic completion.** Write $\mathbf{L}_i\mathcal{A}_{\underline{p}} \colon \mathrm{Mod} \to \mathrm{Mod}$ for the ith left derived functor of $\mathcal{A}_{\underline{p}} \colon \mathrm{Mod} \to \mathrm{Mod}$. Because the inclusion functor $\widehat{\mathrm{Mod}}_{\underline{p}} \to \mathrm{Mod}$ is exact, these coincide with the left derived functors of the analytic completion functor $\overline{\mathcal{A}}_{\underline{p}} \colon \mathrm{Mod} \to \widehat{\mathrm{Mod}}_{\underline{p}}$. In particular, $\mathbf{L}_i\mathcal{A}_{\underline{p}}(M)$ is $\underline{p}$-analytic for all M and i. The basic observation is that if the sequence $\underline{p}$ is tame, then the derived functors of $\mathcal{A}_{\underline{p}}$ coincide with the homology of the complex $\mathcal{K}_\bullet(-, \underline{p})$.

9.5. **Proposition.** *Suppose $\underline{p}$ is a tame sequence in R. Then for any R-module M, we have that*

$$\mathbf{L}_j\mathcal{A}_{\underline{p}}(M) \approx K_j(M, \underline{p}).$$

Thus, if $\underline{p}$ is a tame sequence in R, then M is $\underline{p}$-tame if and only if $\mathbf{L}_j\mathcal{A}_{\underline{p}}(M) = 0$ for $j \geq 1$.

To set up the proof, let $P_\bullet \to M$ be a projective resolution of M, and consider the double complex $\mathcal{K}_\bullet(P_\bullet, \underline{p})$. The spectral sequence (8.3) has E^2-term

$${}^{\mathrm{I}}E^2_{ij} \approx \mathbf{L}_j K_i(M, \underline{p}),$$

where $\mathbf{L}_j K_i(-,\underline{p})$ denotes the jth derived functor of $K_i(-,\underline{p})$. The spectral sequence (8.4) collapses, and thus we obtain a spectral sequence of the form

$$E^2_{ij} = \mathbf{L}_j K_i(M,\underline{p}) \Longrightarrow K_{i+j}(M,\underline{p}). \tag{9.6}$$

Note that since $K_0(-,\underline{p}) \approx \mathcal{A}_{\underline{p}}$, we have that $\mathbf{L}_j\mathcal{A}_{\underline{p}} = \mathbf{L}_j K_0(-,\underline{p})$.

Proof of (9.5). The hypothesis that $\underline{p}$ is a tame sequence means that $K_i(P,\underline{p}) = 0$ for P projective and $i \geq 1$. Therefore $\mathbf{L}_j K_i(-,\underline{p}) = 0$ for $i \geq 1$, and so the spectral sequence (9.6) degenerates to give that $K_j(M,\underline{p}) \approx \mathbf{L}_j K_0(M,\underline{p}) \approx \mathbf{L}_j\mathcal{A}_{\underline{p}}(M)$. The characterization of tameness of M is immediate. □

9.7. **Analytic completion of complexes.**

9.8. **Proposition.** *If $\underline{p}$ is a tame sequence in R, and if $M_\bullet$ is a complex of $\underline{p}$-tame R-modules, there is a strongly convergent spectral sequence*

$$L_i\mathcal{A}_{\underline{p}}(H_j M_\bullet) \Longrightarrow H_{i+j}\mathcal{A}_{\underline{p}}(M_\bullet).$$

In particular, if $M_\bullet$ is degreewise $\underline{p}$-tame and the $H_* M_\bullet$ are $\underline{p}$-analytic, then $\eta\colon M_\bullet \to \mathcal{A}_{\underline{p}}(M_\bullet)$ is a quasi-isomorphism.

Proof. Consider the double complex $\mathcal{K}_\bullet(M_\bullet,\underline{p})$. Because the M_j are tame, the spectral sequence (8.3) collapses at E^2 to give $H_j(\operatorname{Tot}\mathcal{K}_\bullet(M_\bullet,\underline{p})) \approx H_j(\mathcal{A}_{\underline{p}}(M_\bullet))$. Thus the spectral sequence (8.4) takes the form

$${}^{\mathrm{II}}E^2_{ij} = \mathbf{L}_j\mathcal{A}_{\underline{p}}(H_i M_\bullet) \Longrightarrow H_{i+j}(\mathcal{A}_{\underline{p}}(M_\bullet)),$$

using that $\underline{p}$ is a tame sequence to identify $\mathbf{L}_j\mathcal{A}_{\underline{p}}$ with koszul homology. □

9.9. **Projectives in $\widehat{\mathrm{Mod}}_{\underline{p}}$.**

9.10. **Proposition.** *If P is a projective R-module, then $\mathcal{A}_{\underline{p}}(P)$ is a projective object of $\widehat{\mathrm{Mod}}_{\underline{p}}$. The category $\widehat{\mathrm{Mod}}_{\underline{p}}$ has enough projectives.*

Proof. The claims are immediate from the fact that $\mathcal{A}_{\underline{p}}\colon \mathrm{Mod} \to \widehat{\mathrm{Mod}}_{\underline{p}}$ is left adjoint to the fully-faithful inclusion $\widehat{\mathrm{Mod}}_{\underline{p}} \subseteq \mathrm{Mod}$, and the fact that $\mathcal{A}_{\underline{p}}$ is right exact, and so preserves surjections. □

9.11. **Coproducts.** An infinite direct sum of tame modules is not in general tame. However, we do have a result on the vanishing of the "top" derived functor of analytic completion applied to a coproduct, under suitable tameness hypotheses.

9.12. **Proposition.** *Let $\underline{p} = (p_1,\dots,p_r)$ be a tame sequence in R. Let $\{M_i\}$ be an indexed collection of R-modules. If the M_i are $\underline{p}$ tame, then $\mathbf{L}_r\mathcal{A}_{\underline{p}}(\bigoplus M_i) \approx 0$.*

Proof. Since a direct sum of modules injects into the direct product, we have an inclusion of complexes $\mathcal{K}_\bullet(\bigoplus M_i,\underline{p}) \to \mathcal{K}_\bullet(\prod M_i,\underline{p})$. Since these complexes are bounded above at r, we must have that $\mathbf{L}_r\mathcal{A}_{\underline{p}}(\bigoplus M_i) \to \mathbf{L}_r\mathcal{A}_{\underline{p}}(\prod M_i)$ is a monomorphism. But $\mathcal{K}_\bullet(-,\underline{p})$, and hence $\mathbf{L}_i\mathcal{A}_{\underline{p}}$, commute with products, so $\mathbf{L}_r\mathcal{A}_{\underline{p}}(\prod M_i) \approx 0$ by the tamenes hypothesis on the M_i, whence the claim follows. □

The analytic completion functor $\mathcal{A}_{\underline{p}}\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ does not preserve infinite direct sums, and thus infinite coproducts in $\widehat{\mathrm{Mod}}_{R,\underline{p}}$ need not coincide with those in Mod_R. We reserve the symbol $\bigoplus$ for coproducts (i.e., direct sums) in Mod_R, and $\coprod$ for coproducts in $\widehat{\mathrm{Mod}}_{\underline{p}}$. Observe that for $\{M_i\}$ in $\widehat{\mathrm{Mod}}_{\underline{p}}$, we have $\coprod M_i \approx \overline{\mathcal{A}}_{\underline{p}}(\bigoplus M_i)$.

9.13. **Corollary** ([Hov08, Prop. 1.4])**.** *Let $\underline{p} = (p_1, \ldots, p_r)$ be a tame sequence in R. Then the left derived functors $\mathbf{L}_q \coprod$ of the coproduct functor $\coprod\colon \prod \widehat{\mathrm{Mod}}_{\underline{p}} \to \widehat{\mathrm{Mod}}_{\underline{p}}$ vanish for $q \geq r$.*

Proof. It is straightforward that $(\mathbf{L}_q \coprod)(\{M_i\}) \approx \mathbf{L}_q A_{\underline{p}}(\bigoplus M_i)$. Now apply (9.12). □

In particular, we have the following.

9.14. **Corollary.** *If $p \in R$ is such that $\mathrm{Hom}_R(R/p^\infty, R) \approx 0$, then coproducts are exact in $\widehat{\mathrm{Mod}}_p$.*

Proof. Immediate using (9.3) and (9.13). □

For sequences $\underline{p}$ longer than one, coproducts in $\mathrm{Mod}_{\underline{p}}$ can fail to be exact even in the best case (e.g., when $\underline{p}$ is regular); see [Hov08, §1.3] for an example.

9.15. **Sequential colimits and injectives: an example.** Let $p \in R$, and consider the R-module $R/p^\infty = \mathrm{colim}_k\, R/p^k$ as in (8.7), which fits in an exact sequence

$$R \to p^{-1}R \to R/p^\infty \to 0.$$

Each $R/p^n \in \widehat{\mathrm{Mod}}_p$ so $\mathcal{A}_p(R/p^n) \approx R/p^n$, but $\mathcal{A}_p(R/p^\infty) = 0$ by (5.2). Thus when p is not a unit in R we get an example of analytic completion which does not preserve sequential colimits.

Note that the colimit in $\widehat{\mathrm{Mod}}_p$ of the sequence $\cdots \to R/p^k \xrightarrow{p\cdot} R/p^{k+1} \to \cdots$ is $\overline{\mathcal{A}}_p(R/p^\infty) \approx 0$. This gives an example of a sequence of monomorphims whose direct limit is not a monomorphism, and shows that $\widehat{\mathrm{Mod}}_p$ is not an AB5 abelian category. We have shown (9.14) that coproducts are exact in $\widehat{\mathrm{Mod}}_p$ if R is p-tame (e.g., if R is p-regular); this gives a counterexample to the "theorem of Roos", as in [Nee02].

The example also shows that $\widehat{\mathrm{Mod}}_p$ need not have enough injectives; for if R/p embeds in an injective object I of $\widehat{\mathrm{Mod}}_p$, such a map would necessarily extend to a map from the colimit of the R/p^n in $\widehat{\mathrm{Mod}}_p$, which is not possible since the colimit is 0.

9.16. **Analytic completion as an** Ext**.** Given $p \in R$, let $\mathcal{C}_\bullet(p)$ denote the chain complex with

$$\mathcal{C}_0(p) = \mathcal{C}_{-1}(p) = \bigoplus_{k=0}^{\infty} R, \qquad (da)_k = a_{k-1} - pa_k,$$

where we set $a_{-1} = 0$. It is straightforward to show that there is an isomorphism of chain complexes

$$\mathrm{Hom}_R(\mathcal{C}_\bullet(p), M) \approx \mathcal{K}_\bullet(M, p).$$

9.17. **Proposition.** *If $p \in R$ is regular, then*

$$\mathbf{L}_i\mathcal{A}_p(M) \approx \mathrm{Ext}_R^{1-i}(R/p^\infty, M),$$

where $R/p^\infty \approx \mathrm{colim}_k\, R/p^k$ in Mod_R.

Proof. Observe that if p is regular, then $H_0\mathcal{C}_\bullet(p) = 0$ and $H_{-1}\mathcal{C}_\bullet(p) \approx R/p^\infty$. □

Given a sequence $\underline{p} = (p_1, \ldots, p_r)$ in R, let $\mathcal{C}_\bullet(\underline{p}) = \mathrm{Tot}[\mathcal{C}_\bullet(p_1) \otimes_R \cdots \otimes_R \mathcal{C}_\bullet(p_r)]$. It is clear that there is an isomorphism of chain complexes

$$\mathrm{Hom}_R(\mathcal{C}_\bullet(\underline{p}), M) \approx \mathcal{K}_\bullet(M, \underline{p}).$$

9.18. **Proposition.** *If $\underline{p} = (p_1, \ldots, p_r)$ is a regular sequence in R, then*

$$\mathbf{L}_i\mathcal{A}_p(M) \approx \mathrm{Ext}_R^{r-i}(R/(p_1^\infty, \ldots, p_r^\infty), M).$$

Proof. The regularity hypotheses ensures that $H_{-r}\mathcal{C}_\bullet(\underline{p}) \approx R/(p_1^\infty, \dots, p_r^\infty)$ and $H_i\mathcal{C}_\bullet(\underline{p}) \approx 0$ for $i \neq -r$. □

9.19. *Remark.* The object $\mathcal{C}_\bullet(\underline{p})$ is quasi-isomorphic to the "telescope complex" $\mathrm{Tel}(X^r)$ of [GM92], when $\underline{p}$ generates the maximal ideal $\mathfrak{a}$ of a complete local ring. In this case, $\mathcal{K}_\bullet(M, \underline{p})$ is quasi-isomorphic to their "microscope complex" $\mathrm{Mic}(K^\bullet(\mathfrak{a}^r, M))$.

10. Derived category of analytic modules

Recall that the (unbounded) derived category $\mathcal{D}_R$ of R-modules is obtained by inverting quasi-isomorphisms in the category $\mathrm{Ch}\,\mathrm{Mod}_R$ of unbounded chain complexes of R-modules. Likewise, given a sequence $\underline{p} = (p_1, \dots, p_r)$ in R, we can define the (unbounded) derived category $\widehat{\mathcal{D}}_{R,\underline{p}}$ of $\underline{p}$-analytic modules similarly, by inverting quasi-isomorphisms in $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$.

As is well-known, the derived category $\mathcal{D}_R$ can be obtained as the homotopy category of a Quillen model category structure on $\mathrm{Ch}\,\mathrm{Mod}_R$. This model structure has the following properties.

(1) Weak equivalences are quasi-isomorphisms.
(2) Fibrations are degree-wise surjections.
(3) Cofibrations are maps with the left-lifting property with respect to trivial fibrations.
(4) The model category structure is cofibrantly generated, with generating cofibrations $\{S^{n-1}(R) \to D^n(R)\}_{n\in\mathbb{Z}}$ and generating trivial cofibrations $\{0 \to D^n(R)\}_{n\in\mathbb{Z}}$, where $S^n(R)$ is the complex concentrated in degree n with module R, and $D^n(R)$ is the acyclic complex concentrated in degrees n and $n-1$ with values R.
(5) The model category structure is proper.

In this section, we show that if $\underline{p}$ is a *tame sequence* in R, then we can construct a similar model category structure for $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$, and we use this to show that $\widehat{\mathcal{D}}_{R,\underline{p}} \to \mathcal{D}_R$ is fully faithful, and in fact that $\widehat{\mathcal{D}}_{R,\underline{p}}$ is equivalent to the full subcategory in $\mathcal{D}_R$ of complexes whose homology is $\underline{p}$-analytic. Compare with [Val], which carries this out when $\underline{p}$ is a regular sequence.

10.1. Model structure for $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$.

10.2. **Theorem.** *Let $\underline{p} = (p_1, \dots, p_r)$ be a tame sequence in R. Then there exists a Quillen model category structure on $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$ with the following properties.*

(1) *Weak equivalences are quasi-isomorphisms.*
(2) *Fibrations are degree-wise surjections.*
(3) *Cofibrations are maps with the left-lifting property with respect to trivial fibrations.*
(4) *The model category structure is proper.*

Furthermore, the evident adjoint pair

$$\mathcal{A}\colon \mathrm{Ch}\,\mathrm{Mod}_R \rightleftarrows \mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}} : \mathcal{U},$$

where $\mathcal{U}$ is inclusion, and $\mathcal{A}$ is degree-wise analytic completion, is a Quillen pair, with the property that the natural counit map of total derived functors $\mathbf{L}\mathcal{A} \circ \mathbf{R}\mathcal{U} \to \mathrm{id}$ is an isomorphism. An object C of $\mathrm{Ch}\,\mathrm{Mod}_R$ is in the essential image of $\mathbf{R}\mathcal{U}$ if and only if H_*C is $\underline{p}$-analytic.

Thus, if $\underline{p}$ is tame, then the unbounded derived category $\widehat{\mathcal{D}}_{R,\underline{p}}$ of $\underline{p}$-analytic modules is equivalent to the full subcategory of the unbounded derived category $\mathcal{D}_R$ of modules, consisting of modules whose homology is $\underline{p}$-analytic.

We will need the following lemma.

10.3. **Lemma.** *Suppose that $\underline{p}$ is a tame sequence in R, and let $f\colon C \to D$ be a cofibration in the Quillen model structure on* $\operatorname{Ch}\operatorname{Mod}_R$ *described above. Then we have the following.*

(1) *If $f\colon C \to D$ is a quasi-isomorphism, so is $\mathcal{A}_{\underline{p}}(f)\colon \mathcal{A}_{\underline{p}}(C) \to \mathcal{A}_{\underline{p}}(D)$.*
(2) *If H_*D is $\underline{p}$-analytic and if $\eta_{\underline{p}}\colon C \to \mathcal{A}_{\underline{p}}(C)$ is a quasi-isomorphism, then $\eta_{\underline{p}}\colon D \to \mathcal{A}_{\underline{p}}(D)$ is a quasi-isomorphism.*

Proof. Because f is a cofibration, it is injective with degreewise projective cokernel. Thus we have a diagram of complexes

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f"] \arrow[d] & D \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}_{\underline{p}}(C) \arrow[r] & \mathcal{A}_{\underline{p}}(D) \arrow[r] & \mathcal{A}_{\underline{p}}(P) \arrow[r] & 0 \end{tikzcd}$$

in which the top row is exact and P is degreewise projective. Because $\underline{p}$ is tame, the complex P is degreewise tame, whence $\mathbf{L}_1\mathcal{A}_{\underline{p}}(P_i) \approx 0$ for all i, and thus the bottom row is exact.

For statement (1), we have that $H_*P \approx 0$, which is $\underline{p}$-analytic. Thus, (9.8) applies to show that $\eta_{\underline{p}}\colon P \to \mathcal{A}_{\underline{p}}(P)$ is a quasi-isomorphism, and thus $\mathcal{A}_{\underline{p}}(f)$ is a quasi-isomorphism as desired.

For statement (2), note that both H_*C and H_*D are analytic, and therefore H_*P is analytic, being an extension of $\operatorname{Cok}[H_*C \to H_*D]$ and $\operatorname{Ker}[H_{*-1}C \to H_{*-1}D]$. Thus, (9.8) applies to show that $\eta_{\underline{p}}\colon P \to \mathcal{A}_{\underline{p}}(P)$ is a quasi-isomorphism, and the statement follows. □

10.4. **Corollary.** *Suppose $\underline{p}$ is a tame sequence. If P is a cofibrant object in* $\operatorname{Ch}\operatorname{Mod}_R$*, then H_*P is $\underline{p}$-analytic if and only if $\eta_{\underline{p}}\colon P \to \mathcal{A}_{\underline{p}}(P)$ is a quasi-isomorphism.*

Proof. Immediate from case (2) of (10.3), taking $C = 0$ and $D = P$. □

Proof of (10.2). We consider the following classes of morphisms in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$.

- $\mathcal{W}$, the class of quasi-isomorphisms in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$;
- $\mathcal{F}$, the class of degreewise surjections in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$;
- $\mathcal{M}$, the class of degreewise injections in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$;
- $\mathcal{C}$, the class of maps in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$ with the left lifting property with respect to $\mathcal{W}\cap\mathcal{F}$;
- $\mathcal{T}$, the class of maps in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$ with the left lifting property with respect to $\mathcal{F}$.

We will *define* fibrations, cofibrations, and weak equivalences to be the classes $\mathcal{F}$, $\mathcal{C}$, and $\mathcal{W}$ respectively. It is immediate that $\mathcal{T} \subset \mathcal{C}$, that $\mathcal{W}$ is has the 2 of 3 property, and that all five classes $\mathcal{W}$, $\mathcal{F}$, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{T}$ are closed under retracts.

We will show the following, where $\mathcal{A}\colon \operatorname{Ch}\operatorname{Mod}_R \to \operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$ denotes the left adjoint of the inclusion functor, and where "fibration", "cofibration", etc., refer to the those classes in the model category structure on $\operatorname{Ch}\operatorname{Mod}_R$.

(1) The functor $\mathcal{A}$ takes fibrations to morphisms in $\mathcal{F}$.
(2) The functor $\mathcal{A}$ takes trivial cofibrations to morphisms in $\mathcal{W}$.
(3) The functor $\mathcal{A}$ takes cofibrations to morphisms in $\mathcal{C}\cap\mathcal{M}$, and takes trivial cofibrations to morphisms in $\mathcal{T}$.
(4) Any map in $\operatorname{Ch}\widehat{\operatorname{Mod}}_{R,\underline{p}}$ can be factored into a map in $\mathcal{C}\cap\mathcal{M}$ followed by a map in $\mathcal{W}\cap\mathcal{F}$, and into a map in $\mathcal{W}\cap\mathcal{T}$ followed by a map in $\mathcal{F}$.
(5) We have $\mathcal{W}\cap\mathcal{C} \subset \mathcal{T}$.
(6) We have $\mathcal{C} \subset \mathcal{M}$.

The model structure for $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$ follows immediately; statement (4) gives the factorization axiom, and statement (5) gives the remaining lifting part of the lifting axiom. Right properness is immediate, and left properness follows from statement (6) and standard arguments in abelian categories.

It is immediate that $\mathrm{Ch}\,\mathrm{Mod}_R \rightleftarrows \mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$ is a Quillen pair. To show that $\mathbf{L}\mathcal{A}\circ\mathbf{R}\mathcal{U}\to \mathrm{id}$ is an isomorphism, and that the essential image of $\mathbf{R}\mathcal{U}$ consists of complexes with $\underline{p}$-analytic homology, is a straightforward exercise using (10.4).

It remains to prove statements (1)–(6).

Statement (1) is just the fact that analytic completion is right exact, so that $\mathcal{A}$ preserves degreewise surjections.

Statement (2) is precisely case (1) of (10.3).

Most of statement (3) is an immediate consequence of lifting properties in $\mathrm{Ch}\,\mathrm{Mod}_R$, and the fact that the inclusion functor $\mathcal{U}$ preserves fibrations and trivial fibrations. The fact that $\mathcal{A}$ takes cofibrations into $\mathcal{M}$ is immediate from the fact that cofibrations in $\mathrm{Ch}\,\mathrm{Mod}_R$ are degreewise split monomorphisms.

To prove statement (4), let $f\colon M\to N$ be a map in $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$. If $f=pi$ with $i\colon M\to D$ and $p\colon D\to N$ is any factorization in $\mathrm{Ch}\,\mathrm{Mod}_R$ of f, then there exists a unique map q in

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d, "i"'] & \mathcal{A}_{\underline{p}}(D) \arrow[d, "q"] \\ D \arrow[r, "p"'] \arrow[ur, "\eta_{\underline{p}}"] & N \end{tikzcd}$$

making the diagram commute, since N is degreewise analytic. Setting $j=\eta_{\underline{p}}i$, we obtain a factorization $f=qj$ in $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$. Furthermore, up to isomorphism, $j=\mathcal{A}_{\underline{p}}(i)$ and $q=\mathcal{A}_{\underline{p}}(p)$.

If p is a fibration in $\mathrm{Ch}\,\mathrm{Mod}_R$, then $q\in\mathcal{F}$ by statement (1). If i is a cofibration in $\mathrm{Ch}\,\mathrm{Mod}_R$ and either i or p is a quasi-isomorphism, then (10.3) implies that $\eta_{\underline{p}}\colon D\to\mathcal{A}_{\underline{p}}(D)$ is a quasi-isomorphism. Thus, statements (2) and (3) imply that we can obtain the desired factorizations in $\mathrm{Ch}\,\widehat{\mathrm{Mod}}_{R,\underline{p}}$ by starting with an appropriate factorization $f=pi$ in $\mathrm{Ch}\,\mathrm{Mod}_R$ and applying the above construction.

The proof of statement (5) is a standard argument, using the $\mathcal{W}\cap\mathcal{T}/\mathcal{F}$ factorization of statement (4) and the 2 of 3 property of $\mathcal{W}$ to show that any map in $\mathcal{W}\cap\mathcal{C}$ is a retract of a map in $\mathcal{T}$. The proof of statement (6) is similar, using the $\mathcal{C}\cap\mathcal{M}/\mathcal{W}\cap\mathcal{F}$ factorization of statement (4) to show that any map in $\mathcal{C}$ is a retract of a map in $\mathcal{M}$. $\square$

11. Continuous functors and analytic monads

This section addresses the homotopy theory of analytically complete objects equipped with algebraic structure. We focus on commutative R-algebras whose underlying R-module is $\underline{p}$-analytically complete, for some tame sequence $\underline{p}$.

11.1. **Continuous functors.** Fix a commutative ring R, and a sequence $\underline{p}=(p_1,\dots,p_r)$ in R. Let $F\colon \mathrm{Mod}_R\to C$ be a (possibly non-additive) functor to some category C. We say that F is **continuous** if

$$F\eta_{\underline{p}}\colon F\to F\mathcal{A}_{\underline{p}}$$

is a natural isomorphism. It is immediate that if F is continuous, then the functor F factors up to isomorphism as

$$\mathrm{Mod}\xrightarrow{\mathcal{A}_{\underline{p}}}\widehat{\mathrm{Mod}}_{\underline{p}}\xrightarrow{F|\widehat{\mathrm{Mod}}_{\underline{p}}} C.$$

11.2. *Example.* Let M be an R-module. Then the functor $F\colon \mathrm{Mod}_R \to \widehat{\mathrm{Mod}}_{\underline{p}}$ defined by $F(N) = \mathcal{A}_{\underline{p}}(M \otimes N)$ is continuous, by (6.2).

11.3. *Example.* Let M be an R-module. Then the functor $F\colon \mathrm{Mod}_R \to \widehat{\mathrm{Mod}}_{\underline{p}}$ defined by $F(N) = \mathcal{A}_{\underline{p}}\mathrm{Hom}_R(M, N)$ is not necessarily continuous (though it is continuous if, for instance, M is projective). For an example, consider $R = \mathbb{Z}$, $M = \mathbb{Z}/p$, and $N = \mathbb{Z}/p^\infty$, with $\underline{p} = (p)$ for a prime p.

11.4. **Analytic monads.** Given a monad (T, u, m), we write Alg_T for its category of algebras.

Let $\underline{p}$ be a finite sequence in R, and write $\mathcal{A} = \mathcal{A}_{\underline{p}}$ and $\eta = \eta_{\underline{p}}\colon I \to \mathcal{A}$. Suppose that $(T, u\colon I \to T, m\colon TT \to T)$ is a monad on $\mathrm{Mod} = \mathrm{Mod}_R$; we will write Alg_T for the category of T-algebras.

Define

$$\widehat{T} = \mathcal{A}T, \qquad \widehat{u} = (\eta T) \circ u = (\mathcal{A}u) \circ \eta\colon I \to \widehat{T}, \qquad \widehat{m} = (\mathcal{A}m) \circ (\mathcal{A}T\eta T)^{-1}\colon \widehat{T}\widehat{T} \to \widehat{T}.$$

11.5. **Proposition.** *Let (T, u, m) be a monad on* Mod *such that $\mathcal{A}T$ is a continous functor. Then we have the following.*

1. *The data $(\widehat{T}, \widehat{u}, \widehat{m})$ is also a monad on* Mod.
2. *The map $\eta T\colon T \to \mathcal{A}T = \widehat{T}$ is a morphism of monads, and thus induces a forgetful functor $\mathcal{U}\colon \mathrm{Alg}_{\widehat{T}} \to \mathrm{Alg}_T$ on categories of algebras.*
3. *The functor $\mathcal{U}\colon \mathrm{Alg}_{\widehat{T}} \to \mathrm{Alg}_T$ admits a left adjoint $\mathcal{A}\colon \mathrm{Alg}_T \to \mathrm{Alg}_{\widehat{T}}$, which is defined by by sending a T-algebra $(X, \psi\colon TX \to X)$ to $(X, \widehat{\psi}\colon \widehat{T}X \to X)$, where $\widehat{\psi} = (\mathcal{A}\psi) \circ (\mathcal{A}T\eta)^{-1}$.*
4. *The counit map $\mathcal{A} \circ \mathcal{U} \to \mathrm{id}$ is a natural isomorphism, and hence $\mathcal{U}$ induces an equivalence between $\mathrm{Alg}_{\widehat{T}}$ and the full subcategory of Alg_T consisting of objects whose underlying R-module is analytic.*

Proof. For claim (1), it suffices to show that $\widehat{m} \circ (\widehat{T}\widehat{m}) = \widehat{m} \circ (\widehat{m}\widehat{T})$ and $\widehat{m} \circ (\widehat{T}\widehat{u}) = \mathrm{id} = \widehat{m} \circ (\widehat{u}\widehat{T})$, which is a straightforward verification of the commutative diagrams

$$\begin{tikzcd}
 & \mathcal{A}T\mathcal{A}T & \mathcal{A}\mathcal{A}T \arrow[l, "\mathcal{A}u\mathcal{A}T"'] \\
\mathcal{A}T \arrow[r, "\mathcal{A}Tu"] \arrow[dr, "\mathrm{id}"'] & \mathcal{A}TT \arrow[u, "\mathcal{A}T\eta T"] \arrow[d, "\mathcal{A}m"] & \mathcal{A}T \arrow[l, "\mathcal{A}uT"'] \arrow[u, "\mathcal{A}\eta T = \eta\mathcal{A}T"'] \arrow[dl, "\mathrm{id}"] \\
 & \mathcal{A}T &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}T\mathcal{A}T\mathcal{A}T & \mathcal{A}TT\mathcal{A}T \arrow[l, "\mathcal{A}T\eta\mathcal{A}T\mathcal{A}T"'] \arrow[r, "\mathcal{A}m\mathcal{A}T"] & \mathcal{A}T\mathcal{A}T \\
\mathcal{A}T\mathcal{A}TT \arrow[u, "\mathcal{A}T\mathcal{A}T\eta T"'] \arrow[d, "\mathcal{A}T\mathcal{A}m"] & \mathcal{A}TTT \arrow[l, "\mathcal{A}T\eta TT"'] \arrow[r, "\mathcal{A}mT"] \arrow[u, "\mathcal{A}TT\eta T"'] \arrow[d, "\mathcal{A}Tm"] & \mathcal{A}TT \arrow[u, "\mathcal{A}T\eta T"'] \arrow[d, "\mathcal{A}m"] \\
\mathcal{A}T\mathcal{A}T & \mathcal{A}TT \arrow[l, "\mathcal{A}T\eta T"'] \arrow[r, "\mathcal{A}m"] & \mathcal{A}T
\end{tikzcd}$$

Claim (2), that $\eta T \to \widehat{T}$ is map of monads is also a straightforward verification.

To prove claim (3), first note that the given definition of $\mathcal{A}$ actually produces a $\widehat{T}$-algebra, via the commutative diagrams

$$\begin{tikzcd}
\mathcal{A}T\mathcal{A}X & \mathcal{A}\mathcal{A}X \arrow[l, "\mathcal{A}u\mathcal{A}"'] \\
\mathcal{A}TX \arrow[u, "\mathcal{A}T\eta"] \arrow[d, "\mathcal{A}\psi"] & \mathcal{A}X \arrow[l, "\mathcal{A}u"'] \arrow[u, "\mathcal{A}\eta = \eta\mathcal{A}"'] \arrow[dl, "\mathrm{id}"] \\
\mathcal{A}X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}T\mathcal{A}T\mathcal{A}X & \mathcal{A}TT\mathcal{A}X \arrow[l, "\mathcal{A}T\eta\mathcal{A}T\mathcal{A}"'] \arrow[r, "\mathcal{A}m\mathcal{A}"] & \mathcal{A}T\mathcal{A}X \\
\mathcal{A}T\mathcal{A}TX \arrow[u, "\mathcal{A}T\mathcal{A}T\eta"'] \arrow[d, "\mathcal{A}T\mathcal{A}\psi"] & \mathcal{A}TTX \arrow[l, "\mathcal{A}T\eta T"'] \arrow[r, "\mathcal{A}m"] \arrow[u, "\mathcal{A}TT\eta"'] \arrow[d, "\mathcal{A}T\psi"] & \mathcal{A}TX \arrow[u, "\mathcal{A}T\eta"'] \arrow[d, "\mathcal{A}\psi"] \\
\mathcal{A}T\mathcal{A}X & \mathcal{A}TX \arrow[l, "\mathcal{A}T\eta X"'] \arrow[r, "\mathcal{A}\psi"] & \mathcal{A}X
\end{tikzcd}$$

It is straightforward to verify the desired adjunction.

That the counit $\mathcal{A} \circ \mathcal{U} \to \mathrm{id}$ is an isomorphism is also a straightforward verification. It is also clear from the construction of $\mathcal{A}$ that the unit map $\mathcal{U} \circ \mathcal{A} \to \mathrm{id}$ induces an isomorphism on a given T-algebra (X, ψ) if and only if X is analytic. Claim (4) follows. $\square$

We say that (T, η, μ) is a **analytic monad** on Mod_R (with respect to a sequence $\underline{p}$) if the underlying functor $\mathcal{A}T$ is continuous, and therefore satisfies the conclusions of (11.5).

11.6. **Simplicial algebras for an analytic monad.** As above, let $\underline{p}$ be a finite sequence in a commutative ring R, and write $\mathcal{A} = \mathcal{A}_{\underline{p}}$ for the analytic completion functor on $\mathrm{Mod} = \mathrm{Mod}_R$.

Fix a $\underline{p}$-analytic monad (T, η, μ) on Mod. Assume that the functor T preserves reflexive coequalizers in Mod; as a consequence, the functor $\widehat{T} = \mathcal{A}T$ also preserves reflexive coequalizers, since $\mathcal{A}$ is right exact. It is a standard consequence of this assumption that Alg_T and $\mathrm{Alg}_{\widehat{T}}$ are complete and cocomplete, and that the forgetful functors $\mathrm{Alg}_T \to \mathrm{Mod}$ and $\mathrm{Alg}_{\widehat{T}} \to \mathrm{Mod}$ preserve and reflect reflexive coequalizers.

A morphism $X \to Y$ in a complete and cocomplete category C is called an **effective epimorphism** if the projection pair $X \times_Y X \rightrightarrows X \to Y$ is a coequalizer in C. A **projective** object in C is an object P such that $\mathrm{Hom}_C(P, f)$ is surjective whenever f is an effective epimorphism. We say C has **enough projectives** if for every object A in C there exists an effective epimorphsm $P \to A$ where P is projective.

It is clear that effective epimorphisms in Mod are precisely the surjections, and the projectives are the projective modules. Because the forgetful functors $\mathrm{Alg}_T \to \mathrm{Mod}$ and $\mathrm{Alg}_{\widehat{T}} \to \mathrm{Mod}$ preserve and create reflexive coequalizers and pullbacks, this implies that effective epimorphisms in Alg_T and $\mathrm{Alg}_{\widehat{T}}$ are precisely maps which are surjective. Projectives in Alg_T are retracts of free T-algebras $T(P)$ on projective R-modules P, and projectives in $\mathrm{Alg}_{\widehat{T}}$ are retracts of free $\widehat{T}$-algebras $\widehat{T}P$ on projective R-modules P. It is clear that both Alg_T and $\mathrm{Alg}_{\widehat{T}}$ have enough projectives.

11.7. **Proposition.** *Suppose (T, u, m) is an analytic monad on Mod_R such that the functor $T\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ preserves reflexive coequalizers. Then there exist simplical closed model category structures on $s\mathrm{Alg}_T$ and $s\mathrm{Alg}_{\widehat{T}}$, so that a morphism f in either category is a fibration (resp. weak equivalence) if the underlying map of simplicial sets is a fibration (resp. a weak equivalence). The evident adjoint functors*

$$A\colon s\mathrm{Alg}_T \rightrightarrows s\mathrm{Alg}_{\widehat{T}} : U$$

define a Quillen pair.

Proof. This is immediate from case (*) of [Qui67, Theorem II.4.4]. $\square$

11.8. *Remark.* We note here that the proof of (11.7) implies that any cofibrant object in $s\mathrm{Alg}_T$ is degreewise projective in Alg_T, and therefore in each degree is a retract of an free T-algebra $T(P)$, where P is a projective R-module. Likewise, any cofibrant object in $s\mathrm{Alg}_{\widehat{T}}$ is degreewise projective in $\mathrm{Alg}_{\widehat{T}}$, and therefore in each degree is a retract of an free $\widehat{T}$-algebra $\widehat{T}(P)$, where P is a projective R-module.

11.9. **Proposition.** *Suppose that $\underline{p}$ is a tame sequence in R, and that (T, u, m) is a $\underline{p}$-analytic monad on Mod_R which preserves reflexive coequalizers, and is such that the free T-algebra $T(P)$ is a projective R-module for every projective R-module P. Then the Quillen pair of the above proposition has the property that the right adjoint of*

$$\mathbf{L}A\colon h(s\mathrm{Alg}_T) \rightleftarrows h(s\mathrm{Alg}_{\widehat{T}}) : \mathbf{R}U$$

is fully faithful, with essential image the simplicial T-algebras B such that $\pi_* B$ is $\underline{p}$-analytic.

Proof. It suffices to show that $\mathbf{L}A \circ \mathbf{R}U \to I$ is a natural isomorphism. Given Y in $s\mathrm{Alg}_{\widehat{T}}$, choose a cofibrant object X in $s\mathrm{Alg}_T$ together with a weak equivalence $f\colon X \to U(Y)$. We want to show that $A(X) \xrightarrow{A(f)} AU(Y) \approx Y$ is a quasi-isomorphism, which amounts to showing that the underlying map $\mathcal{A}(X) \to \mathcal{A}(Y) \approx Y$ of simplicial R-modules is a quasi-isomorphism.

Because $\underline{p}$ is tame and X is degreewise a projective R-module, and $\pi_* X \approx \pi_* Y$ are $\underline{p}$-analytic, (9.8) implies that $X \to \mathcal{A}(X)$ is a quasi-isomorphism. Thus the claim follows. ☐

11.10. **Commutative rings.** Fix a commutative ring R, together with a tame sequence $\underline{p}$.

Let T be the "non-unital commutative R-algebra" monad on Mod_R, which satisfies $T(M) = \bigoplus_{k\geq 1} M^{\otimes_R k}_{\Sigma_k}$. The category Alg_T is precisely the category of non-unital commutative R-algebras. Note that this category is equivalent to the category of augmented unital commutative R-algebras; to a non-unital algebra B, associate the unital algebra $R \times B$. We write Comm_R for this category.

It follows from (6.2) that T is a $\underline{p}$-analytic monad, and therefore we obtain a monad $\widehat{T} = \mathcal{A}T$ whose algebras are non-unital commutative R-algebras whose underlying R-module is analytic. We write $\widehat{\mathrm{Comm}}_R$ for this category.

Furthermore, T preserves reflexive coequalizers, and $T(P)$ is a projective R-module when P is projective. Therefore by (11.7), we obtain simplicial model categories related by a Quillen pair

$$A\colon s\mathrm{Comm}_R \rightleftarrows s\widehat{\mathrm{Comm}}_R : U.$$

It is convenient to write $\mathcal{A}$ for the composite $UA\colon s\mathrm{Comm}_R \to s\widehat{\mathrm{Comm}}_R$, and $\eta\colon I \to \mathcal{A}$ for the unit map, since this describes how the functor behaves on the underlying R-modules.

Let $Q\colon \mathrm{Comm}_R \to \mathrm{Mod}_R$ be the indecomposables functor. It is left adjoint to the "square-zero" functor $\mathrm{Mod}_R \to \mathrm{Comm}_R$.

11.11. **Proposition.** *For any non-unital commutative R-algebra B, we have that*

$$\mathcal{A}Q\eta\colon \mathcal{A}Q(B) \to \mathcal{A}Q\mathcal{A}(B)$$

is an isomorphism.

Proof. We have a diagram

$$\begin{tikzcd}
B \otimes B \arrow[r] \arrow[d, "\eta\otimes\eta"] & B \arrow[r] \arrow[d, "\eta"] & Q(B) \arrow[r] \arrow[d, "Q(\eta)"] & 0 \\
\mathcal{A}B \otimes \mathcal{A}B \arrow[r] & \mathcal{A}B \arrow[r] & Q(\mathcal{A}B) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. After appyling $\mathcal{A}$ to this diagram, the rows remain exact, and the maps $\mathcal{A}(\eta\otimes\eta)$ and $\mathcal{A}(\eta)$ are isomorphisms by (6.2). ☐

In other words, the functor $\widehat{Q} = \mathcal{A}Q\colon \mathrm{Comm}_R \to \widehat{\mathrm{Mod}}_R$ is continuous. We write $\widehat{Q}'\colon \widehat{\mathrm{Comm}} \to \widehat{\mathrm{Mod}}_R$ for the factorization of $\widehat{Q}$ through the completion functor $\mathcal{A}$.

11.12. **Proposition.** *We have a commutative square of Quillen pairs, whose diagram of left adjoints is*

$$\begin{tikzcd}
s\mathrm{Comm}_R \arrow[r, "\mathcal{A}"] \arrow[d, "Q"'] & s\widehat{\mathrm{Comm}}_R \arrow[d, "\widehat{Q}'"] \\
s\mathrm{Mod}_R \arrow[r, "\mathcal{A}"'] & s\widehat{\mathrm{Mod}}_R
\end{tikzcd}$$

Proof. Commutativity up to natural isomorphism is given by (11.11). ☐

11.13. **Quillen homology.** The **Quillen homology** groups of a simplicial augmented commutative R-algebra are the left derived functors of the indecomposables functor $s\mathrm{Comm}_R \to s\mathrm{Mod}_R$:

$$H_m^Q(B) = \pi_m(\mathbf{L}Q(B)).$$

The **$\underline{p}$-analytic Quillen homology** of a simplicial augmented commutative R-algebra are the left derived functors of the completed indecomposables functor $\widehat{Q}\colon s\mathrm{Comm} \to s\widehat{\mathrm{Mod}}_{R,\underline{p}}$:

$$H_m^{\widehat{Q}}(B) = \pi_m(\mathbf{L}\widehat{Q}(B)).$$

We can also consider the derived functors of $\widehat{Q}'\colon s\widehat{\mathrm{Comm}}_R \to s\widehat{\mathrm{Mod}}_R$, but these are not any different.

11.14. **Proposition.** *If $B \in s\widehat{\mathrm{Comm}}_R$ and if $\mathcal{U}(B) \in s\mathrm{Comm}_R$ denotes the underlying object, then $(\mathbf{L}\widehat{Q}')(B) \approx (\mathbf{L}\widehat{Q})(\mathcal{U}B)$.*

Proof. If $P \to \mathcal{U}(B)$ is a cofibrant resolution in $s\mathrm{Comm}_R$, then $\mathcal{A}(P) \to B$ is a cofibrant resolution in $s\widehat{\mathrm{Comm}}_R$. The result follows from (11.11). ☐

11.15. **Proposition.** *There is a composite functor spectral sequence*

$$E^2_{ij} = \mathbf{L}_i\mathcal{A}(H_j^Q(B)) \Longrightarrow H_{i+j}^{\widehat{Q}}(B).$$

Proof. If $P \to B$ is a cofibrant resolution, then $Q(P)$ is degreewise projective, and hence degreewise tame, in which case we have the spectral sequence (9.8). ☐

11.16. **Corollary.** *If B a simplicial R-algebra such that $H_*^Q(B)$ is tame, then*

$$H_*^{\widehat{Q}}(B) \approx \mathcal{A}(H_*^Q(B)).$$

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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL

E-mail address: `rezk@math.uiuc.edu`