LOOIJENGA LINE BUNDLES IN COMPLEX ANALYTIC ELLIPTIC COHOMOLOGY

CHARLES REZK

Abstract. We present a calculation, which shows how the moduli of complex analytic elliptic curves arises naturally from the Borel cohomology of an extended moduli space of $U(1)$-bundles on a torus. Furthermore, we show how the analogous calculation, applied to a moduli space of principal bundles for a $K(\mathbb{Z},2)$ central extension of $U(1)^d$ give rise to Looijenga line bundles. We then speculate on the relation of these calculations to the construction of complex analytic equivariant elliptic cohomology.

1. Introduction

In this note, we describe some aspects of how complex analytic elliptic curves arise naturally from the cohomology of certain spaces which parameterize principal bundles on orientable genus 1 surfaces. This suggests how elliptic cohomology emerges from certain derived complex analytic spaces associated to dimensional reduction applied to 2-dimensional field theories.

1.1. **Complex analytic elliptic cohomology.** Complex analytic equivariant elliptic cohomology was first defined by Grojnowski [Gro07][1]. In its most basic formulation, given

- a compact connected abelian Lie group G (i.e., $G \approx U(1)^d$), with cocharacter lattice $B = \mathrm{Hom}(U(1), G)$, and
- an elliptic curve $C_\tau = \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ for $\mathrm{Im}\,\tau > 0$,

he obtains an equivariant coholomolgy theory

$$\mathrm{Ell}_G^*\colon h\mathrm{Top}_G^{\mathrm{fin}} \to \mathrm{Coh}(C_\tau \otimes B)$$

on G-spaces homotopy equivalent to finite G-CW-complexes, taking values in coherent sheaves of $\mathcal{O}_{C_\tau \otimes B}$-modules on the complex analytic abelian variety $C_\tau \otimes B \approx C_\tau^d$.

Grojnowski descibes his construction as "declocalized". That is, $\mathrm{Ell}_G^*(X)$ is produced by gluing together certain localizations of the values of Borel equivariant cohomology rings $H^*(X^H \times_G EG; \mathbb{C})$ for various subgroups H of G. Conceptually, one can regard this as a "reverse engineered" version of a character sheaf, by analogy with the interpretation of $\mathbb{C} \otimes K_G(X)$ as a sheaf over the multiplicative group $\mathbb{G}_m$, whose localizations at various points of $\mathbb{G}_m$ are computed, in terms of standard localization theorems, in terms of Borel cohomology (e.g., as in [BBM85]).

Grojnowski's theory has been extended and used to explain aspects of elliptic genera, notably the rigidity of the Ochanine genus [Ros01], and the modularity of the Witten genus [AB02], [And03]. A significant feature of this theory is the ablility to twist by a *level*, which in the above formulation is described in terms of tensoring sheaves with the *Looijenga line bundle* associated to a quadratic form on the cocharacter lattice B [Gro07, §3.3]. Looijenga's theta functions appear explicitly in the Kac character formula, which can be identified with the calculation of a Gysin map in elliptic cohomology [And00], [Gan14].

This construction of analytic elliptic cohomology, though productive, is somewhat ad hoc, and technically rather intricate. Furthermore, we should expect more from the theory. In particular,

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[1] Originally circulated as a preprint in 1994; see [AM07].

(1) it should take values not (merely) in sheaves on a scheme or complex analytic space, but rather in sheaves on a *derived* scheme or complex analytic space, and
(2) it should in some sense classify, or at least give invariants of, *two-dimensional reductions* of certain kinds 2-dimensional field theories.

These should nowadays be much more approachable goals than was the case when Grojnowski originally defined the theory. For point (1), there is well-developed machinery for constructing cohomology theories from *derived* geometric objects [Lur09]. Furthermore, there is a direct construction of a derived algebraic scheme realizing rational equivariant elliptic cohomlogy for $G = U(1)$ following Grojnowski's delocalized approach [Gre05]. Point (2) is more difficult; however, there has been partial success in relating elliptic cohomology to field theory (following the program of Segal [Seg88]), and many features of the relationship are understood (see, e.g., [ST11]).

1.2. **This purpose and results of this paper.** We are motivated by the observation that elliptic cohomology at the Tate curve should be associated to *one-dimensional reduction* of 1-dimensional field theories. Very roughly, Tate elliptic cohomology should arise as some kind of equivariant K-theory for extended loop groups. By the "extended loop group" $\mathcal{L}^{\mathrm{ext}}G$ of G, we really mean the topological groupoid whose objects are certain principal G-bundles $P \to \mathbb{T}$, over a circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and whose morphisms are maps $(P \to \mathbb{T}) \to (P' \to \mathbb{T})$ of G-bundles covering a rotation of the circle.

Our point of view is inspired by that of [Gan07], [Gan13], which considers the special case of finite groups G, in which case $\mathcal{L}^{\mathrm{ext}}G$ is a Lie groupoid, and thus comes with a well-defined equivariant K-theory. Furthermore, Kitchloo has defined a version of equivariant K-theory for certain Kac-Moody groups [Kit09]. Using this, he constructs [Kit14], for loop groups on simple and simply connected G, a version of G-equivariant elliptic cohomology associated to the Tate curve. It turns out that Looijenga line bundles arise naturally in this framework.

The purpose of this note is to describe calculations inspired by the idea of *two-dimensional reduction*. Thus, (i) the circle $\mathbb{T}$ is replaced with an orientable genus 1 surface Σ (e.g., $\mathbb{T}^2$), and (ii) equivariant K-theory is replaced with Borel cohomology with complex coefficients. We restrict attention to a limited class of equivariance groups G, namely (i) tori $G = U(1)^d$, or (ii) "central extensions" $\widetilde{G} = U(1)^d \times_\phi K(\mathbb{Z}, 2)$ of a torus G by $K(\mathbb{Z}, 2)$, according to a class $\phi \in H^4(BG; \mathbb{Z})$.

We summarize our calculations as follows; precise statements are given in §§2–3. Fix

$$\Sigma = \text{orientable genus 1 surface}, \qquad G = \text{topological group},$$

and consider the "wreath product" group

$$\mathcal{W}(G) = \mathcal{W}^{\Sigma}(G) := \mathrm{Map}(\Sigma, G) \rtimes \mathrm{Diff}(\Sigma),$$

where $\mathrm{Diff}(\Sigma)$ is the group of diffeomorphisms[2] (not necessarily orientation preserving). Note that $\mathrm{Map}(\Sigma, G)$ is the gauge group of $\Sigma \times G \to \Sigma$, the trivial G-bundle over Σ, and thus $\mathcal{W}(G)$ is an "extended gauge group". Its classifying space $B\mathcal{W}(G)$ is thus a homotopy theoretic moduli space for the data (smooth genus 1 surface, principal G-bundle).

Let $\mathcal{W}_0(G) \subseteq \mathcal{W}(G)$ denote the identity component, with discrete quotient $\overline{\mathcal{W}}(G) = \mathcal{W}(G)/\mathcal{W}_0(G)$. Thus there is a natural action $\overline{\mathcal{W}}(G) \curvearrowright B\mathcal{W}_0(G)$ on the classifying space of the connected subgroup. For the G we will consider, the cohomology ring $H^*(B\mathcal{W}_0(G); \mathbb{C})$ is concentrated in even degree, whence we obtain an action

$$(\overline{\mathcal{W}}(G) \times \mathbb{C}^\times)^{\mathrm{op}} \curvearrowright \operatorname{Spec} H^*(B\mathcal{W}_0(G); \mathbb{C})$$

on an affine complex variety, where $\mathbb{C}^\times$ acts linearly on H^2. Let

$$\mathcal{X}_G := [\operatorname{Spec} H^*(B\mathcal{W}_0(G); \mathbb{C})]_{\mathrm{an}} \smallsetminus \{\mathrm{bad}\},$$

[2] That we use the diffeomorphism group here is not essential, since we will only use homotopy invariant features of this action. Thus, in its place we could use the homeomorphism group of Σ, or even the monoid of self-homotopy equivalences of Σ, each of which have the same homotopy type as $\mathrm{Diff}(\Sigma)$.

which is a complex analytic space obtained as the "analytification" of the complex variety, with a certain closed subset (described in §2.10) removed. In our examples $\mathcal{X}_G$ is always smooth. The object we are interested in is

$$\mathcal{M}_G := (\overline{\mathcal{W}}(G) \times \mathbb{C}^\times) \,\backslash\backslash\, \mathcal{X}_G,$$

the stacky quotient in complex manifolds. We compute that

$$\begin{aligned}
\mathcal{M}_e &\approx \mathcal{M} = \text{the moduli stack of (complex analytic) elliptic curves},\\
\mathcal{M}_{U(1)} &\approx \mathcal{E} = \text{the universal elliptic curve over } \mathcal{M},\\
\mathcal{M}_{U(1)^d} &\approx \mathcal{E}^d = \mathcal{E} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{E} = \text{the } d\text{-fold product of } \mathcal{E},\\
\mathcal{M}_{K(\mathbb{Z},2)} &\approx \mathbb{G}_m \times \mathcal{M} = \text{the multiplicative group as a trivial bundle of groups over } \mathcal{M},\\
\mathcal{M}_{U(1)^d \times_\phi K(\mathbb{Z},2)} &\approx \mathcal{P}_\phi = \text{principal } \mathbb{G}_m\text{-bundle associated to } \mathcal{L}_\phi
\end{aligned}$$

where $\mathcal{L}_\phi \to \mathcal{E}^d$ is the "Looijenga line bundle" associated to $\phi \in H^4(BU(1)^d, \mathbb{Z})$, regarded as a quadratic function $\phi\colon H_2BU(1)^d = \mathbb{Z}^d \to \mathbb{Z}$. The first three cases of the computation are easy observations, and are described in §2. The main purpose of this paper is prove the last two cases, which are stated in §3.

1.3. **Organization of this paper.** The basic observation is the following: the universal complex analytic elliptic curve arises *naturally* from the cohomology of certain spaces. We present this observation in §2. I have not seen this observation stated in this way before; however, it is closely related to an observation by Etingof and Frenkel about coadjoint actions in double loop groups, a relationship we describe briefly in §2.12.

In §3, we replace $G = U(1)^d$ with $\widetilde{G} = U(1)^d \times_\phi K(\mathbb{Z}, 2)$, the extension associated to a class $\phi \in H^4(BG; \mathbb{Z})$, and observe that our formulation naturally gives Looijenga-type line bundles. This is stated as (3.7), which is our main result.

In §4 we observe how isogenies of complex analytic elliptic curves fit naturally into this story, via finite covering maps of genus 1 surfaces.

In §5 we speculate as to how these constructions might give rise to *derived* elliptic curves (in an analytic setting) following the pattern described in [Lur09], and to elliptic cohomology theories of Grojnowski type. We only sketch a picture here; setting this up formally would involve confronting a definition of derived complex analytic space, which is beyond the scope of this note. We also describe the "stacky" dependence of our constructions on the group G, and note what happens in the simpler 1-dimensional case (where Σ is a circle).

The remainder of the paper (§6–§9) is taken up with the proof of the main result (3.7), which is itself a derived from a more general and coordinate invariant formulation (7.6).

1.4. **Conventions.** At various points we need to consider the action of a group on another group (always from the left). We will sometimes use the notation $g \propto h$ for such an action, so as to typographically distinguish it from gh a product of group elements. When G acts on H from the left, a semidirect product K is always a group with subgroups G and H that $GH = HG = K$ and $G \cap H = \{1\}$, and such that $ghg^{-1} = g \propto h$. There are two distinct but canonically isomorphic constructions of such: $G \ltimes H$ and $H \rtimes G$ with group laws $(g, h) \cdot (g', h') = (gg', (g'^{-1} \propto h)h')$ and $(h, g) \cdot (h', g') = (h(g \propto h'), gg')$ respectively. In §10, we describe the homotopy theoretic conventions we use, primarily in order to establish the sign conventions we need in (§6–§9).

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2. Analytic moduli of elliptic curves, vs. homotopic moduli of genus 1 surfaces

2.1. **Moduli of elliptic curves over $\mathbb{C}$.** The classical uniformization theory of Weierstrass says that

(1) every elliptic curve is isomorphic, as a complex manifold, to $\mathbb{C}/\Lambda$ for some lattice Λ, (i.e., a subgroup $\Lambda = \mathbb{Z}t_1 + \mathbb{Z}t_2$ such that $\mathbb{R} \otimes \Lambda = \mathbb{C}$), with neutral element at the origin, and
(2) every map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ between such complex manifolds fixing the neutral element is given by multiplication by a complex scalar.

That is, such curves correspond to lattices in $\mathbb{C}$ up to scaling by a non-zero complex number.

This can be enriched to a desciption of the moduli stack of such curves. Let

$$\mathcal{X} := \{ (t_1, t_2) \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \} \subset \mathbb{C}^2.$$

We have a group action

$$GL_2(\mathbb{Z}) \times \mathbb{C}^\times \curvearrowright \mathcal{X}$$

by

$$A \propto (t_1, t_2) = (at_1 + bt_2, ct_1 + dt_2), \qquad A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}), \tag{2.2}$$

and

$$\lambda \propto (t_1, t_2) = (\lambda t_1, \lambda t_2), \qquad \lambda \in \mathbb{C}^\times. \tag{2.3}$$

Points of the quotient space $(GL_2(\mathbb{Z}) \times \mathbb{C}^\times)\backslash\mathcal{X}$ are in bijective correspondence to homothety-equivalence classes ($\Lambda \sim \lambda\Lambda$) of lattices, i.e., to isomorphism classes of elliptic curves. It turns out that the moduli stack is in fact the stack quotient

$$\mathcal{M} := (GL_2(\mathbb{Z}) \times \mathbb{C}^\times) \,\backslash\!\backslash\, \mathcal{X}.$$

For our purposes, we do not need to worry about the general notion of stacks. It is sufficient to remember that information defining $\mathcal{M}$ is precisely contained in the group action, so that (for instance), sheaves on the stack $\mathcal{M}$ are precisely equivariant sheaves on $\mathcal{X}$.

2.4. *Remark.* The stack $\mathcal{M}$ is an orbifold, though the above does not present it as such. The continuous group $\mathbb{C}^\times$ acts freely on $\mathcal{X}$, so that $\mathbb{C}^\times\backslash\mathcal{X} \approx \mathbb{C} \smallsetminus \mathbb{R}$ defined by $(t_1, t_2) \mapsto \tau = t_1/t_2$ gives an identification with the double-half plane. The residual $GL_2(\mathbb{Z})$-action descends to an action on $\mathbb{C} \smallsetminus \mathbb{R}$ with finite isotropy, whence $\mathcal{M} \approx GL_2(\mathbb{Z}) \,\backslash\!\backslash\, (\mathbb{C} \smallsetminus \mathbb{R})$.

2.5. *Remark.* Instead of $\mathcal{X}$ we could use $\mathcal{X}^+ = \{ (t_1, t_2) \in \mathcal{X} \mid \operatorname{Im}(t_1/t_2) > 0 \}$, so $\mathcal{M} \approx (SL_2(\mathbb{Z}) \times \mathbb{C}^\times) \,\backslash\!\backslash\, \mathcal{X}^+ \approx SL_2(\mathbb{Z}) \,\backslash\!\backslash\, \mathcal{H}$ where $\mathcal{H} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \}$.

2.6. **The universal elliptic curve.** The universal elliptic curve $\mathcal{E} \to \mathcal{M}$ can be modelled by a map $C \to \mathcal{X}$, with fiber $C_{(t_1,t_2)} = \mathbb{C}/(\mathbb{Z}t_1 + \mathbb{Z}t_2)$ over $(t_1, t_2) \in \mathcal{X}$, together with a lift of the group action on $\mathcal{X}$. Since the fibers are themselves quotients by a free action, we can decribe the universal curve as a stack quotient, via the action

$$(GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathbb{C}^\times \curvearrowright \mathcal{X} \times \mathbb{C} = \{ (t_1, t_2, y) \in \mathbb{C}^2 \times \mathbb{C} \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \}$$

defined by

$$\begin{aligned} A \propto (t_1, t_2, y) &= (at_1 + bt_2, ct_1 + dt_2, y), & A &\in GL_2(\mathbb{Z}), \\ (m_1, m_2) \propto (t_1, t_2, y) &= (t_1, t_2, y + m_1 t_1 + m_2 t_2), & (m_1, m_2) &\in \mathbb{Z}^2, \\ \lambda \propto (t_1, t_2, y) &= (\lambda t_1, \lambda t_2, \lambda y), & \lambda &\in \mathbb{C}^\times. \end{aligned} \tag{2.7}$$

Thus, the stack quotient

$$\mathcal{E} := ((GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathbb{C}^\times) \,\backslash\!\backslash\, \mathcal{X} \times \mathbb{C}$$

presents the universal curve over $\mathcal{M}$.

2.8. **Moduli of genus 1 surfaces.** Fix a smooth surface Σ, closed and orientable of genus 1. We write

$$\mathrm{Diff}(\Sigma) \supset \mathrm{Diff}_0(\Sigma),$$

for the group of diffeomorphisms and its identity component. The classifying space $B\,\mathrm{Diff}(\Sigma)$ can be viewed as a homotopy-theoretic moduli space of orientable (but not oriented) genus 1-surfaces.

For convenience in describing calculations, we use the model $\Sigma := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then $\mathrm{Diff}(\Sigma)$ is weakly equivalent, as a topological group, to the subgroup $\mathbb{T}^2 \rtimes GL_2(\mathbb{Z})$ (acting on $\mathbb{T}^2$ in the evident way from the left) [EE67].

Therefore $B\,\mathrm{Diff}_0(\Sigma) \approx B\mathbb{T}^2$, which carries an evident action by $GL_2(\mathbb{Z}) = \mathrm{Diff}_+(\Sigma)/\,\mathrm{Diff}_0(\Sigma)$. We may thus consider the induced action

$$GL_2(\mathbb{Z})^{\mathrm{op}} \curvearrowright H^*(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}).$$

It is immediate that

$$H^*(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}) = H^*(B\mathbb{T}^2; \mathbb{C}) \approx \mathbb{C}[t_1, t_2], \qquad t_1, t_2 \in H^2,$$

with $GL_2(\mathbb{Z})$ action given by the precisely the formula (2.2). The cohomology also carries a natural $\mathbb{C}^\times$ action, determined by the grading, which coincides with (2.3).

2.9. **Universal degree 0 line bundle on a genus 1 surface.** Now consider the group

$$\mathcal{W}(U(1)) := \mathrm{Map}(\Sigma, U(1)) \rtimes \mathrm{Diff}(\Sigma),$$

which has identity component $\mathcal{W}_0(U(1)) := \mathrm{Map}_0(\Sigma, U(1)) \rtimes \mathrm{Diff}_0(\Sigma)$, and set $\overline{\mathcal{W}}(U(1)) := \pi_0\mathcal{W}(U(1)) = \mathcal{W}(U(1))/\mathcal{W}_0(U(1))$. The classifying space $B\mathcal{W}(U(1))$ carries the universal example of a smooth orientable genus one surface together with a degree 0 complex line bundle over it.

Using $\Sigma = \mathbb{T}^2$, we obtain an explicit finite dimensional model for $\mathcal{W}(U(1))$ (up to homotopy equivalence), namely

$$\big(\mathrm{Hom}(\mathbb{T}^2, U(1)) \times U(1)\big) \rtimes \big(GL_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big).$$

That is, the homomorphism $\mathrm{Hom}(\mathbb{T}^2, U(1)) \times U(1) \to \mathrm{Map}(\Sigma, U(1))$ defined by[3] $(m, y) \mapsto ((s_1, s_2) \mapsto y + m_1 s_1 + m_2 s_2)$ is a homotopy equivalence, and is invariant under the evident action of $GL_2(\mathbb{Z}) \rtimes \mathbb{T}^2 \subset \mathrm{Diff}(\Sigma)$. We can rebracket this as

$$\big(GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\big) \ltimes \big(\mathbb{T}^2 \times U(1)\big),$$

using the left action $GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{T}^2 \times U(1)$ given by $(A, m) \propto (t, y) = (At, y + m_1 t_1 + m_2 t_2)$.

The induced action $\overline{\mathcal{W}}(U(1))^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(U(1)); \mathbb{C})$ thus has the form

$$(GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)^{\mathrm{op}} \curvearrowright H^*(B(\mathbb{T}^2 \times U(1)); \mathbb{C}).$$

We easily read off that

$$H^*(B(\mathbb{T}^2 \times U(1)); \mathbb{C}) \approx \mathbb{C}[t_1, t_2, y], \qquad t_1, t_2, y \in H^2,$$

with $GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ action given precisely by the first two formulas from (2.7)[4]. The grading of cohomology corresponds to the $\mathbb{C}^\times$-action from (2.7).

[3] We write the group laws on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $U(1) \approx \mathbb{R}/\mathbb{Z}$ additively, and use the evident isomorphism $\mathbb{Z}^2 = \mathbb{Z}^{1\times 2} \approx \mathrm{Hom}(\mathbb{T}^2, U(1))$.

[4] We can regard cohomology classes "t_1", "t_2" and "y" as coordinate functions on the space $\mathcal{X} \times \mathbb{C} = \{(t_1, t_2, y)\}$, so the formulas of (2.7) also describe how to pull back such functions.

2.10. **The geometric picture.** As in the introduction (§1.2) we write

$$\mathcal{W}(G) = \mathcal{W}^{\Sigma}(G) := \mathrm{Map}(\Sigma, G) \rtimes \mathrm{Diff}(\Sigma),$$

with group law $(\psi, \phi) \cdot (\psi', \phi') = (\psi \cdot (\psi' \circ \phi^{-1}), \phi \circ \phi')$, for the extended gauge group of a trivial principal G-bundle over Σ; hence the classifying space $B\mathcal{W}(G)$ carries the universal example of a trivializable principal G-bundle over a genus 1 surface. We let $\mathcal{W}_0(G) \subseteq \mathcal{W}(G)$ denote the identity component, and set $\overline{\mathcal{W}}(G) = \mathcal{W}(G)/\mathcal{W}_0(G) = \pi_0\mathcal{W}(G)$.

Now assume that we restrict to groups G for which $H^*(B\mathcal{W}_0(G); \mathbb{C})$ is concentrated in even degrees. We obtain an action

$$(\overline{\mathcal{W}}(G) \times \mathbb{C}^\times)^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(G); \mathbb{C})$$

where $\mathbb{C}^\times$ acts by scalar multiplication on H^2, and note that this action is *functorial* with respect to the group G and homomorphisms, i.e., $\phi\colon G \to G'$ induces a map of cohomology rings which is compatible with the the group actions in the evident way. In particular, the tautological homomorphism $G \to e$ induces a map $\pi\colon B\mathcal{W}_0(G) \to B\,\mathrm{Diff}_0(\Sigma)$ which is invariant under the action of $\overline{W}(G)$.

We can now take the analyticification of the resulting affine scheme over $\mathbb{C}$. Define

$$\mathcal{X}_G := [\mathrm{Spec}\, H^*(B\mathcal{W}_0(G); \mathbb{C})]_{\mathrm{an}} \smallsetminus B_G,$$

where B_G is the closed (in the analytic topology) subset consisting of $\mathbb{C}$-points p such that the composite

$$H^2(B\,\mathrm{Diff}_0(\Sigma); \mathbb{R}) \to H^2(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}) \xrightarrow{\pi^*} H^2(B\mathcal{W}_0(G); \mathbb{C}) \xrightarrow{p} \mathbb{C}$$

is *not* a bijection. Thus $\mathcal{X}_G$ is the preimage of $\mathcal{X} = \mathcal{X}_e \subset [\mathrm{Spec}\, H^*(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C})]_{\mathrm{an}} \approx \mathbb{C}^2$ with respect to the map induced by π, and is invariant under the action of $\overline{\mathcal{W}}(G) \times \mathbb{C}^\times$. Hence we obtain

$$\mathcal{M}_G := \overline{\mathcal{W}}(G) \times \mathbb{C}^\times \,\backslash\!\backslash\, \mathcal{X}_G.$$

2.11. **Products of elliptic curves vs. degree 0 torus-bundles.** Consider $G = U(1)^d$, with $d \geq 1$. As in the case of $d = 1$, we have a finite dimensional model

$$\big(\mathrm{Hom}(\mathbb{T}^2, U(1)^d) \times U(1)^d)\big) \rtimes \big(GL_2(\mathbb{Z}) \ltimes \mathbb{T}^2)\big) \xrightarrow{\sim} \mathcal{W}(U(1)^d)$$

which can be rebracketed as

$$\big(GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{d\times 2}\big) \ltimes \big(\mathbb{T}^2 \times U(1)^d\big).$$

Thus

$$H^*(B\mathcal{W}_0(U(1)^d); \mathbb{C}) \approx H^*(B(\mathbb{T}^2 \times U(1)^d); \mathbb{C}) \approx \mathbb{C}[t_1, t_2, y_1, \dots, y_d],$$

with induced action by $(\overline{\mathcal{W}}(G) \times \mathbb{C}^\times)^{\mathrm{op}}$ described much as in (2.7), except that we have

$$m \propto (t_1, t_2, y) = (t_1, t_2, y + m_1 t_1 + m_2 t_2), \qquad m = (m_1, m_2) \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^d) \approx (\mathbb{Z}^d)^2,$$

where $y = (y_1, \dots, y_d)$. Geometrically, this gives

$$GL_2(\mathbb{Z}) \rtimes \mathbb{Z}^{d\times 2} \times \mathbb{C}^\times \curvearrowright \mathcal{X}_{U(1)^d} = \mathcal{X} \times \mathbb{C}^d = \{\, (t_1, t_2, y_1, \dots, y_d) \in \mathbb{C}^2 \times \mathbb{C}^d \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \,\},$$

whence $\mathcal{M}_{U(1)^d} \to \mathcal{M}_e$ describes the the d-fold fiber product of $\mathcal{E}$ over $\mathcal{M}$.

2.12. **Remarks on the relation to double loop groups.** The construction we just described seems to be a variant of one described in [EF94]. Here we will briefly describe how to relate the two.

Fix a compact and simply connected Lie group G, with maximal torus T and Weyl group W. By analogy with loop groups, one has the *double loop group*

$$LLG := \mathrm{Map}(\mathbb{T}^2, G)$$

(where now we consider smooth maps), and also the *extended double loop group*

$$LL^{\mathrm{ext}}G := \mathrm{Map}(\mathbb{T}^2, G) \rtimes \mathbb{T}^2,$$

where the $\mathbb{T}^2$ acts by rotations. The group $LL^{\mathrm{ext}}G$ itself has an action of $\mathrm{Aut}(\mathbb{T}^2) = GL_2(\mathbb{Z})$. This describes a subgroup $LL^{\mathrm{ext}}G \rtimes GL_2(\mathbb{Z})$ of $\mathcal{W}(G)$.

The extended double loop group contains a finite dimensional torus

$$T^{\mathrm{ext}} = T_G^{\mathrm{ext}} := T \times \mathbb{T}^2,$$

where the T corresponds to constant maps $\mathbb{T}^2 \to \{*\} \to T \subseteq G$. The Weyl group of $T^{\mathrm{ext}} \subset LL^{\mathrm{ext}}G$ is the *elliptic Weyl group*

$$W_{\mathrm{Ell}} = W \ltimes \mathrm{Hom}(\mathbb{Z}^2, \check{T})$$

of G, where $\check{T}$ = the cocharacter lattice of T. The Lie algebra $\mathrm{Lie}(T^{\mathrm{ext}}) = \mathrm{Lie}(T \times \mathbb{T}^2)$ inherits an action by W_{Ell}, as well as an action by $GL_2(\mathbb{Z})$.

For the trivial group e we have $T_e^{\mathrm{ext}} = \mathbb{T}^2$, and $\mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C} \approx \mathbb{C}^2$. Let $\mathcal{X}_e \subseteq \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C}$ denote the subset consisting of pairs of elements in $\mathbb{C}$ which generate a lattice, and define $\mathcal{X}_G$ as the preimage with repsect to the evident projection π:

$$\begin{tikzcd} \mathcal{X}_G \arrow[r, tail] \arrow[d] & \mathrm{Lie}(T \times \mathbb{T}^2) \otimes \mathbb{C} \arrow[d, "\pi"] \\ \mathcal{X}_e \arrow[r, tail] & \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C} \end{tikzcd}$$

The action by $GL_2(\mathbb{Z}) \ltimes W_{\mathrm{Ell}}$ restricts to one $\mathcal{X}_G$, and acts fiberwise with respect to π, so that for each $t \in \mathcal{X}_e$ we obtain $W_{\mathrm{Ell}} \curvearrowright \pi^{-1}(t)$. When $G = T$ is itself a torus, this is evidently the same action as the one we described in the previous section, related via the Chern-Weil isomorphism

$$\mathrm{Sym}(\mathrm{Lie}(T \times \mathbb{T}^2)^* \otimes \mathbb{C}) \xrightarrow{\sim} H^*(B(T \times \mathbb{T}^2); \mathbb{C}) = H^*(B\mathcal{W}_0(G); \mathbb{C}).$$

Etingof and Frenkel [EF94] describe the following construction. Given a *simply connected* G with complexification $G_{\mathbb{C}}$, together with a choice of holomorphic structure (=complex structure + invariant holomorphic 1-form) on Σ, they describe a "coadjoint action" of $\mathrm{Map}(\Sigma, G_{\mathbb{C}})$ on $\mathrm{Lie}(\mathrm{Map}(\Sigma, G_{\mathbb{C}}))$ (actually a twisted version of the usual coadjoint action which depends on the chosen holomorphic structure on Σ). They show that orbits for this action correspond to isomorphism classes of holomorphic principal G-bundles on Σ. A generic class of orbits are given by the restriction to the maximal torus: the orbits of W_{Ell} acting on $\mathrm{Lie}(T_{\mathbb{C}})$ correspond to the "flat and unitary" holomorphic G-bundles on Σ.

Examining the formulas in Etingof and Frenkel, one sees that the holomorphic data for Σ corresponds to a choice of point $t \in \mathcal{X}_e \subset \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C}$, and that their action $W_{\mathrm{Ell}} \curvearrowright \mathrm{Lie}(T_{\mathbb{C}})$ coincides with the action of W_{Ell} on the fiber $\pi^{-1}(t) \subset \mathrm{Lie}(T \times \mathbb{T}^2) \otimes \mathbb{C}$ that we described above. (Note: in the formulation of Etingof and Frenkel, they do not identify holomorphic structures on Σ with points in $\mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C}$; rather, the use the holomorphic structure to construct a central $\mathbb{C}^\times$-extension of $\mathrm{Map}(\Sigma, \mathbb{C}^\times)$, so that their coadjoint action is the natural one on a slice of the Lie algebra of their central extension [EF94, §3].)

3. Looijenga line bundles

We now describe the main result of this paper: if in our construction we replace $G = U(1)^d$ with $\widetilde{G} = U(1)^d \times_\phi K(\mathbb{Z},2) =$ a "central" extension of $U(1)^d$ by $K(\mathbb{Z},2)$, we get the total space of the principal bundle of a Looijenga line bundle. We start with the special case of $d = 0$, i.e., $\widetilde{G} = K(\mathbb{Z},2)$.

3.1. $\widetilde{G} = K(\mathbb{Z},2)$ **gives the multiplicative group.** We describe our results in the case that $\widetilde{G} = K(\mathbb{Z},2)$. We have that

$$\overline{\mathcal{W}}(K(\mathbb{Z},2)) \approx GL_2(\mathbb{Z}) \ltimes \mathbb{Z}$$

where $GL_2(\mathbb{Z})$ acts on $\mathbb{Z} = H^2(\Sigma;\mathbb{Z})$ via the determinant. We have

$$H^*(B\mathcal{W}_0(K(\mathbb{Z},2));\mathbb{C}) \approx \mathbb{C}[t_1,t_2,x_1,x_2]/(t_1x_1+t_2x_2), \qquad t_i, x_k \in H^2.$$

The resulting action

$$(\overline{\mathcal{W}}(K(\mathbb{Z},2)) \times \mathbb{C}^\times)^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(K(\mathbb{Z},2));\mathbb{C})$$

is described by

$$\begin{aligned} &n \propto (t_1,t_2,x_1,x_2) = (t_1,t_2,x_1-nt_2,x_2+nt_1) && n\in\mathbb{Z},, \\ &A \propto (t_1,t_2,x_1,x_2) = (at_1+bt_2, ct_1+dt_2, \frac{dx_1-cx_2}{\det A}, \frac{-bx_1+ax_2}{\det A}), && A \in GL_2(\mathbb{Z}), \\ &\lambda \propto (t_1,t_2,x_1,x_2) = (\lambda t_1,\lambda t_2,\lambda x_1,\lambda x_2), && \lambda\in\mathbb{C}^\times. \end{aligned} \tag{3.2}$$

The associated geometric object is

$$\mathcal{X}_{K(\mathbb{Z},2)} = \{\, (t,x) \in \mathbb{C}^2 \mid t_1x_2 + t_2x_2 = 0,\ \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \,\} \subset \mathcal{X} \times \mathbb{C}^2.$$

The projection $\mathcal{X}_{K(\mathbb{Z},2)} \to \mathcal{X}$ is a trivial line bundle over $\mathcal{X}$, via the nowhere vanishing section $(t_1,t_2) \mapsto (t_1,t_2,-t_2,t_1)$. The action $\mathbb{Z} \curvearrowright \mathcal{X}_{K(\mathbb{Z},2)}$ is fiber-by-fiber, via translation along this section, and so acts freely. Thus

$$\mathbb{Z}\backslash\backslash\mathcal{X}_{K(\mathbb{Z},2)} \approx \mathbb{Z}\backslash\mathcal{X}_{K(\mathbb{Z},2)} \approx \mathcal{X} \times \mathbb{C}^\times.$$

Explicitly, $\mathbb{Z}\backslash\mathcal{X}_{K(\mathbb{Z},2)} \xrightarrow{\sim} \mathcal{X}\times\mathbb{C}^\times$ is given by

$$(t_1,t_2,x_1,x_2) \mapsto (t_1,t_2,e^{2\pi i(x_1/t_2)}).$$

The $GL_2(\mathbb{Z}) \times \mathbb{C}^\times$ action descends to an action on $\mathcal{X}\times\mathbb{C}^\times$ of the form

$$(A,\lambda) \propto (t_1,t_2,u) = (\lambda(at_1+bt_2), \lambda(ct_1+dt_2), u^{1/\det A}).$$

Since elements $A \in GL_2(\mathbb{Z})$ with $\det A = -1$ switch the two components of $\mathcal{X}$, we see that

$$\mathcal{M}_{K(\mathbb{Z},2)} \approx \mathcal{M} \times \mathbb{C}^\times.$$

This is naturally a group object over $\mathcal{M}$, via the group structure on $K(\mathbb{Z},2)$.

3.3. *Remark.* This will follow from the general theorem (7.6). To see how it arises, consider the Serre spectral sequence for $B\,\mathrm{Map}_0(\Sigma, K(\mathbb{Z},2)) \to B\mathcal{W}_0(K(\mathbb{Z},2)) \to B\,\mathrm{Diff}_0(\Sigma)$, which has $E_2^{p,q} = \mathbb{C}[t_1,t_2] \otimes \mathbb{C}[x_1,x_2,\epsilon]$, with $|\epsilon| = (0,3)$. The only differential is $d_2(\epsilon) = \pm(t_1x_1+t_2x_2)$. The terms $(\dots, \dots - nt_2, \dots + nt_1)$ in the first line of (3.2) ultimately derive from the non-degenerate pairing $H_1\mathbb{T}^2 \xrightarrow{\sim} H^1\mathbb{T}^2$ adjoint to the Pontryagin product on $H_*\mathbb{T}^2$.

3.4. **Quadratic functions.** Let B and C be finitely generated free abelian groups. A **quadratic function** $\phi\colon B \to C$ is a function such that

- $\beta(b,b') := \phi(b+b') - \phi(b) - \phi(b')$ is bilinear, and
- $\phi(nb) = n^2\phi(b)$ for $n \in \mathbb{Z}$.

The symmetric bilinear form $\beta\colon B \otimes B \to C$ is called the **Hessian form** of ϕ. Note that $\phi(b) = \frac{1}{2}\beta(b,b)$, so β determines ϕ.

Let $\Gamma_2 B$ be the second degree part of the divided power algebra on B; since B is 2-torsion free, $\Gamma_2 B \approx (B \otimes B)^{\Sigma_2}$. The function $\gamma_2\colon B \to \Gamma_2 B$ given by $b \mapsto b \otimes b$ is the universal quadratic function out of B, so that

$$\operatorname{Hom}(\Gamma_2 B, C) \xrightarrow[\sim]{\widetilde{\phi} \mapsto \widetilde{\phi}\circ\gamma_2} \{\text{quadratic } B \xrightarrow{\phi} C\}$$

is a bijection. We will use the notation $\widetilde{\phi}$ for the homomorphism associated to a quadratic function ϕ.

A **bilinear extension** of ϕ is any bilinear (but not necessarily symmetric) map $\omega\colon B \times B \to C$ such that $\phi(b) = \omega(b,b)$. Such extensions always exist (because the exact sequence $0 \to \Gamma_2 B \to B \otimes B \to \Lambda^2 B \to 0$ splits), and any two such extensions differ by an alternating form.

In terms of a choice of coordinates $B \approx \mathbb{Z}^d$, we have

$$\phi(y) = \tfrac{1}{2}\sum_{i,j} c_{ij} y_i y_j, \qquad \beta(y,y') = \sum_{i,j} c_{ij} y_i y'_j, \qquad \omega(y,y') = \sum_{i,j} d_{ij} y_i y'_j, \tag{3.5}$$

where (c_{ij}) is a symmetric integer matrix with $c_{ii} \in 2\mathbb{Z}$, and (d_{ij}) any integer matrix such that $c_{ij} = d_{ij} + d_{ji}$.

3.6. **Case of $\widetilde{G}$ = extension of $U(1)^d$ by $K(\mathbb{Z},2)$.** Given a topological group G and a map $\widetilde{\phi}\colon BG \to K(\mathbb{Z},4)$, we have a fibration sequence of the form

$$B\widetilde{G} \to BG \xrightarrow{\widetilde{\phi}} K(\mathbb{Z},4).$$

We define $\widetilde{G}$ to be the (based) loop space of the fiber $B\widetilde{G}$, modelled as a topological group. We call this $\widetilde{G}$ the $K(\mathbb{Z},2)$**-central extension of** G corresponding to ϕ (though as realized above the extension might not be central).

Given $G = U(1)^d$, set $B := \pi_1 G = H_2(BG,\mathbb{Z}) = \mathbb{Z}^d$, so that up to homotopy maps $\widetilde{\phi}$ correspond to elements

$$\widetilde{\phi} \in H^4(BU(1)^d;\mathbb{Z}) \approx \operatorname{Sym}^2 H^2(BU(1)^d;\mathbb{Z}) \approx \operatorname{Hom}(\Gamma_2 B,\mathbb{Z}),$$

and thus to quadratic functions $\phi\colon B \to \mathbb{Z}$.

3.7. **Theorem.** *Let $\widetilde{G}$ be a $K(\mathbb{Z},2)$-central extension of $G = U(1)^d$ associated to a quadratic function ϕ with Hessian form β, and choose a bilinear extension ω of ϕ.*

(1) *We have*

$$\overline{\mathcal{W}}(\widetilde{G}) \approx GL_2(\mathbb{Z}) \ltimes E,$$

where E is a central extension

$$0 \to \mathbb{Z} \to E \to \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d) \to 0, \tag{3.8}$$

defined so that the group law on $E = \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d) \times \mathbb{Z}$ takes the form

$$(m_1,m_2,n)\cdot(m'_1,m'_2,n') = \Big(m_1+m'_1,\ m_2+m'_2,\ n+n' + \big(\omega(m_1,m'_2) - \omega(m_2,m'_1)\big)\Big)$$

where $n,n' \in \mathbb{Z}$ and $m,m' \in \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d) \approx (\mathbb{Z}^d)^2$.

The group $GL_2(\mathbb{Z})$ acts on E (from the left) by

$$A \propto (m_1,m_2,n) = \Big(\frac{dm_1 - cm_2}{\det A},\ \frac{-bm_1 + am_2}{\det A},\ n\Big).$$

(2) *We have*

$$H^*(B\mathcal{W}_0(\widetilde{G});\mathbb{C}) \approx \mathbb{C}[t_1,t_2,y_1,\dots,y_d,x_1,x_2]/(\phi(y)+(t_1x_1+t_2x_2)), \qquad t_i,y_j,x_k\in H^2.$$

(3) *The action* $\overline{\mathcal{W}}(\widetilde{G})^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(\widetilde{G});\mathbb{C})$ *is given (in terms of the description in (1)) by*

$$\begin{aligned} n \propto (t,y,x) &= (t_1,t_2,\ y,\ x_1-nt_2, x_2+nt_1), && n\in\mathbb{Z},\\ m \propto (t,y,x) &= (t,\ y+mt,\ x-\beta(y,m)-\omega(mt,m)) && m\in \mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z}^d),\\ A \propto (t,y,x) &= (at_1+bt_2, ct_1+dt_2,\ y,\ \frac{dx_1-cx_2}{\det A}, \frac{-bx_1+ax_2}{\det A}), && A\in GL_2(\mathbb{Z}) \end{aligned} \tag{3.9}$$

where $t=(t_1,t_2)$, $y=(y_1,\dots,y_d)$, $x=(x_1,x_2)$.

3.10. *Remark.* The second line of (3.9) is in compressed form. In full it means

$$\begin{aligned}(m_1,m_2)\propto(t,y,x) = (t_1,t_2,\ & y+m_1t_1+m_2t_2,\\ & x_1-\beta(y,m_1)-\omega(m_1t_1+m_2t_2,m_1),\\ & x_2-\beta(y,m_2)-\omega(m_1t_1+m_2t_2,m_2)),\end{aligned} \tag{3.11}$$

where $m=(m_1,m_2)\in\mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z}^d)\approx(\mathbb{Z}^d)^2$.

3.12. *Remark.* Up to isomorphism, the central extension (3.8) depends only on the antisymmetrization of the 2-cocycle $\gamma(m,m')=\omega(m_1,m_2')-\omega(m_2,m_1')$, which is $\gamma_{\mathrm{antisym}}(m,m')=\gamma(m,m')-\gamma(m',m)=\beta(m_1,m_2')-\beta(m_2,m_1')$, and thus depends only on ϕ, not on ω.

The corresponding geometric object $\mathcal{X}_{\widetilde{G}}\subseteq\mathcal{X}\times\mathbb{C}^d\times\mathbb{C}^2$ is the locus of $t_1x_1+x_2t_2=-\phi(y)$, subject to $\mathbb{R}t_1+\mathbb{R}t_2=\mathbb{C}$. The free quotient $\mathbb{Z}\backslash\mathcal{X}_{\widetilde{G}}$ is a principal $\mathbb{C}^\times$-bundle over $\mathcal{X}\times\mathbb{C}^d$. Thus, $\mathcal{M}_{\widetilde{G}}$ is the total space of a principal $\mathbb{C}^\times$-bundle over $\mathcal{E}^d$.

In fact, let us consider the quotient of $\mathcal{X}_{\widetilde{G}}$ under the free action by $\mathbb{Z}\times\mathbb{C}^\times$. Explicitly,

$$(\mathbb{Z}\times\mathbb{C}^\times)\backslash\mathcal{X}_{\widetilde{G}} \xrightarrow{\sim} (\mathbb{C}\smallsetminus\mathbb{R})\times\mathbb{C}^d\times\mathbb{C}^\times$$

is given by

$$(t_1,t_2,\ y_1,\dots,y_d,\ x_1,x_2)\mapsto(\tfrac{t_1}{t_2},\ \tfrac{y_1}{t_2},\dots,\tfrac{y_d}{t_2},\ e^{2\pi i(x_1/t_2)})=(\tau,\ z_1,\dots,z_d,\ u).$$

The $\overline{\mathcal{W}}(G)$-action descends to an action by $\mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z}^d)\rtimes GL_2(\mathbb{Z})$ on the quotient, given by

$$\begin{aligned} m\propto(\tau,z,u) &= (\tau,\ z+m_1\tau+m_2,\ ue^{2\pi i[-\beta(z,m_1)-\phi(m_1)\tau]}), && m\in\mathrm{Hom}(\mathbb{Z}^2,\mathbb{Z}^d),\\ A\propto(\tau,z,u) &= (A\tau,\ (c\tau+d)^{-1}z,\ u^{1/\det A}e^{2\pi i(1/\det A)[c(c\tau+d)^{-1}\phi(z)]}), && A\in GL_2(\mathbb{Z}), \end{aligned} \tag{3.13}$$

where $A\tau=(a\tau+b)/(c\tau+d)$ and $z=(z_1,\dots,z_d)$. This describes the principal $\mathbb{C}^\times$-bundle over $\mathcal{E}^{\times d}$ whose associated line bundle has as sections $\theta(\tau,z)$ such that (for $\mathrm{Im}(\tau)>0$ and $A\in SL_2(\mathbb{Z})$)

$$\begin{aligned}\theta(\tau,\ z+m_1\tau+m_2) &= \theta(\tau,z)\,e^{2\pi i[-\beta(z,m_1)-\phi(m_1)\tau]},\\ \theta(A\tau,\ (c\tau+d)^{-1}z) &= \theta(\tau,z)\,e^{2\pi i[c(c\tau+d)^{-1}\phi(z)]}.\end{aligned}$$

In other words, we obtain the *Looijenga line bundle* associated to the quadratic form ϕ [Loo76].

3.14. *Remark.* Suppose $\phi\colon B=\mathbb{Z}^d\to\mathbb{Z}$ is a non-degenerate quadratic function. Then with our conventions, the line bundle L_ϕ associated to ϕ, admits a non-trivial *holomorphic* section over C_τ^d (for any chosen $C_\tau:=\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ with $\mathrm{Im}(\tau)>0$) if and only if ϕ is *positive definite.* The main example of interest is the positive definite quadratic function ϕ associated to the Killing form on the coroot lattice of a simply connected compact Lie group; in this case ϕ is invariant under the action of the Weyl group, so the bundle L_ϕ is equivariant for the Weyl group.

To see the existence of sections in this case, we use [Mum70, I.2 and I.3]. In the notation of [Mum70, I.2, p. 15–16; I.3, p. 24–25], the line bundle $L_\phi|C_\tau^d$ is described by a 1-cocycle e on $U = \mathbb{Z}^d\tau + \mathbb{Z}^d \subseteq \mathbb{C}^d$ with coefficients in holomorphic functions $\mathbb{C}^d \to \mathbb{C}^\times$, given by

$$e_u(z) = e^{2\pi i f_u(z)}, \qquad f_{m_1\tau+m_2}(z) = -\beta(z, m_1) - \tfrac{1}{2}\beta(m_1, m_1)\tau, \qquad m_1, m_2 \in \mathbb{Z}^d.$$

By [Mum70, I.2, p. 18, Proposition],

$$E(u, u') := f_{u'}(z + u) + f_u(z) - f_u(z + u') - f_{u'}(z), \quad \text{any } z \in \mathbb{C}^d,$$

defines an alternating 2-form $E \colon U \times U \to \mathbb{Z}$ which represents the Chern class of $L_\phi|C_\tau^d$. We calculate that in our case,

$$E(m_1\tau + m_2, m_1'\tau + m_2') = \beta(m_1, m_2') - \beta(m_2, m_1').$$

Extend E to an $\mathbb{R}$-linear form $\mathbb{C}^d \times \mathbb{C}^d \to \mathbb{R}$ and set $H(x, y) := E(ix, y) + iE(x, y)$. Then H is a Hermitian form with $\operatorname{Im} H = E$. By ([Mum70, I.3, p. 26], proposition and preceding discussion), if H is non-degenerate, then $L_\phi|C_\tau^d$ admits non-zero holomorphic sections if and only if H is positive definite, in which case $\dim H^0(C_\tau^d, L_\phi|C_\tau^d) = \sqrt{\det E}$ (express E as a matrix using a $\mathbb{Z}$-basis of U). We calculate that in our case,

$$H(x, x) = (\operatorname{Im}\tau)^{-1}\beta(x, \overline{x}), \qquad x \in \mathbb{C}^d.$$

As $\operatorname{Im}\tau > 0$ and $\beta(x, y) = \sum c_{ij}x_iy_j$ is a symmetric form on $\mathbb{C}^d$ with $c_{ij} \in \mathbb{Z} \subseteq \mathbb{R}$, we see that H is non-degenerate/positive definite on $\mathbb{C}^d$ if and only if β is non-degenerate/positive definite on $\mathbb{R}^d$, and if so we have $\sqrt{\det E} = \det(c_{ij})$.

3.15. *Remark.* For $\phi\colon B \approx \mathbb{Z}^d \to \mathbb{Z}$ positive definite, sections θ_u of $L_\phi|C_\tau^d$ are given by $\theta_u(\tau, z) = \sum_{v\in B} e^{2\pi i[-\beta(z,u+v)+\phi(u+v)\tau]}$ for $u \in B \otimes \mathbb{R}$ such that $\beta(u, B) \subseteq \mathbb{Z}$, [Loo76, §4].

3.16. **Proof of the theorem.** We will derive (3.7) from a more general (and coordinate invariant) statement (7.6), whose setup and proof takes up §§6–10. It is entirely calculational, and amounts to completely describing the homotopy type of the spaces $B\mathcal{W}(\widetilde{G})$. In particular, the key is to compute all Whitehead products in the homotopy groups of this space.

We note that one can instead regard $\widetilde{G}$ as arising from a *Lie 2-group*, specifically as a 2-group extension as considered in [Gan15]. It seems likely that 2-group methods should lead to a more informative proof of the results shown here.

4. Isogenies

We describe how, according to the picture of the previous sections, finite coverings of genus 1 surfaces correspond to isogenies of elliptic curves.

Fix a finite covering map $f\colon \Sigma' \to \Sigma$ between two surfaces. Let $\operatorname{Diff}(f) \subset \operatorname{Diff}(\Sigma) \times \operatorname{Diff}(\Sigma')$ denote the group of pairs of diffeomorphisms compatible with f. We note that the projection map $\operatorname{Diff}(f) \xrightarrow{t} \operatorname{Diff}(\Sigma)$ is a finite covering map, while the projection map $\operatorname{Diff}(f) \xrightarrow{s} \operatorname{Diff}(\Sigma')$ is injective and induces a homotopy equivalence between $\operatorname{Diff}(f)$ and a union of path components of $\operatorname{Diff}(\Sigma')$, corresponding to a finite index subgroup of $\pi_0 \operatorname{Diff}(\Sigma')$.

Given any group G, we can form a diagram as follows

$$\begin{tikzcd}[column sep=small]
B\mathcal{W}^{\Sigma}(G) \arrow[d] & (Bt)^*B\mathcal{W}^{\Sigma}(G) \arrow[l] \arrow[r, bend left, "f^*"] \arrow[dr] & & (Bs)^*B\mathcal{W}^{\Sigma'}(G) \arrow[r] \arrow[dl] & B\mathcal{W}^{\Sigma'}(G) \arrow[d] \\
B\operatorname{Diff}(\Sigma) & & B\operatorname{Diff}(f) \arrow[ll, "Bt"'] \arrow[rr, "Bs"'] & & B\operatorname{Diff}(\Sigma')
\end{tikzcd} \tag{4.1}$$

where the trapezoids are homotopy pullbacks. That is, $(Bt)^*B\mathcal{W}^\Sigma(G) \approx B(\mathrm{Map}(\Sigma, G) \rtimes \mathrm{Diff}(f))$ and $(Bs)^*B\mathcal{W}^{\Sigma'}(G) \approx B(\mathrm{Map}(\Sigma', G) \rtimes \mathrm{Diff}_+(f))$, while the map labeled f^* is obtained from the map $\mathrm{Map}(\Sigma, G) \to \mathrm{Map}(\Sigma', G)$ given by restriction along f.

The observation is that, after applying the construction of §2.10, the map f^* presents an isogeny of curves, of degree equal to the degree of f. To see this, we consider an explicit example.

4.2. *Example.* Fix $\Sigma = \Sigma' = \mathbb{T}^2$, and let $f\colon \Sigma' \to \Sigma$ be the map induced by left multiplication by some integer matrix B. Set $\Gamma_B := GL_2(\mathbb{Z}) \cap B^{-1}\, GL_2(\mathbb{Z})\, B$. Note that, using a suitable choice of bases of $H_1\Sigma$ and $H_1\Sigma'$, the matrix B can be given the form $B = \left(\begin{smallmatrix} M & 0 \\ 0 & MN \end{smallmatrix}\right)$ for some $M, N \geq 1$, in which case $\Gamma_B = \Gamma_0(N) = \{\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}) \mid c \equiv 0 \mod N \,\}$.

Then there is a weak equivalence of topological groups

$$\Gamma_B \ltimes \mathbb{T}^2 \xrightarrow{\sim} \mathrm{Diff}(f),$$

so that the projections $\mathrm{Diff}(\Sigma) \leftarrow \mathrm{Diff}(f) \to \mathrm{Diff}(\Sigma')$ correspond to

$$GL_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xleftarrow{(BAB^{-1}, Bt) \leftarrow\!\shortmid (A,t)} \Gamma_B \ltimes \mathbb{T}^2 \xrightarrow{(A,t) \mapsto (A,t)} GL_2(\mathbb{Z}) \ltimes \mathbb{T}^2.$$

Let $G = U(1)$, form $\mathrm{Spec}_{\mathrm{an}}$ of the cohomology of universal covers of objects in (4.1), and restrict to the subset $\mathcal{X} = \{\, (t_1, t_2) \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \,\}$. Together with actions of fundamental groups and the grading action by $\mathbb{C}^\times$, the middle triangle of (4.1) is seen to have the form

$$\begin{tikzcd} (\Gamma_B \ltimes' \mathbb{Z}^2) \times \mathbb{C}^\times \backslash\!\backslash \mathcal{X} \times \mathbb{C} \arrow[rr, "f^*"] \arrow[dr] & & (\Gamma_B \ltimes \mathbb{Z}^2) \times \mathbb{C}^\times \backslash\!\backslash \mathcal{X} \times \mathbb{C} \arrow[dl] \\ & \Gamma_B \times \mathbb{C}^\times \backslash\!\backslash \mathcal{X} & \end{tikzcd}$$

where f^* is induced by the identity on $\mathcal{X} \times \mathbb{C}$, and both maps $\mathcal{X} \times \mathbb{C} \to \mathcal{X}$ are the evident projection. The semidirect product $\Gamma_B \ltimes \mathbb{Z}^2$ is induced by the tautological action $\Gamma_B \subset GL_2(\mathbb{Z})$, while the semidirect product $\Gamma_B \ltimes' \mathbb{Z}^2$ is induced by the homomorphism $A \mapsto BAB^{-1}\colon \Gamma_B \to GL_2(\mathbb{Z})$. The action in the upper-right corner is

$$A \propto (t, y) = (At, y), \quad m \propto (t, y) = (t, y + mt), \quad \lambda \propto (t, y) = (\lambda t, \lambda y),$$

while the action in the upper-left corner is

$$A \propto (t, y) = (At, y), \quad m \propto (t, y) = (t, y + mBt), \quad \lambda \propto (t, y) = (\lambda t, \lambda y)$$

where $A \in \Gamma_B$, $m \in \mathbb{Z}^2$ (treated as a row vector), $t \in \mathcal{X}$ (treated as a column vector), $y \in \mathbb{C}$, and $\lambda \in \mathbb{C}^\times$.

Thus, in the "fibers" over $(t_1, t_2) \in \mathcal{X}$ we obtain (after taking quotients by $\mathbb{Z}^2$-actions) the projection $\mathbb{C}/\big((Bt)_1\mathbb{Z} + (Bt)_2\mathbb{Z}\big) \to \mathbb{C}/\big(t_1\mathbb{Z} + t_2\mathbb{Z}\big)$, an isogeny of degree $\det B$. E.g., for $B = \left(\begin{smallmatrix} M & 0 \\ 0 & MN \end{smallmatrix}\right)$ we get $\mathbb{C}/\big(Mt_1\mathbb{Z} + MNt_2\mathbb{Z}\big) \to \mathbb{C}/(t_1\mathbb{Z} + t_2\mathbb{Z})$.

5. Remarks on the formalism

5.1. **Remarks on the construction of an equivariant cohomology theory.** We can easily produce for each group G that we consider an equivariant cohomology theory of the form

$$E_G^*\colon \big(G\text{-CW-complexes}\big)^{\mathrm{op}} \to \big(\overline{\mathcal{W}}(G)\text{-equivariant } H^*(B\mathcal{W}_0(G); \mathbb{C})\text{-algebras}\big).$$

Given a G-space X let

$$\mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, X) \subseteq \mathrm{Map}_G(\Sigma \times G, X)$$

be the subspace consisting of **ghost maps**, i.e., G-equivariant maps $f\colon \Sigma \times G \to X$ such that $f(\Sigma \times G)$ is contained in a single G-orbit. The ghost maps are invariant under the evident action of $\mathcal{W}(G)$ on $\mathrm{Map}_G(G \times \Sigma, X)$, so we can define

$$E_G^*(X) := \big(\overline{\mathcal{W}}(G)^{\mathrm{op}} \curvearrowright H^*(\mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, X)_{h\mathcal{W}_0(G)}; \mathbb{C})\big).$$

That this is a cohomology theory amounts to the observations that (i) $X \mapsto \mathrm{Map}^{\mathrm{gh}}_G(\Sigma \times G, X)$ preserves pushouts along cofibrations and (ii) $\mathrm{Map}^{\mathrm{gh}}_G(\Sigma \times G, T \times X) \approx T \times \mathrm{Map}^{\mathrm{gh}}_G(\Sigma \times G, X)$ when T has trivial G-action.

5.2. *Example.* Let $G = U(1)$ and $X = U(1)/\mu_N$. Then

$$E^*_{U(1)}(U(1)/\mu_N) \approx \prod_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{C}[t_1, t_2, y]/(y - (n_1/N)t_1 - (n_2/N)t_2),$$

which is an algebra over $H^*(B\mathcal{W}_0(U(1)); \mathbb{C}) \approx \mathbb{C}[t_1, t_2, y]$ in the obvious way and which carries an evident compatible action by $\overline{\mathcal{W}}(U(1)) = GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

Ideally one would like to "analytify" the equivariant module $E^*_G(X)$, to obtain a sheaf of $\mathcal{O}_{\mathcal{M}_G}$-algebras on $\mathcal{M}_G$, to be coherent at least if X is a finite G-CW-complex; we would then hope to take it as a model for Grojnowski's equivariant elliptic cohomology. Unfortunately, the most obvious way to do this (e.g., by tensoring up from algebraic to holomorphic functions), though exact, behaves poorly on most $E^*_G(X)$ (which are often non-Noetherian, even when X is a G-orbit).

5.3. **Remarks on derived constructions.** In this paper we have been content to produce examples of "classical" geometric objects, e.g., complex analytic spaces. However, we know that elliptic cohomology wants to take values in sheaves on a *derived* geometric object, along the lines of [Lur09]. I don't know how to make such a derived construction; however, I'll give some speculation here.

Fix a commutative dga[5] $\mathbb{C}[u^{\pm}]$, where $|u| = 2$ with $du = 0$. This admits an evident grading coaction by the Hopf algebra $\mathbb{C}[\lambda^{\pm}]$ with $|\lambda| = 0$ and $d(\lambda) = 0$, by $u \mapsto \lambda \otimes u$. Thus $\mathbb{G}_m := \operatorname{Spec}^{\mathrm{der}} \mathbb{C}[\lambda^{\pm}]$ acts on $\operatorname{Spec}^{\mathrm{der}} \mathbb{C}[u^{\pm}]$.

For a space X, let C^*X denote a functorial commutative dga model for the cochains on X with $\mathbb{C}$ coefficients; e.g., we could take C^*X to be the PL-de Rham forms on X. Thus $\operatorname{Spec}^{\mathrm{der}} C^*X \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm}]$ inherits an action by $\mathbb{G}_m$. We can then plug in $\overline{W}(G) \curvearrowright B\mathcal{W}_0(G)$ as above to obtain

$$\overline{W}(G) \times \mathbb{G}_m \curvearrowright \operatorname{Spec}^{\mathrm{der}} C^*B\mathcal{W}_0(G) \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm}],$$

a derived scheme equipped with an action by a group scheme.

At this point we posit the existence of a *derived analytification* functor $\operatorname{Spec}^{\mathrm{der}}_{\mathrm{an}}$, which takes as input a commutative dga A^* over $\mathbb{C}$, as gives as output a derived complex analytic space $\operatorname{Spec}^{\mathrm{der}}_{\mathrm{an}} A^* = (X, \mathcal{O})$, in some suitable ∞-category $\mathrm{An}^{\mathrm{der}}$ of derived analytic spaces. It should have the property that (at least in the examples we care about), the underlying complex analytic space $(X, H^0(\mathcal{O}))$ is equivalent to $[\operatorname{Spec} H^0 A^*]_{\mathrm{an}}$. Given this, we would could then proceed to construct derived versions of $\mathcal{X}_G$ and $\mathcal{M}_G$ as desired.

An ∞-category $\mathrm{An}^{\mathrm{der}}$ has been constructed in work of Lurie [Lur11] and Porto [Por15], and in fact comes equipped with an analytification functor. A significant issue in carrying out this program (as pointed out to me by Mauro Porto) is that the "rings" which appear in this model are fundamentally (-1)-connected objects, whereas the rings we want to consider are naturally non-connected, and in fact are generally 2-periodic.

5.4. **Remarks on functoriality.** As we have described it, our construction $G \mapsto \mathcal{M}_G$ is functorial with respect to homomorphisms of groups. For a derived version of this construction it is highly desirable to have an enhanced "stacky" version of this functoriality, where homomorphisms are enriched to maps between classifying spaces (not necessarily basepoint preserving), i.e., we should have

$$\mathrm{Map}_{\mathrm{Top}}(BG, BG') \to \mathrm{Map}_{\mathrm{An}^{\mathrm{der}}}(\mathcal{M}^{\mathrm{der}}_G, \mathcal{M}^{\mathrm{der}}_{G'}).$$

[5] We use homological grading here, so $x \in C^q$ has $|x| = -q$.

This extneded functoriality should apply not just to tori but to $K(\mathbb{Z},2)$-extensions of them, and thus should be consistent with Lurie's notion of *2-equivariance* [Lur09, §5]. I'll briefly indicate how to achieve this; it may be enlightening even in the non-derived case.

Consider $\operatorname{Map}(\Sigma, BG)$, the space parameterizing principal G-bundles over Σ. There is a distinguished path component $\operatorname{Map}_0(\Sigma, BG) \subseteq \operatorname{Map}(\Sigma, BG)$ corresponding to trivializable bundles, which is equivalent to $B\operatorname{Map}(\Sigma, G)$. There is a corresponding path component

$$\mathcal{P}(BG) \subseteq \operatorname{Map}(\Sigma, BG)_{h\operatorname{Diff}(\Sigma)}$$

equivalent to $B\mathcal{W}(G)$. Note that a map $BG \to BG'$ sends $\mathcal{P}(BG) \to \mathcal{P}(BG')$.

Thus, "enhanced functoriality" follows once we describe how to functorially obtain a derived stack from a suitable connected space X (such as $X = \mathcal{P}(BG)$).

For each path connected space X, make an arbitrary choice of universal cover $p\colon \widetilde{X} \to X$, and write G for its group of deck transformations. Note that p is a principal G-bundle. We get a topological quotient stack $G \backslash\backslash \widetilde{X}$ which is equivalent to X. After taking cohomology we obtain a total quotient stack $G \backslash\backslash \operatorname{Spec} H^*\widetilde{X}$; replacing cohomology with cochains gives the corresponding derived object. When $X = \mathcal{P}(BG)$ this recovers the construction $\overline{W}(G) \backslash\backslash H^*B\mathcal{W}_0(G)$.

Consider a map $f\colon X \to Y$ to another path connected space, we and write $(\widetilde{Y}, q, H)$ for the analogous choices for Y. Then f induces a map $G \backslash\backslash \operatorname{Spec} H^*\widetilde{X} \to H \backslash\backslash \operatorname{Spec} H^*\widetilde{Y}$ of stacks which is represented by a *bibundle*, as follows. Consider

$$\widetilde{X} \xleftarrow{\pi} \operatorname{Lift}(f) \times \widetilde{X} \xrightarrow{\epsilon} \widetilde{Y}$$

where $\operatorname{Lift}(f) = \{\, \widetilde{f}\colon \widetilde{X} \to \widetilde{Y} \mid q\widetilde{f} = fp \,\}$ is the set of lifts of f to the universal covers, π is the projection map, and ϵ is the evaluation map. We have

- G acts on $\operatorname{Lift}(f) \times \widetilde{X}$ from the left by $g \propto (\widetilde{f}, \widetilde{x}) = (\widetilde{f}g^{-1}, g\widetilde{x})$,
- H acts on $\operatorname{Lift}(f) \times \widetilde{X}$ from the left by $h \propto (\widetilde{f}, \widetilde{x}) = (h\widetilde{f}, \widetilde{x})$,
- the group actions on $\operatorname{Lift}(f) \times \widetilde{X}$ commute,
- π is equivariant with respect to G and H (where H acts trivially on $\widetilde{X}$),
- ϵ is equivariant with respect to G and H (where G acts trivially on $\widetilde{Y}$), and
- π describes a *G-equivariant principal H-bundle* over $\widetilde{X}$.

That is, the diagram describes a bibundle from the topological groupoid $G \backslash\backslash \widetilde{X}$ to $H \backslash\backslash \widetilde{Y}$; i.e, it is a "stacky" presentation of f in terms of the chosen covers.

Taking cohomology (or in the derived context, cochains) gives

$$\operatorname{Spec} H^*\widetilde{X} \xleftarrow{\pi} \operatorname{Lift}(f) \times \operatorname{Spec} H^*\widetilde{X} \xrightarrow{\epsilon} \operatorname{Spec} H^*\widetilde{Y},$$

exhibiting a bibundle between groupoid schemes, i.e., representing a map $G \backslash\backslash \operatorname{Spec} H^*\widetilde{X} \to H \backslash\backslash \operatorname{Spec} H^*\widetilde{Y}$ of stacks.

5.5. *Example.* Applying this to our set-up in the case of a map $f\colon * \approx Be \to BU(1)$ gives

$$\mathcal{X} \xleftarrow{\pi} (GL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathcal{X} \xrightarrow{\epsilon} \mathcal{X} \times \mathbb{C}$$

with $\epsilon((B,n),t) = (Bt, nBt)$. The groups $G = GL_2(\mathbb{Z})$ and $H = GL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$ act on $\mathbb{Z}^2 \times \mathcal{X}$ by $A \cdot ((B,n),t) = ((BA^{-1}, n), At)$ and $((B,n),t) \cdot (A', m) = ((A'B, (n+m)A'^{-1}), t)$.

5.6. **Remarks on the 1-dimensional case.** We can carry out the analogue of our constructions in the case that Σ is a circle rather than a torus. The relevant calculations can be read off from (7.6). The main differences are that in this case we take

$$\mathcal{X} = \mathcal{X}_e = \{\, t \in \mathbb{C} \mid t \neq 0 \,\},$$

the set of vectors which generate a rank 1 lattice in $\mathbb{C}$. Then we easily discover that $\mathcal{M}_{U(1)}$ is the "universal multiplicative group" living over $\mathcal{M}_e \approx (\mathrm{Aut}(\mathbb{G}_m) \,\backslash\backslash\, *) \approx (\{\pm 1\} \,\backslash\backslash\, *)$. The central extension groups turn out to be invisible from this point of view, since $\mathcal{M}_{K(\mathbb{Z},2)} \approx \mathcal{M}_e$.

6. Some spaces and groups

The spaces $X = B\mathcal{W}_0(\widetilde{G})$ that we need to deal with are simply connected 3-types such that π_2 and π_3 are finitely generated and free. We first discuss some general facts and conventions about such spaces, concluding with the calculation of $H^*(X;\mathbb{Q})$ in terms of the Whitehead product in $\pi_* X$ in good cases; all of this material is surely standard. We next describe an explicit topological group model for the central extensions $\widetilde{G} = U(1)^d \times_\phi K(2,\mathbb{Z})$ that we need to consider.

6.1. **Simply connected 3-types with all homotopy groups finitely generated and free.** Let $\mathcal{C}$ denote the full subcategory of spaces X which are (i) simply connected, (ii) have $\pi_k X \approx 0$ for $k \geq 4$, and (iii) have $\pi_2 X$ and $\pi_3 X$ which are finitely generated free abelian groups. Write $h\mathcal{C}$ for the associated homotopy category.

For $X \in \mathcal{C}$ with $\pi_2 X = B$ and $\pi_3 X = C$, the Whitehead product (10.7) $[-,-]\colon \pi_2 X \times \pi_2 X \to \pi_3 X$ defines a bilinear symmetric form

$$\beta\colon B \otimes B \to C.$$

Precomposition $\circ\eta\colon \pi_2 X \to \pi_3 X$ with the Hopf map $\eta \in \pi_3 S^2$ is a function

$$\phi\colon B \to C,$$

quadratic in the sense of §3.6, which satisfies

$$\phi(y + y') = \phi(y) + \beta(y, y') + \phi(y').$$

(This identity fixes our preferred choice of generator η of $\pi_3 S^2$.) We call ϕ the **quadratic invariant** of X, and β the associated **Hessian form**.

It is classical that the data of (B, C, ϕ) is a complete invariant for the homotopy type of $X \in \mathcal{C}$. In fact, $X \mapsto \phi$ defines an equivalence between the homotopy category $h\mathcal{C}$ of such spaces, and the category of quadratic functions between finitely generated free groups.

Let $i\colon K(C,3) \to X$ and $j\colon X \to K(B,2)$ be maps, unique up to homotopy, which induce identity on the relevant homotopy groups. We can furthermore extend to a fibration sequence

$$X \xrightarrow{j} K(B,2) \xrightarrow{\psi} K(C,4),$$

i.e., so that j is identified with the tautological map from the homotopy fiber of ψ.

6.2. **Proposition.** *Let $X \in \mathcal{C}$ with quadratic invariant ϕ and Hessian form β, and consider $b, b' \in B = \pi_2 X$. There exists a homotopy commutative diagram*

$$\begin{tikzcd} S^3 \arrow[r, "w"] \arrow[d, "f"'] & S^2 \vee S^2 \arrow[r, tail] \arrow[d, "(b,b')"'] & S^2 \times S^2 \arrow[r, two heads] \arrow[d, "g"'] & S^4 \arrow[d, "\widetilde{f}"'] \\ K(C,3) \arrow[r, "i"'] & X \arrow[r, "j"'] & K(B,2) \arrow[r, "\psi"'] & K(C,4) \end{tikzcd}$$

where w is the universal Whitehead product, and $\widetilde{f}\colon S^4 \approx S^3 \wedge S^1 \to K(C,4)$ is adjoint to $f\colon S^3 \to \Omega K(C,4) \approx K(C,3)$. Furthermore,

(1) *$g_*\colon H_4(S^2 \times S^2) \to H_4 K(B,2) \approx \Gamma_2 B \subseteq (B \otimes B)^{\Sigma_2}$ sends $[S^2] \times [S^2] \mapsto b \otimes b' + b' \otimes b$,*
(2) *$\widetilde{f}_*\colon H_4 S^4 \to H_4 K(C,4) \approx C$ sends $[S^4] \mapsto \beta(b,b')$, and*
(3) *$\psi_*\colon H_4 K(B,2) \approx \Gamma_2 B \to H_4 K(C,4) \approx C$ coincides with $\widetilde{\phi}\colon \Gamma_2 B \to C$, the homomorphism associated to ϕ as defined in* (3.4).

Proof. We are using the tautological identification $H_4K(B,2) \approx (B\otimes B)^{\Sigma_2}$ dual to $H^4K(B,2) \approx (B\otimes B)_{\Sigma_2}$ defined by the cup product. With respect to this identification, the H-space structure on $K(B,2)$ induces a Pontryagin product $H_2K(B,2)\otimes H_2K(B,2) \to H_4K(B,2)$ given by $b\otimes b' \mapsto b\otimes b' + b'\otimes b\colon B\otimes B \to \Gamma_2(B)$.

The construction of the diagram is straightforward. In particular, we can use the H-space structure on $K(B,2)$ to define g as the composite $S^2\times S^2 \xrightarrow{b\times b'} K(B,2)\times K(B,2) \to K(B,2)$, from which statement (1) follows immediately. Statement (2) is immediate from the fact that f and $\widetilde{f}$ are adjoint, and that $if = [b,b']\colon S^3 \to X$. Statement (3) then follows from the commutativity of the diagram. □

Thus, any $X \in \mathcal{C}$ is the homotopy fiber of the characteristic class in $H^4(K(B,2),C)$ corresponding to its quadratic invariant.

6.3. **Rational cohomology ring of $X \in \mathcal{C}$.** Say that a quadratic function $\phi\colon B \to C$ is **regular** if the function

$$\widetilde{\phi}^*\colon C^*\otimes\mathbb{Q} \to \operatorname{Hom}(\Gamma_2 B,\mathbb{Q}) \approx \operatorname{Sym}^2(B^*\otimes\mathbb{Q})$$

dual to $\widetilde{\phi}\colon \Gamma_2 B \to C$ sends some basis of $C^*\otimes\mathbb{Q}$ to a regular sequence in the ring $\operatorname{Sym}(B\otimes\mathbb{Q})^*$.

6.4. *Remark.* If $B = \mathbb{Z}^d$, $C = \mathbb{Z}^e$, and $\phi(y) = (\phi_1(y),\dots,\phi_e(y))$ with $\phi_k(y) = \frac{1}{2}\sum_{i,j} c^k_{ij} y_i y_j$, then ϕ is regular if and only if the sequence of polynomials $\phi_1(y),\dots,\phi_e(y)$ form a regular sequence in $\mathbb{Q}[y_1,\dots,y_d]$.

In particular, if $e=1$, then ϕ is regular if and only if $\phi \neq 0$.

6.5. **Proposition.** *Let $X \in \mathcal{C}$ with quadratic invariant ϕ. Then the map $j^*\colon H^*(K(B,2);\mathbb{Q}) \to H^*(X;\mathbb{Q})$ factors through*

$$\operatorname{Sym}(B^*\otimes\mathbb{Q})/(\widetilde{\phi}^*(C^*\otimes\mathbb{Q})) \to H^*(X;\mathbb{Q})$$

where $\widetilde{\phi}^*\colon C^* \to (\Gamma_2 B)^*$ is the $\mathbb{Z}$-dual to $\widetilde{\phi}$. Furthermore, the above map is an isomorphism of rings when ϕ is regular.

Proof. The Serre spectral sequence for the fibration sequence $K(C,3) \xrightarrow{i} F \xrightarrow{j} K(B,2)$ has

$$E_2 = E_4 = H^*(K(B,2); H^*(K(C,3);\mathbb{Q})) \approx \operatorname{Sym}(B^*\otimes\mathbb{Q})\otimes\Lambda(C^*\otimes\mathbb{Q}).$$

The first non-trivial differential is $d_4\colon E_4^{0,3} \to E_4^{4,0}$, which must be $\pm\widetilde{\phi}^*$ by (6.2). The regularity condition is what is needed for $E_5^{p,q}$ with $q>0$ to vanish, so that the spectral sequence collapses to $E_\infty^{*,*} = E_5^{*,0}$.

□

6.6. **An explicit group model for central extensions.** Every space $X \in \mathcal{C}$ is equivalent to the classifying space of a topological group. We give an explicit construction of such a group as a central extension. In particular, given a bilinear map $\omega\colon B\otimes B \to C$ between finitely generated free groups, we construct a topological group G_ω so that $X = BG_\omega \in \mathcal{C}$ has quadratic invariant ϕ with ω as its bilinear extension, and thus sits in a fiber sequence $K(C,3) \to X \to K(B,2) \xrightarrow{\widetilde{\phi}} K(C,4)$. In particular, this produces an explicit model for our extension groups $U(1)^d \times_\phi K(\mathbb{Z},2)$.

Let $K(B,1)_\bullet$ and $K(C,2)_\bullet$ be simplicial abelian groups, degreewise free, together with identifications $B \approx \pi_1 K(B,1)_\bullet$ and $C \approx \pi_2 K(C,2)_\bullet$, and all other homotopy groups trivial. There exists a map

$$\kappa\colon K(B,1)_\bullet \otimes K(B,1)_\bullet \to K(C,2)_\bullet$$

of simplicial abelian groups inducing ω on π_2, which is unique up to homotopy. We fix such a choice of κ.

Consider the composite map of simplicial sets

$$K(B,1)_\bullet \times K(B,1)_\bullet \xrightarrow{(x,y)\mapsto x\otimes y} K(B,1)_\bullet \otimes K(B,1)_\bullet \xrightarrow{\kappa} K(C,2)_\bullet.$$

Taking geometric realization produces a map of spaces which we also denote

$$\kappa\colon K(B,1)\times K(B,1)\to K(C,2),$$

which is a bilinear map between topological abelian groups (and so factors through $K(B,1)\wedge K(B,1)$). Let G_ω be the space $K(B,1)\times K(C,2)$ with group law[6]

$$(y,x)\cdot(y',x') := (y+y', -\kappa(y,y')+x+x'), \qquad (y,x),(y',x')\in G_\omega.$$

Note that inversion in G_ω is given by

$$(y,x)^{-1} = (-y, -\kappa(y,y)-x),$$

while the commutator is given by

$$(y,x)\cdot(y',x')\cdot(y,x)^{-1}\cdot(y',x')^{-1} = (0, -\kappa(y,y')+\kappa(y',y)). \tag{6.7}$$

Thus G_ω is a central extension of $K(B,1)$ by $K(C,2)$, and we have evident isomorphisms $\pi_1 G_\omega \approx B$ and $\pi_2 G_\omega \approx C$.

The commutator $G_\omega\wedge G_\omega\to G_\omega$ defines the Samelson product

$$\langle -,-\rangle\colon \pi_p G_\omega\times\pi_q G_\omega\to\pi_{p+q}G_\omega.$$

6.8. **Proposition.** *The Samelson product $\pi_1G_\omega\times\pi_1G_\omega\to\pi_2G_\omega$ is given by*

$$\langle b,b'\rangle = -\omega_{\mathrm{sym}}(b,b') := -\omega(b,b')-\omega(b',b).$$

Proof. The map $\kappa\colon K(B,1)\wedge K(B,1)\to K(C,2)$ induces ω on homotopy groups by construction, and therefore $(y,y')\mapsto\kappa(y',y)$ induces $(b,b')\mapsto-\omega(b',b)$ on homotopy groups, with sign introduced by switching the order of the two classes in π_1G_ω. The result follows from (6.7). □

6.9. **Proposition.** *Let $X=BG_\omega$. Then the Whitehead product $\pi_2X\times\pi_2X\to\pi_3X$ is given by*

$$[b,b'] = \omega_{\mathrm{sym}}(b,b') = \omega(b,b')+\omega(b',b).$$

Proof. This is a special case of the relation between the Whitehead and Samelson products (10.9); in this dimension, the two differ by a sign. □

Thus, the space $X=BG_\omega$ has quadratic invariant $\phi\colon B\to C$, with associated Hessian form $\beta=\omega_{\mathrm{sym}}\colon B\otimes B\to C$.

7. The main theorem

In this section we restate our main theorem (3.7), but in terms of our explicit models for G, and in somewhat more generality, in that we allow for a torus of rank other than 2.

7.1. **The group $W^T(G)$.** Fix a finitely generated free abelian group L, and let $T:=L\otimes\mathbb{T}$ be the associated torus. We consider the semidirect product group $D(T):=\mathrm{Aut}(T)\ltimes T$ with group law

$$(A,t)\cdot(A',t') := (AA', (A')^{-1}t+t').$$

Note that $D(T)$ acts on the space T by

$$(A,t)\cdot s = A(s+t).$$

Given a topological group G, we define a group

$$W^T(G) := \mathrm{Map}(T,G)\rtimes D(T)$$

[6]The group G_ω really depends on the choice of κ, but all our computations about it will only depend on ω.

with group law given by

$$(g,f)\cdot(g',f') = \big(\big(s\mapsto g(s)\cdot g'(f^{-1}(s))\big), ff'\big).$$

We write $W_0^T(G)\subseteq W^T(G)$ for the identity component, and $\overline{W}^T(G) := W^T(G)/W_0^T(G)$ for the quotient.

Given $\omega\colon B\otimes B\to C$, we will compute the homotopy type of the classifying space $BW_0^T(G_\omega)$, together with the evident action of $\overline{W}^T(G_\omega)$ on its homotopy groups.

7.2. **Homology and cohomology of T.** Because T is an abelian group, H_*T is naturally a graded commutative Hopf algebra. The iterated coproduct $\psi\colon H_pT\to H_1T\otimes\cdots\otimes H_1T$ gives an identification of H_pT with the antisymmetric invariants $\Lambda_pL\subseteq L^{\otimes p}$. In terms of this identification the Pontryagin product $H_1T\otimes H_1T\to H_2T$ is given by $t\otimes t'\mapsto t\wedge t' := t\otimes t' - t'\otimes t\in\Lambda_2L$ (a direct consequence of the fact that H_*T is a *graded* Hopf algebra: $\psi(tt')=\psi(t)\psi(t')=(t\otimes 1+1\otimes t)(t'\otimes 1+1\otimes t') = tt'\otimes 1 + (t\otimes t'-t'\otimes t)+1\otimes tt'$).

The Kronecker pairing $(-,-)\colon H^*(T;B)\otimes H_*T\to B$ then gives an identification $H^p(T;B)\xrightarrow{\sim}\mathrm{Hom}(H_pT,B)=\mathrm{Hom}(\Lambda_pL,B)$. We note the following formula for the cup product in these terms, which involves a tricky sign.

7.3. **Proposition.** *Let $f\in H^1(T;B)$ and $f'\in H^1(T;B')$ be cohomology classes corresponding to $m\in\mathrm{Hom}(L,B)$ and $m'\in\mathrm{Hom}(L,B')$ via the Kronecker pairing. Then with respect to the Kronecker pairing, the cup product $f\smile f'\in H^2(T;B\otimes B')$ corresponds to*

$$(\Lambda_2L\hookrightarrow L\otimes L\xrightarrow{-m\otimes m'} B\otimes B')\in\mathrm{Hom}(\Lambda_2L, B\otimes B').$$

Thus, $f\smile f'$ corresponds to the function $t\wedge t'\mapsto(-m\otimes m')(t\wedge t') = -m(t)\otimes m'(t')+m(t')\otimes m'(t)$.

Proof. Using the *graded* Kronecker pairing $(-,-)\colon (H^*(T;B)\otimes H^*(T;B'))\otimes(H_*T\otimes H_*T)\to B\otimes B'$ we have

$$(f\smile f',u) = (f\otimes f',\sum v\otimes v') = -\sum(f,v)(f',v'),$$

for $u\in H_2T$ where $\psi(u)=\sum v\otimes v'\in H_1T\otimes H_1T$ (the component of the coproduct in degree $(1,1)$). In terms of our identifications, $\psi\colon H_2T\to H_1T\otimes H_1T$ is the inclusion $\Lambda_2L\to L\otimes L$, and the formula follows.

(The additional sign here comes from a conflict of two sign conventions: the graded Kronecker pairing $H^1(T;B)\otimes H^1(T;B')\otimes H_1T\otimes H_1T\to B\otimes B'$, which is what is used to identify the coproduct ψ as dual to cup product, and which introduces a sign, vs. the evaluation pairing $\mathrm{Hom}(\Lambda_2L,B)\otimes\mathrm{Hom}(\Lambda_2L,B')\otimes L\otimes L\to B\otimes B'$, which does not introduce a sign.) $\square$

The Kronecker pairing generalizes to the "slant product"

$$f,v\mapsto f\nabla v\colon H^{p+q}(T;M)\times H_qT\to H^p(T;M)$$

by

$$f\nabla v = \sum f'(f'',v),$$

where $f\in H^{p+q}(T;M)$, $v\in H^qT$, and $\sum f'\otimes f'' = \mathrm{Mult}_* f$, the image of f under the map $H^*(T;M)\to H^*(T\times T;M)\approx H^*(T;M)\otimes H^*T$ induced by multiplication in T. Thus if $|f|=|v|$ then $f\nabla v=(f,v)$.

Given $n\in\mathrm{Hom}(\Lambda_kL,M)$ and $t\in L$, we define the contraction operation $n\nabla t\in\mathrm{Hom}(\Lambda_{k-1}L,M)$ by

$$(n\nabla t)(\tau) := n(t\wedge\tau).$$

7.4. **Proposition.** *With respect to the usual identifications $H_1T = L$ and $H^k(T;M) = \mathrm{Hom}(\Lambda_kL,M)$, the slant product coincides with the contraction pairing.*

Proof. Let $f \in H^k(T;M)$ so $n = (f,-)\colon H_kT = \Lambda_k L \to M$. For $t \in H_1T = L$ and $u \in H_{k-1}T = \Lambda_{k-1}L$ we have

$$\begin{aligned}(f\nabla t, u) &= (\sum f'(f'',t),u) = \sum (f',u)(f'',t) = (-1)^{k-1}\sum (f'\otimes f'', u\otimes t)\\ &= (-1)^{k-1}(f, \mathrm{Mult}_*(u\otimes t)) = (-1)^{k-1}(f, u\wedge t) = (f, t\wedge u) = n(t\wedge u) = (n\nabla t)(u),\end{aligned}$$

so $f\nabla v$ corresponds to $n\nabla v$. □

For instance, if $k=2$, then $H^2(T;M)\otimes H_1T \to H^1(T;M)$ is described by $(n\nabla t)(t') = n(t\wedge t') = n(t\otimes t' - t'\otimes t)$.

Finally, given a bilinear map $\gamma\colon L\otimes L \to C$, we will use the same symbol γ for its restriction $\Lambda_2 L \to C$ (e.g., $\gamma = \omega(m\otimes m')$ in the statement of the (7.6) below).

7.5. **The homotopy groups of $BW_0^T(G_\omega)$.** We now describe $\pi_* BW^T(G_\omega)$, its quadratic invariant, the evident action of $\overline{W}(G_\omega)$ on homotopy groups, and its cohomology ring. After this we briefly explain how (3.7) is read off from this calculation.

7.6. **Theorem.** *Let $\omega\colon B\otimes B \to C$ be a bilinear function with associated quadratic function ϕ and Hessian form β, and let G_ω be a topological group associated with ω as in* (6.6)*. The space $X = BW_0^T(G_\omega)$ is an object of $\mathcal{C}$, with*

$$\begin{aligned}\pi_3 X &\approx C,\\ \pi_2 X &\approx L\times B\times \mathrm{Hom}(L,C),\end{aligned}$$

and with quadratic invariant $\phi^\sharp\colon \pi_2X \to \pi_3X$ given by

$$\phi^\sharp(t,y,x) = \phi(y) + xt, \qquad t\in L,\ y\in B,\ x\in \mathrm{Hom}(L,C).$$

Furthermore we have

$$\overline{W}^T(G_\omega) \approx \mathrm{Aut}(L)\ltimes E,$$

where E is a group with underlying set

$$E = \mathrm{Hom}(L,B)\times \mathrm{Hom}(\Lambda_2 L, C)$$

and group law

$$(m,n)\cdot(m',n') = (m+m', \omega(m\otimes m') + n + n'), \qquad m,m'\in \mathrm{Hom}(L,B),\ n,n'\in \mathrm{Hom}(\Lambda_2L,C).$$

while the semi-direct product is defined via the action of $\mathrm{Aut}(L)$ on E given by

$$A \propto (m,n) = (mA^{-1}, n(\Lambda_2 A^{-1})).$$

The action of $E \subseteq \overline{W}^T(G_\omega)$ on π_*X is given by

$$\begin{aligned}(m,n)\propto c &= c, && c\in \pi_3X\\ (m,n)\propto (t,y,x) &= (t, y+mt, x - \beta(y,m) - \omega(mt,m) + n\nabla t), && (t,y,x)\in \pi_2X,\end{aligned}$$

while the action of $\mathrm{Aut}(L)\subseteq \overline{W}^T(G_\omega)$ on π_*X is given by

$$\begin{aligned}A\propto c &= c, && c\in\pi_3X,\\ A\propto (t,y,x) &= (At, y, xA^{-1}), && (t,y,x)\in \pi_2X.\end{aligned}$$

7.7. *Remark.* The central extension E corresponds to the cocycle $\gamma\colon \mathrm{Hom}(L,B)\times\mathrm{Hom}(L,B) \to \mathrm{Hom}(\Lambda^2 L, C)$ defined by $(m,m')\mapsto \omega(m\otimes m')$. Up to isomorphism, this central extension depends only on the antisymmetrization of γ, which satisfies $\gamma_{\mathrm{antisym}}(m,m') = \beta(m\otimes m') - \beta(m'\otimes m)$ and so depends only on the quadratic function ϕ.

The proof of (7.6) is given in the next several sections, culminating in (9.5).

It is straightforward to check that $\phi^\sharp$ is regular. For instance, if we choose coordinates and write $t = (t_1, \dots, t_r) \in L = \mathbb{Z}^r$, $(y_1, \dots, y_d) \in B = \mathbb{Z}^d$, $C = \mathbb{Z}^e$, and $(x_{11}, \dots, x_{re}) \in \operatorname{Hom}(L, C) = \mathbb{Z}^{r\times e}$, and write $\phi = (\phi_1, \dots, \phi_e)$, then $\phi^\sharp = (\phi_1^\sharp, \dots, \phi_e^\sharp)$ where $\phi_k^\sharp(t, y, x) = \phi_k(y) + t_1 x_{1k} + \cdots + t_r x_{rk}$. This sequence of polynomials is easily seen to be regular (when $r \geq 1$); in fact, it remains regular after passing to the quotient ring in which $y_1 = \cdots = y_d = 0$. From (6.5) we obtain the following.

7.8. **Corollary.** *If the rank of T is positive, then*

$$\begin{aligned} H^*(BW_0^T(G_\omega); \mathbb{Q}) &\approx \operatorname{Sym}((L \times B \times \operatorname{Hom}(L, C))^* \otimes \mathbb{Q})/(\textit{image of } (\widetilde{\phi^\sharp})^*) \\ &\approx \mathbb{Q}[t_1, \dots, t_r, y_1, \dots, y_d, x_{11}, \dots, x_{re}]/(\phi_1^\sharp, \dots, \phi_e^\sharp), \end{aligned}$$

with the evident action by $\overline{W}^T(G_\omega)^{\mathrm{op}}$.

We obtain the statement of the main theorem (3.7) by:

- setting $B = \mathbb{Z}^d$, so $K(B, 1) \approx U(1)^d$,
- setting $C = \mathbb{Z}$,
- taking $\omega\colon B \otimes B \to C$ to be any quadratic refinement of $\phi\colon B \to C$,
- setting $T = \mathbb{T}^2$ so $L = H_1\mathbb{T}^2 = \mathbb{Z}^2$, and
- identifying $\mathbb{Z} \approx \Lambda_2 L$ via the generator $e_1 \wedge e_2 \in \Lambda_2 L$, where $e_1 = (1, 0), e_2 = (0, 1) \in \mathbb{Z}^2 = L$.

This last identification implies that for $n \in \mathbb{Z} \approx \operatorname{Hom}(\Lambda^2 L, \mathbb{Z})$ and $t = (t_1, t_2) \in L$, we have that $(n\nabla t)(e_1) = -t_2$ and $(n\nabla t)(e_2) = t_1$. With these choices, $W^T(G_\omega)$ is a model for $\mathcal{W}^\Sigma(U(1)^d \times_\phi K(\mathbb{Z}, 2))$, and the results of (3.7) follow.

8. Computation of $\pi_* \operatorname{Map}(\Sigma, G_\omega)$

We fix a topological group $G = G_\omega$ associated to a homomorphism $\omega\colon B \otimes B \to C$. In this section we compute invariants of the space $\operatorname{Map}(\Sigma, G_\omega)$ for an arbitrary space Σ.

8.1. **Bilinear cohomology operations.** Any bilinear map $\alpha\colon B \otimes B' \to C$ induces a bilinear cohomology operation

$$\widetilde{\alpha}\colon H^p(-; B) \times H^q(-; B') \to H^{p+q}(-; C)$$

by

$$\widetilde{\alpha}(x, y) := \alpha_*(x \smile y),$$

where the cup product is $\smile\colon H^p(-; B) \times H^q(-; B') \to H^{p+q}(-; B \otimes B')$, and $\alpha_*\colon H^*(-; B \otimes B') \to H^*(-; C)$ is the natural map induced by the homomorphism on coefficients.

8.2. *Remark.* We note that

$$\widetilde{\alpha}(x, y) = (-1)^{|x||y|}(\widetilde{\alpha\tau})(y, x),$$

where $\tau\colon B' \otimes B \xrightarrow{\sim} B \otimes B'$ is the evident transposition map. In particular, if $\alpha\colon B \otimes B \to C$ is symmetric, then $\widetilde{\alpha}(x, y) = (-1)^{|x||y|}\widetilde{\alpha}(y, x)$.

8.3. **The functors $\pi_* \operatorname{Map}(-, G_\omega)$.** We are going to describe $\pi_k \operatorname{Map}(-, G_\omega)$ (basepoint at the identity element) as functors on the homotopy category of spaces, together with their Samelson products.

8.4. **Proposition.** *Let Σ be an arbitrary space.*

(1) *We have natural bijections of sets*

$$\pi_k \operatorname{Map}(\Sigma, G_\omega) \approx H^{1-k}(\Sigma; B) \times H^{2-k}(\Sigma; C)$$

for all k.

(2) *The group law on* $\pi_k \operatorname{Map}(\Sigma, G_\omega)$ *for* $k \geq 1$ *is the evident additive one. The group law on* π_0 *is given by*

$$(u, v) \cdot (u', v') = (u + u', -\widetilde{\omega}(u, u') + v + v').$$

(3) *Samelson products* $\pi_p \times \pi_q \to \pi_{p+q}$ *on* $\operatorname{Map}(\Sigma, G_\omega)$ *are given, in terms of the above bijection, by*

$$\langle (y, x), (y', x') \rangle = (0, -(-1)^{p(1-q)} \widetilde{\omega_{\mathrm{sym}}}(y, y')).$$

Of course, $\pi_k \operatorname{Map}(\Sigma, G_\omega) \approx 0$ for $k \geq 3$, and the first factor is always 0 in $\pi_2 \operatorname{Map}(\Sigma, G_\omega)$. In low dimensions, the Samelson products take the form

$$\begin{aligned}
\langle (u, v), (y, x) \rangle &= (0, -\widetilde{\omega_{\mathrm{sym}}}(u, y)) = (0, -\widetilde{\omega_{\mathrm{sym}}}(y, u)), && \pi_0 \times \pi_1 \to \pi_0, \\
\langle (y, x), (u, v) \rangle &= (0, \widetilde{\omega_{\mathrm{sym}}}(y, u)) = (0, \widetilde{\omega_{\mathrm{sym}}}(u, y)), && \pi_1 \times \pi_0 \to \pi_1, \\
\langle (y, x), (y', x') \rangle &= -\widetilde{\omega_{\mathrm{sym}}}(y, y') = \widetilde{\omega_{\mathrm{sym}}}(y', y), && \pi_1 \times \pi_1 \to \pi_2.
\end{aligned}$$

The Samelson product $\pi_0 \times \pi_2 \to \pi_2$ is trivial.

8.5. *Remark.* We record here the inversion formula for $\pi_0 \operatorname{Map}(\Sigma, G_\gamma)$:

$$(u, v)^{-1} = (-u, -\widetilde{\omega}(u, u) - v).$$

8.6. *Remark.* Suppose $\Sigma = T$ is a torus so that as in (7.2) we have identifications $L = H_1 T$, and

$$H^1(\Sigma; B) \xrightarrow{\sim} \operatorname{Hom}(L, B), \qquad H^2(\Sigma; C) \xrightarrow{\sim} \operatorname{Hom}(\Lambda_2 L, C).$$

Then

$$\pi_0 \operatorname{Map}(T, G_\omega) \approx \operatorname{Hom}(L, B) \times \operatorname{Hom}(\Lambda_2 L, C)$$

with group law given by

$$(m, n) \cdot (m', n') = (m + m', \omega(m \otimes m') + n + n'), \qquad m, m' \in \operatorname{Hom}(L, B),\ n, n' \in \operatorname{Hom}(\Lambda_2 L, C),$$

where $\omega(m \otimes m')$ represents the composite $\Lambda_2 L \hookrightarrow L \otimes L \xrightarrow{m \otimes m'} B \otimes B \xrightarrow{\omega} C$. The change sign in the formula is a consequence of (7.3).

8.7. *Remark.* Let $X = B \operatorname{Map}(T, G_\omega)$. Then we have $\pi_k X = \pi_{k-1} \operatorname{Map}(T, G_\omega)$. The identifications of (8.4) give

$$\pi_1 X \approx \operatorname{Hom}(L, B) \times \operatorname{Hom}(\Lambda_2 L, C), \qquad \pi_2 X \approx B \times \operatorname{Hom}(L, C), \qquad \pi_3 X \approx C,$$

with group law on $\pi_1 X = \pi_0 \operatorname{Map}(T, G_\omega)$ given as above.

Tracing through: the relation between Samelson products and Whitehead products (10.8), the expression of $\pi_1 \curvearrowright \pi_n$ in terms of Whitehead products (10.7), and the relation between Whitehead products and the quadratic invariant (6.1), together with the identifications and formulas of (8.4), we find that:

- The action of $\pi_1 X$ on $\pi_2 X$ is given by
$$(m, n) \propto (y, x) = (y, x) + [(m, n), (y, x)] = (y, x) + \langle (m, n), (y, x) \rangle = (b, x - \beta(y, m)).$$
- The action of $\pi_1 X$ on $\pi_3 X$ is trivial.
- The quadratic invariant $\phi^\sharp \colon \pi_2 X \to \pi_3 X$ is given by $\phi^\sharp(y, x) = \phi(y)$.

Proof of (8.4)*(1) and (2).* The bijections of (1) are immediate from the description of G_ω. The formula for the group law in (2) amounts to the fact that the topological cocycle $-\kappa \colon K(B, 1) \times K(B, 1) \to K(C, 2)$ represents the cohomology operation $-\widetilde{\omega}$. That the group structure for $\pi_{*\geq 1}$ is the additive one is straightforward; i.e., $\Omega G \approx K(B, 0) \times K(C, 1)$ as an H-space. $\square$

It remains to prove part (3) of (8.4), for which we need a suspension map.

8.8. **The suspension map.** There is a natural **suspension map**

$$S_k\colon \pi_p \operatorname{Map}(\Sigma, X) \to \pi_{p-k} \operatorname{Map}(\Sigma \times S^k, X)$$

induced by the evident inclusion

$$\operatorname{Map}_*(S^k, \operatorname{Map}(\Sigma, X)) \subseteq \operatorname{Map}(S^k, \operatorname{Map}(\Sigma, X)) \approx \operatorname{Map}(\Sigma \times S^k, X).$$

We compute the effect of suspension for $X = G_\omega$ in terms of the isomorphisms (8.4)(1).

8.9. **Proposition.** *When $X = G_\omega$, the suspension map S_k is given by*

$$S_k(x, y) = (x \times \epsilon_k, y \times \epsilon_k), \qquad x \in H^{1-p}(\Sigma; B),\ y \in H^{2-p}(\Sigma; C),$$

where $\epsilon_k \in H^k S^k$ is the canonical generator. In particular, the suspension map is injective.

Proof. Using $G_\omega = K(B, 1) \times K(C, 2)$, we see that S_k is the same as the suspension map in cohomology. (See (10.5).) □

8.10. **Computation of Samelson products, and proof of** (8.4)**(3).** When $p = q = 0$, the Samelson product is just the commutator on $\pi_0 \operatorname{Map}(\Sigma, G_\omega)$. Thus, in terms of (8.4)(1),

$$\langle (y, x), (y', x') \rangle = (0, -\widetilde{\omega}(y, y') + \widetilde{\omega}(y', y)) = (0, -\widetilde{\omega_{\mathrm{sym}}}(y, y')).$$

The last equality is because $\widetilde{\gamma_{\mathrm{sym}}}\colon H^1(\Sigma; B) \times H^1(\Sigma; B) \to H^2(\Sigma; C)$ is the *antisymmetrization* of $\widetilde{\gamma}$, for degree reasons (8.2).

8.11. **Proposition.** *Let G be a topological group. There is a commutative diagram of the form*

$$\begin{tikzcd}
\pi_p \operatorname{Map}(\Sigma, G) \times \pi_q \operatorname{Map}(\Sigma, G) \arrow[r, "\langle -,- \rangle"] \arrow[d, "S_p \times S_q"'] & \pi_{p+q} \operatorname{Map}(\Sigma, G) \arrow[d, "S_{p+q}"] \\
\pi_0 \operatorname{Map}(\Sigma \times S^p, G) \times \pi_0 \operatorname{Map}(\Sigma \times S^q, G) \arrow[d, "\pi_p^* \times \pi_q^*"'] & \pi_0 \operatorname{Map}(\Sigma \times S^{p+q}, G) \arrow[d, "\pi_{p,q}^*"] \\
\pi_0 \operatorname{Map}(\Sigma \times S^p \times S^q, G) \times \pi_0 \operatorname{Map}(\Sigma \times S^p \times S^q, G) \arrow[r, "\langle -,- \rangle"'] & \pi_0 \operatorname{Map}(\Sigma \times S^p \times S^q, G)
\end{tikzcd}$$

where $\pi_p\colon S^p \times S^q \to S^p$, $\pi_q\colon S^p \times S^q \to S^q$, and $\pi_{p,q}\colon S^p \times S^q \to S^p \wedge S^q \approx S^{p+q}$ are the evident projections.

Proof. The underlying point-set diagram commutes. □

Proof of (8.4)*(3).* We have already proved the commutator formula for π_0 above. Inserting $(y, x) \in \pi_p \operatorname{Map}(\Sigma, G)$ and $(y', x') \in \pi_q \operatorname{Map}(\Sigma, G)$, going around the upper/right side of the square of (8.11) gives

$$\langle (y, x), (y', x') \rangle \times \epsilon_p \times \epsilon_q,$$

while going around the left/lower side gives

$$(0, -\widetilde{\omega_{\mathrm{sym}}}(y \times \epsilon_p \times 1, y' \times 1 \times \epsilon_q)) = (0, -(-1)^{p(1-q)} \widetilde{\omega_{\mathrm{sym}}}(y, y') \times \epsilon_p \times \epsilon_q),$$

since $y' \in H^{1-q}(\Sigma; B)$, using the formula for π_0. Therefore we arrive at the formula

$$\langle (y, x), (y', x') \rangle = (0, -(-1)^{p(1-q)} \widetilde{\omega_{\mathrm{sym}}}(y, y')), \qquad (y, x) \in \pi_p \operatorname{Map}(\Sigma, G),\ (y', x') \in \pi_q \operatorname{Map}(\Sigma, G).$$

□

8.12. **A desuspension map.** Consider the basepoint preserving map

$$D\colon \operatorname{Map}(\Sigma \times S^1, G) \to \Omega \operatorname{Map}(\Sigma, G)$$

defined by

$$(Df)(t)(s) := f(s,t) \cdot f(s,*)^{-1}, \qquad t \in S^1,\ s \in \Sigma,$$

using the group law of G; here $* \in S^1$ represents the basepoint. This induces a map on homotopy groups

$$D_*\colon \pi_k \operatorname{Map}(\Sigma \times S^1, G) \xrightarrow{\pi_k D} \pi_k \Omega \operatorname{Map}(\Sigma, G) \underset{\sim}{\xrightarrow{\nu}} \pi_{k+1} \operatorname{Map}(\Sigma, G).$$

8.13. **Proposition.** *For $k = 0, 1$ the map $D_*\colon \pi_k \operatorname{Map}(\Sigma \times S^1, G_\omega) \to \pi_{k+1} \operatorname{Map}(\Sigma, G_\omega)$ is given by*

$$D_*(y \times 1 + y' \times \epsilon,\ x \times 1 + x' \times \epsilon) = (y',\ -\widetilde{\omega}(y', y) + x'),$$

where $y \times 1 + y' \times \epsilon \in H^{1-k}(\Sigma \times S^1; B)$, $x \times 1 + x' \times \epsilon \in H^{2-k}(\Sigma \times S^1; C)$, and where $\epsilon \in H^1 S^1$ is such that $(\epsilon, [S^1]) = 1$.

In this case $k = 1$ we must have $y' = 0$ and this simplifies to

$$D_*(y \times 1,\ x \times 1 + x' \times \epsilon) = x'.$$

Proof. Consider the composite $S_1 \circ D$ with the suspension map, which is the self-map of $\operatorname{Map}(\Sigma \times S^1, G)$ which sends f to $(s,t) \mapsto f(s,t) \cdot f(s,*)^{-1}$.

For $k = 0$, the effect of $S_1 \circ D$ on $(y + y' \times \epsilon, x + x' \times \epsilon)$ (we omit "$\times 1$" from the notation), using the formula for the group law on π_0, including the formula (8.5) for inversion, is

$$\begin{aligned}
(y + y' \times \epsilon, x + x' \times \epsilon) \cdot (y, x)^{-1} &= (y + y' \times \epsilon, x + x' \times \epsilon) \cdot (-y, -\widetilde{\omega}(y,y) - x) \\
&= (y' \times \epsilon, \widetilde{\gamma}(y,y) - \widetilde{\omega}(y + y' \times \epsilon, -y) + x' \times \epsilon) \\
&= (y' \times \epsilon, \big(-\widetilde{\omega}(y', y) + x'\big) \times \epsilon).
\end{aligned}$$

Note that $\widetilde{\omega}(y' \times \epsilon, y) = \omega_*((y' \times \epsilon) \smile y) = -\omega_*(y' \smile y) \times \epsilon = -\widetilde{\omega}(y', y) \times \epsilon$, since $y \in H^1$. We read off the formula we need using (8.9).

For $k = 1$, the calculation of $S_1 \circ D$ on $(y, x + x' \times \epsilon)$ is as expected:

$$(y, x + x' \times \epsilon) \cdot (y, x)^{-1} = (y, x + x' \times \epsilon) - (y, x) = (0, x' \times \epsilon),$$

whence the desired formula. □

9. Computation of $\pi_* W^T(G_\omega)$ and proof of (7.6)

9.1. **Certain Samelson products in $\pi_* W^T(G_\omega)$.** For a based map $v\colon S^1 \to T$, define

$$C_v\colon T \times S^1 \to T, \qquad C_v(t,s) = v(s)^{-1} \cdot t = t \cdot v(s)^{-1}.$$

That is, C_v is the composite

$$T \times S^1 \xrightarrow{\mathrm{id} \times v} T \times T \xrightarrow{\mathrm{id} \times \mathrm{Inv}} T \times T \xrightarrow{\mathrm{Mult}} T.$$

9.2. **Lemma.** *The induced map $C_v^*\colon H^k(T; M) \to H^k(T \times S^1; M)$ is given by*

$$C_v^*(f) = f \times 1 - (f \nabla v) \times \epsilon,$$

where we also write v for $v_*[S^1] \in H_1 T$.

Proof. If $\mathrm{Mult}^* f = \sum f' \otimes f''$, then

$$(\mathrm{id} \times \mathrm{Inv})^* \, \mathrm{Mult}^* f = \sum (-1)^{|f''|} f' \otimes f''.$$

Since $v^* f = (f, v_*[*]) \times 1 + (f, v_*[S^1]) \times \epsilon$, we have

$$(\mathrm{id} \times v \, \mathrm{Inv})^* \, \mathrm{Mult}^* f = \sum (-1)^{|f''|} f'(f'', [*]) \times 1 + (-1)^{|f''|} f'(f'', v) \times \epsilon = f \nabla [*] \times 1 - f \nabla v \times \epsilon.$$

□

Given elements $(1,f),(g,1)\in \operatorname{Map}(T,G)\rtimes T\subset W^T(G)$, their commutator is given by

$$(1,f)\cdot(g,1)\cdot(1,f^{-1})\cdot(g^{-1},1)=(1,t\mapsto g(f^{-1}\cdot t)\cdot g^{-1}(t)).$$

9.3. **Lemma.** *For $v\in\pi_1T$ and $w\in\pi_k\operatorname{Map}(T,G)$ we have the Samelson product*

$$\langle(0,v),(w,0)\rangle=\big((D\circ C_v^*)_*(w),0\big)\in\pi_{k+1}\operatorname{Map}(T,G)\rtimes T=\pi_{k+1}\operatorname{Map}(T,G)\times\pi_{k+1}T,$$

where $D\colon \operatorname{Map}(T\times S^1,G)\to\Omega\operatorname{Map}(T,G)$ is the desuspension map, and $C_v^*=\operatorname{Map}(C_v,\mathrm{id})\colon \operatorname{Map}(T,G)\to\operatorname{Map}(T\times S^1,G)$.

Proof. Evidently $D\circ C_v^*$ sends $g\in\operatorname{Map}(T,G)$ to $h\in\Omega\operatorname{Map}(T,G)$ defined by $h(s):=\big(t\mapsto g(v(s)^{-1}\cdot t)\cdot g(t)^{-1}\big)$. That is, $h(s)=v(s)\cdot g\cdot v(s)^{-1}\cdot g^{-1}$, the commutator in $\operatorname{Map}(T,G)\rtimes T$ of $v(s)\in T$ and $g\in\operatorname{Map}(T,G)$. Thus for a based map $w\colon S^k\to\operatorname{Map}(T,G)$, the composite $D\circ C_v^*\circ w$ is adjoint to the desired Samelson product. □

Now fix $G=G_\omega$. We compute the Samelson product $\langle-,-\rangle\colon\pi_1W^T(G_\omega)\times\pi_kW^T(G_\omega)\to\pi_{k+1}W^T(G_\omega)$ on elements which come from the subgroups T and $\operatorname{Map}(T,G)$.

9.4. **Proposition.** *For $t\in L=\pi_1T\subset\pi_1D(T)$ and $(y,x)\in H^{1-k}(T;B)\times H^{2-k}(T;C)=\pi_k\operatorname{Map}(T,G_\omega)\subset\pi_1W^T(G_\omega)$, then for $k=0,1$ we have*

$$\langle(0,t),((y,x),0)\rangle=(-y\nabla t,\ \widetilde{\omega}(y\nabla t,y)-x\nabla t)\in\pi_{k+1}\operatorname{Map}(T,G_\omega)\subset\pi_{k+1}W^T(G_\omega).$$

For $k=1$ this simplifies to

$$\langle(0,t),((y,x),0)\rangle=(0,-x\nabla t).$$

Proof. Combine (9.3), the formula (8.13) for the desuspension map, and the effect of C_t^* on $\pi_*\operatorname{Map}(T,G_\omega)$ (for $t\in L=\pi_1T$), which is computed by (9.2). Explicitly, in terms of the isomorphism $\pi_k\operatorname{Map}(\Sigma,G_\omega)\approx H^{1-k}(\Sigma;B)\times H^{2-k}(\Sigma;C)$ of (8.4), the formula of (9.2) gives

$$C_t^*(y,x)=(y\times1-(y\nabla t)\times\epsilon,x\times1-(x\nabla t)\times\epsilon),$$

whence (8.13) gives

$$D(C_t^*(y,x))=(-y\nabla t,\ -\widetilde{\omega}(-y\nabla t,y)-x\nabla t)=(-y\nabla t,\ \widetilde{\omega}(y\nabla t,y)-x\nabla t).$$

□

9.5. **Structure of $\pi_*W^T(G_\omega)$.** We now put this together to compute the homotopy groups of $\operatorname{Map}(T,G_\omega)\rtimes T\subset W^T(G_\omega)$ together with its Samelson products.

9.6. **Theorem.** *Fix $\omega\colon B\otimes B\to C$, and set $L=H_1T=\pi_1T$.*

(1) *We have set bijections*

$$\begin{aligned}\pi_2(\operatorname{Map}(T,G_\omega)\rtimes T)&\approx H^0(T;C),\\ \pi_1(\operatorname{Map}(T,G_\omega)\rtimes T)&\approx H_1T\times H^0(T;B)\times H^1(T;C),\\ \pi_0(\operatorname{Map}(T,G_\omega)\rtimes T)&\approx H^1(T;B)\times H^2(T;C).\end{aligned}$$

(2) *The group structure on π_1 and π_2 is the evident additive one, while the group law in π_0 is given by*

$$(m,n)\cdot(m',n')=(m+m',-\widetilde{\omega}(m,m')+n+n').$$

(3) *Samelson products on π_* are given in terms of the above bijections by*

$$\langle(m,n),(t,y,x)\rangle=(0,\ m\nabla t,\ -\widetilde{\beta}(m,y)-\widetilde{\omega}(m\nabla t,m)+n\nabla t),\qquad \pi_0\times\pi_1\to\pi_1,\tag{9.7}$$

$$\langle(t,y,x),(m,n)\rangle=(0,\ -m\nabla t,\ \widetilde{\beta}(m,y)+\widetilde{\omega}(m\nabla t,m)-n\nabla t),\qquad \pi_1\times\pi_0\to\pi_1,\tag{9.8}$$

$$\langle(t,y,x),(t',y',x')\rangle=-\widetilde{\beta}(y,y')-x\Delta t'-x'\Delta t,\qquad \pi_1\times\pi_1\to\pi_2,\tag{9.9}$$

while $\pi_0\times\pi_2\to\pi_2$ is trivial.

Proof. This is a combination of what we have proved up to now. The set bijections are from the evident product decomposition of $\mathrm{Map}(T, G_\omega) \rtimes T$ and (8.4)(1). The group structure in π_0 is by (8.4)(2), while in higher degrees it is follows easily from (8.4)(2) since the groups must be abelian.

Part (3) is a consequence of the bilinearity of the Samelson product, combined with (8.4)(3), (9.4), and the fact that Samelson products in $\pi_* T$ vanish since T is abelian. □

We can add in the action of $\mathrm{Aut}(L)$.

9.10. **Proposition.** *With respect to the identification of the homotopy groups of* $\mathrm{Map}(T, G_\omega) \rtimes T$ *described in* (9.6)*, the (left) action of* $\mathrm{Aut}(L)$ *on* $\mathrm{Map}(T, G_\omega)$ *induces the following action on homotopy groups, written in terms of the evident left actions of* $\mathrm{Aut}(L) = \mathrm{Aut}(T)$ *on* $H^*(T; M)$*.*

$$\begin{aligned} A \propto (m, n) &= ((A^{-1})^* m, (A^{-1})^* n), && (m, n) \in H^1(T; B) \times H^2(T; C), \\ A \propto (t, y, x) &= (At, y, (A^{-1})^* x), && (t, y, x) \in L \times H^0(T; B) \times H^1(T; C), \\ A \propto c &= c, && c \in H^0(T; C). \end{aligned}$$

Proof. Immediate by functoriality. □

If we introduce the isomorphisms

$$H^0(T; M) \approx M, \qquad H^1(T; M) \approx \mathrm{Hom}(L, M), \qquad H^2(T; M) \approx \mathrm{Hom}(\Lambda_2 L, M),$$

then we have

$$\begin{aligned} \pi_2(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx C, \\ \pi_1(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx L \times B \times \mathrm{Hom}(L, C), \\ \pi_0(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx \mathrm{Hom}(L, B) \times \mathrm{Hom}(\Lambda_2 L, C), \end{aligned}$$

and the formulas take the form

$$\begin{aligned} (m, n) \cdot (m', n') &= (m + m',\ \omega(m \otimes m') + n + n'), \\ \langle (m, n), (t, y, x) \rangle &= (0,\ mt,\ -\beta(m, y) - \omega(mt, m) + n\nabla t), \\ \langle (t, y, x), (m, n) \rangle &= (0,\ -mt,\ \beta(m, y) + \omega(mt, m) - n\nabla t), \\ \langle (t, y, x), (t', y', x') \rangle &= -\beta(y, y') - x\nabla t' - x'\nabla t, \\ A \propto (m, n) &= (mA^{-1}, n(\Lambda_2 A^{-1})) \\ A \propto (t, y, x) &= (At, y, xA^{-1}). \end{aligned}$$

Here:

- the pairings $m, t \mapsto m\nabla t$ and $x, t \mapsto x\nabla t$ are examples of Kronecker pairings $H^1(T; M) \times H_1 T \to H^0(T; M)$, and so become evaluation maps $\mathrm{Hom}(L, M) \otimes L \to M$;
- the pairing $n, t \mapsto n\nabla t\colon H^2(T; C) \times H_1 T \to H^1(T; C)$ becomes the contraction pairing $\nabla\colon \mathrm{Hom}(\Lambda_2 L, C) \times L \to \mathrm{Hom}(L, C)$ (7.4);
- the expression $-\widetilde{\omega}(m, m')$ becomes $\omega(m \otimes m')$ as explained in (8.6);
- the action of $\mathrm{Aut}(L) = \mathrm{Aut}(T)$ on $H^k(T; M)$ becomes the evident action on $\mathrm{Hom}(\Lambda_k L, M)$ defined by functoriality of Λ_k and composition.

Proof of (9.10). This is read off from the calculations (9.6), (9.10), together with the relation between Whitehead and Samelson products (10.9), and the fact that Whitehead products $\pi_1 \times \pi_k \to \pi_k$ describe the action $\pi_1 \curvearrowright \pi_k$ (10.7). □

We can now give the proof of the general result.

Proof of (7.6). We read off the results using the isomorphism $\pi_* BW^T(\Sigma) \approx \pi_{*-1} W^T(\Sigma)$. The Samelson products become Whitehead products, up to a sign as described in (10.9). □

10. Conventions

10.1. **Orientations.** Let $I = [0,1]$. We use $S^n = I^n/\partial I^n$. We orient I^n, and thus S^n, using the Künneth map, e.g., using $I^n/\partial I^n \approx (I/\partial I)^{\wedge n}$. The boundary ∂I^n is homeomorphic to S^{n-1}, and its orientation is fixed by the boundary map in singular homology. When necessary, we take $(0,0)$ as the base point of ∂I^n.

Examining the Eilenberg-Zilber map shows that the face $\{ (t_1, \dots, t_n) \mid t_i = 1 \} \subset I^n$ receives a positive orientation when i is odd, and a negative orientation when i is even. In particular, we see that an orientation preserving equivalence $I/\partial I \to \partial I^2$ goes counterclockwise around the square; e.g., $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$.

10.2. **Loops.** We compose paths in temporal order: thus $\gamma \cdot \delta$ is defined when $\gamma(1) = \delta(0)$.

The standard action of the fundamental group $\pi_1 X \curvearrowright \pi_p X$ is from the *left*. We write it as $\gamma \propto \alpha$, $\gamma \in \pi_1 X$ and $\alpha \in \pi_p X$. It is necessarily defined, in terms of the "balloon on a string" picture, by putting the *end* of the string $\gamma(1)$ at the basepoint of the p-sphere: i.e., via $S^p \to I \cup_{\{1\}} S^p \xrightarrow{(\gamma,\alpha)} X$.

With this convention, we may extend this action to a functor $\pi_p \colon \Pi_1 X \to \mathrm{Ab}$ from the fundamental groupoid, for $p \geq 2$.

We have for $k \geq 1$ a standard isomorphism

$$\nu \colon \pi_k X \xrightarrow{\sim} \pi_{k-1} \Omega X,$$

defined so that $f \colon S^k \to X$ corresponds to $\nu f \colon S^{k-1} \to X$ given by $(\nu f)(x_1, \dots, x_{k-1})(t) = f(x_1, \dots, x_{k-1}, t)$. This convention matches the most lexically convenient convention for the tensor-hom adjunction, i.e., $\mathrm{Map}_*(X \wedge Y, Z) \approx \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$. In particular, the standard isomorphism $\mathrm{Map}_*(X \wedge S^p, Z) \approx \mathrm{Map}_*(X, \Omega^p Z)$ is determined in this way.

10.3. *Remark.* If instead we consider $\nu' \colon \pi_k X \xrightarrow{\sim} \pi_{k-1} \Omega X$, so that $\nu(f)(x_1, \dots, x_{k-1})(t) = f(t, x_1, \dots, x_{k-1})$, then we have $(\nu' f) \sim (-1)^{k-1}(\nu f)$, with the sign coming from the evident transposition $S^{k-1} \wedge S^1 \to S^1 \wedge S^{k-1}$. (The map ν' is the standard identification used in [Whi78], for instance.)

10.4. **Eilenberg-Mac Lane spaces.** For us, an Eilenberg-Mac Lane space consists of a based space K *equipped* with a choice of isomorphism $A \xrightarrow{\sim} \pi_n K$ to its only non-trivial homotopy group. Such an object is unique up to canonical homotopy equivalence (and in fact, up to contractible choice), and so can be unambiguously denoted $K(A, n)$.

We thus obtain a canonical weak equivalence $\Omega K(A, n) \approx K(A, n-1)$, using the standard isomorphism $\nu \colon \pi_n \xrightarrow{\sim} \pi_{n-1} \Omega$ to fix the identification of the homotopy group.

The identification $\widetilde{H}^n(X, A) \approx \pi_0 \mathrm{Map}_*(X, K(A, n))$ is fixed in the usual way, via a choice of tautological class $\iota \in \widetilde{H}^n(K(A, n); A) \approx \mathrm{Hom}(H_n(K(A, n); A), A) \approx \mathrm{Hom}(\pi_n K(A, n), A)$ corresponding to the identity map of A. (Ultimately, this depends on the choice of the fundamental class in $H_n S^n$, which defines the Hurewicz map.)

Using the standard identification $\Omega^p K(A, n) \approx K(A, n-p)$, we obtain a standard natural isomorphism

$$\pi_p \mathrm{Map}(X, K(A, n)) \approx \pi_0 \mathrm{Map}(X, \Omega^p K(A, n)) \approx H^{n-p}(X; A).$$

Similarly, we obtain a natural suspension isomorphism

$$\widetilde{H}^{n-p}(X; A) \approx [X, \Omega^p K(A, n)]_* \approx [X \wedge S^p, K(A, n)]_* \approx \widetilde{H}^n(X \wedge S^p; A).$$

10.5. **Cup product.** The cup product $\smile \colon H^m(X; A) \otimes H^n(X; B) \to H^{m+n}(X; A \otimes B)$ is represented by a map $\smile \colon K(A, m) \wedge K(B, n) \to K(A \otimes B, m+n)$, characterized by the property that $\pi_m K(A, m) \otimes \pi_n K(B, n) \to \pi_{m+n} K(A \otimes B, m+n)$ induces the identity map of $A \otimes B$.

Let $\epsilon \in H^1 S^1$ be the tautological class, corresponding to the map $S^1 \to K(\mathbb{Z}, 1)$ sending the tautological class $\iota_1 \in \pi_1 S^1$ to $1 \in \mathbb{Z} = \pi_1 K(\mathbb{Z}, 1)$. The map

$$- \times \epsilon \colon \widetilde{H}^{n-1}(X; A) \to \widetilde{H}^n(X \wedge S^1; A)$$

coincides with the standard isomorphisms

$$[X, K(A, n-1)]_* \approx [X, \Omega K(A, n)]_* \approx [X \wedge S^1, K(A, n)]_*,$$

and thus with the suspension isomorphism described above.

More precisely: using our standard identifications $K(A, n-1) \approx \Omega K(A, n)$ and $H^k(X, B) \approx [X, K(B, k)]$, the composite

$$[X, K(A, n-1)] \approx [X, \Omega K(A, n)] \to [X \times S^1, K(A, n)]$$

coincides with $- \times \epsilon \colon H^{n-1}(X; A) \to H^n(X \times S^1; A)$, whereas the composite

$$[X, K(A, n-1)] \approx [X, \Omega K(A, n)] \to [S^1 \times X, K(A, n)]$$

coincides with $(-1)^{n-1}(\epsilon \times -) \colon H^{n-1}(X; A) \to H^n(S^1 \times X; A)$ (and not with $\epsilon \times -$ as one might naively think).

Cohomology operations. A based map $\psi \colon K(A, m) \to K(B, n)$ induces a cohomology operation $\psi \colon H^m(X; A) \to H^n(X; B)$. Taking loops gives a map $\Omega\psi \colon K(A, m-1) \to K(B, n-1)$ and a corresponding operation $\Omega\psi \colon H^{m-1}(X; A) \to H^{n-1}(X; B)$. The diagram

$$\begin{tikzcd} H^{m-1}(X; A) \arrow[r, "\Omega\psi"] \arrow[d, "\times\epsilon"'] & H^{n-1}(X; B) \arrow[d, "\times\epsilon"] \\ H^m(X \wedge S^1; A) \arrow[r, "\psi"'] & H^n(X \wedge S^1; B) \end{tikzcd}$$

commutes, i.e.,

$$\Omega\psi(x) \times \epsilon = \psi(x \times \epsilon).$$

Note that the analogous diagram involving $\epsilon\times$ only commutes *up to sign* depending on $m - n$, i.e., $(-1)^{n-1}\epsilon \times \Omega\psi(x) = \psi((-1)^{m-1}\epsilon \times x)$.

A based map $\psi \colon K(A, p) \wedge K(B, q) \to K(C, n)$ determines a cohomology operation $\psi \colon H^p(X; A) \times H^q(X; B) \to H^n(X; C)$ in two variables. Using the evident maps $\Omega X \wedge Y \to \Omega(X \wedge Y)$ and $X \wedge \Omega Y \to \Omega(X \wedge Y)$, we can loop ψ in either of its two inputs, obtaining

$$\Omega_1\psi \colon K(A, p-1) \wedge K(B, q) \to K(C, n-1), \quad \Omega_2\psi \colon K(A, p) \wedge K(B, q-1) \to K(C, n-1).$$

The relation between these are given by

$$\Omega_1\psi(x, y) \times \epsilon = (-1)^{|y|}\psi(x \times \epsilon, y), \quad \Omega_2\psi(x, y) \times \epsilon = \psi(x, y \times \epsilon),$$

10.6. **Groups and loop spaces.** Let G be a topological group, and let $EG \to BG$ be the universal principal bundle. We may regard EG as having a *left* action by G.

For a based space (X, x_0), the path fibration $PX \to X$ may be defined by

$$PX = \{ \gamma \in \mathrm{Map}([0, 1], X) \mid \gamma(0) = x_0 \}.$$

With this definition (with the free end of the path at $t = 1$), the composition law on ΩX extends naturally to an "action" $* \colon \Omega X \times PX \to PX$.

Taking $X = BG$, any lift $EG \to P(BG)$ of the two projections gives rise to a weak equivalence $G \to \Omega G$, which furthermore is an H-space map.

10.7. **Whitehead products.** The Whitehead product $[-,-]\colon \pi_p \times \pi_q \to \pi_{p+q-1}$ is defined via precomposition with

$$\big(\partial I^{p+q} \to \partial I^{p+q}/\sim\big) = \big(I^p \times \partial I^q \cup \partial I^p \times I^q \to I^p/\partial I^p \vee I^q/\partial I^q\big),$$

using the orientation convention of (10.1).

- If $p = q = 1$, the Whitehead product is the loop commutator $[\gamma,\delta] = \gamma\delta\gamma^{-1}\delta^{-1}$.
- If $\gamma \in \pi_1$ and $\alpha \in \pi_{q\geq 2}$, then the Whitehead product is $[\gamma,\alpha] = (\gamma \propto \alpha) - \alpha$.
- On inputs in dimensions ≥ 2, $[-,-]$ is bilinear.
- In general, we have the commutation relation $[\beta,\alpha] = (-1)^{pq}[\alpha,\beta]$ for $\alpha \in \pi_p$ and $\beta \in \pi_q$.
- We only need to care about the cases when $p,q \leq 2$:

$$\begin{aligned} &[\delta,\gamma] = [\gamma,\delta]^{-1}, && \gamma,\delta \in \pi_1, \\ &[\alpha,\gamma] = [\gamma,\alpha], && \gamma \in \pi_1,\ \alpha \in \pi_2, \\ &[\beta,\alpha] = [\alpha,\beta], && \alpha,\beta \in \pi_1. \end{aligned}$$

  These conventions agree with those of [Whi78, §7.4][7].

10.8. **Samelson products.** For a topological group G, the Samelson product $\langle -,-\rangle\colon \pi_p \times \pi_q \to \pi_{p+q}$ is defined in the evident way using the commutator map $G \wedge G \to G$ sending $(x,y) \mapsto xyx^{-1}y^{-1}$.

The definition admits an extension to loop spaces. For such spaces, the Samelson product agrees with the Whitehead product up to a sign. In fact, for $\alpha \in \pi_p X$ and $\beta \in \pi_q X$ we have that

$$\nu[\alpha,\beta] = (-1)^{p-1}\langle \nu(\alpha), \nu(\beta)\rangle, \tag{10.9}$$

according to [Whi78, 7.10][8]. In particular, we have

$$\begin{aligned} &\langle \nu(\gamma), \nu(\delta)\rangle = \nu[\gamma,\delta], && \gamma,\delta \in \pi_1, \\ &\langle \nu(\gamma), \nu(\alpha)\rangle = \nu[\gamma,\alpha], && \gamma \in \pi_1,\ \alpha \in \pi_2, \\ &\langle \nu(\alpha), \nu(\gamma)\rangle = -\nu[\alpha,\gamma] = -\nu[\gamma,\alpha], && \gamma \in \pi_1,\ \alpha \in \pi_2, \\ &\langle \nu(\alpha), \nu(\beta)\rangle = -\nu[\alpha,\beta], && \alpha,\beta \in \pi_2. \end{aligned}$$

We note the commutation relation

$$\langle \alpha,\beta\rangle = -(-1)^{pq}\langle \beta,\alpha\rangle, \qquad \alpha \in \pi_p G,\ \beta \in \pi_q G.$$

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[7] Although he defines the products using disks, not cubes. It is also unclear to me where he puts his basepoint; when $p,q \geq 2$, the choice of basepoint "shouldn't matter".

[8] In fact, his formula uses ν' instead of ν. But this makes no difference, as can be shown using (10.3).

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Department of Mathematics, University of Illinois, Urbana, IL
E-mail address: `rezk@illinois.edu`