1. Convenient category of topological spaces

We write $\text{Map}(X, Y)$ for the set of continuous maps from $X$ to $Y$. We want to give this a topology, so that there is a bijective correspondence between sets of continuous maps

$$\{T \times X \to Y\} \approx \{T \to \text{Map}(X, Y)\}$$

so that $f: T \times X \to Y$ corresponds to $\tilde{f}: T \to \text{Map}(X, Y)$ defined by $\tilde{f}(t)(x) = f(t, x)$.

If this is so, then taking $T = I = [0, 1]$ we see there is a bijective correspondence

$$\{\text{homotopies of maps } X \to Y\} \approx \{\text{paths in } \text{Map}(X, Y)\}.$$

I.e., homotopies are just paths in function space.

If $X$ is a space such that such a topology on $\text{Map}(X, Y)$ exists for every space $Y$, then we have an adjoint pair of functors

$$-	imes X: \dashv: \text{Map}(X, -).$$

Here is a necessary condition for the existence of such a topology.

1.1. Proposition. Suppose $X$ is a space such that there exists a topology on $\text{Map}(X, Y)$ with the above property, for every space $Y$. Then product with $X$ preserves quotient maps. That is, if $p: A \to B$ is a quotient map, so is $p \times \text{id}: A \times X \to B \times X$.

Proof. Exercise. By “$f$ is a quotient map”, we mean that $f$ is surjective and $f^{-1}V$ is open in $A$ iff $V$ is open in $B$. The following are equivalent:

1. $f$ is a quotient map.
2. Given any function (not necessarily continuous) $g: B \to S$ to some space $S$, the function $g$ is continuous if and only if the composite function $gf: A \to S$ is continuous.

(Exercise.)

Given this, to show $f = p \times \text{id}$ is a quotient map, we need to show that if $g: B \times X \to S$ is a function such that $h = g(p \times \text{id})$ is continuous, then $g$ is continuous.

Let $g: B \times X \to S$. Given $b \in B$, write $g_b: X \to S$ for the “slice function” of $g$ at $b$, so $g_b(x) := g(b, x)$. Note that if $h = g(p \times \text{id})$, we have

$$h_a(x) = g((p \times \text{id})(a, x)) = g(p(a), x) = g_{p(a)}(x),$$

i.e., $g_{p(a)} = h_a$. Since $h$ is continuous so are its slice functions $h_a$, and since $p$ is surjective we have $b = p(a)$ for some $a \in A$, whence $g_b = h_a$. Thus, each slice $g_b: X \to S$ is continuous.

This means that the adjoint function $\tilde{g}: B \to \text{Map}(X, S)$ sending $b \mapsto g_b$ is a well defined function. Note that $\tilde{h} = \tilde{g}p$, (since $\tilde{h}(a)(x) = h(a, x) = g(p(a), x) = \tilde{g}(p(a))(x)$). Since $h$ is continuous so is its adjoint $\tilde{h}$, whence $\tilde{g}$ is continuous since $p$ is a quotient map.

Thus we have proved that $g(p \times \text{id})$ continuous implies $g$ continuous, as desired.

\[\square\]
A standard argument (using the “tube lemma”) says that any locally compact\textsuperscript{1} space $X$ has the property that product with $X$ preserves quotient maps. However, not every space $X$ has this property, so the topology we want cannot exist universally.

There is a topology called the **compact-open topology** on $\text{Map}(X,Y)$. One can show that if $X$ is locally compact then $\text{Map}(X,-)$, defined using the compact-open topology, is right adjoint to $-\times X$.

This is not ideal: many spaces of interest are not locally compact, including some CW-complexes like $\mathbb{C}P^{\infty}$. Also, $\text{Map}(X,Y)$ is rarely locally compact.

The fix is to identify some full subcategory $\mathcal{C}$ of $\text{Top}$, which (i) includes the spaces we are usually interested in, (ii) has all limits and colimits, and (iii) for every object $X$ of $\mathcal{C}$, the functor $-\times X: \mathcal{C} \to \mathcal{C}$ has a right adjoint $\text{Map}(X,-): \mathcal{C} \to \mathcal{C}$.

**Observation.** If such a right adjoint exists, the underlying point-set of $\text{Map}(X,Y)$ is necessarily the set of continuous maps, because $\ast \to \text{Map}(X,Y)$ correspond to $X \approx \ast \times X \to Y$.

**Problem.** Even if $\mathcal{C}$ has all limits and colimits, it is not likely that such limits and colimits in $\mathcal{C}$ will be computed the same way as they are in $\text{Top}$. At best, we may hope that the underlying point sets of such limits and colimits are the same as they are in Top. However, even this is not always possible.

There are a number of solutions $\mathcal{C}$, which often go by the name of **compactly generated spaces**, or just **CG-spaces**. **Warning.** The definition of “compactly generated space” can vary depending on where you look. Some definitions have slightly better properties than others.

People seem to be standardizing on defining CG-spaces to be the **weak Hausdorff k-spaces\textsuperscript{2}**. These are defined as follows.

- A subset $C \subseteq X$ of a space is **k-closed** if $f^{-1}C$ is closed in $K$ for every continuous map $f: K \to X$ from a compact Hausdorff $K$. Complements of k-closed subsets are **k-open**.
- A space $X$ is a **k-space** if every k-closed subset is closed.
- A k-space $X$ is **weak Hausdorff** if the diagonal subset $\Delta_X = \{(x,x) | x \in X\}$ is k-closed in $X \times X$, with the usual product topology. (This is not the general definition of “weakly Hausdorff” for arbitrary spaces.)

A reference for CG spaces which relies on a minimal number of prerequisites are my notes: https://faculty.math.illinois.edu/~rezk/cg-spaces-better.pdf.

I will implicitly assume that we are working with CG spaces, though I’ll try to note the places where we actually need this. The main thing is to understand how limits and colimits work in CG spaces, which is different than in Top. Here are the general rules:

- To compute a limit in CG spaces (e.g., product), take the usual limit topology, and then modify the topology by forcing all k-closed sets to be closed (the “kification”). Thus, limits in CG spaces have the same underlying point set as in Top, but with a (potentially) finer topology. In a few cases that we meet, the new topology is the same as the old one.
- To compute a limit in CG spaces (e.g., pushouts), take the usual colimit topology, and then take the minimal quotient space which is weak Hausdorff (this does exist). Thus, colimits in CG spaces are always quotients of the analogous colimit in Top, and so can have a different point set. In many cases that we will meet, the CG colimit turns out to be the same as that in Top.

A few key facts:

- All locally compact spaces are CG, as are all CW-complexes.
- More generally, a product of a CG space with a locally compact space (with the usual product topology) is CG. In particular, if $X$ is CG, so is $X \times [0,1]$.

---

\textsuperscript{1}I.e., $X$ is Hausdorff and every point is contained in the interior of some compact subspace.

\textsuperscript{2}Introduced in McCord, “Classifying spaces and infinite symmetric products”, TAMS (1969)
A subspace of a CG space need not be CG. However, a closed subspace of a CG space is always CG.

The pushout of a diagram of CG spaces along a closed inclusion (as computed in Top) is also CG, and in fact computes the pushout in CG spaces.

2. Mapping cylinder and mapping cone

Given a map \( f : X \to Y \) of spaces, the **mapping cylinder** \( \text{Cyl}(f) \) is the pushout in the diagram

\[
\begin{array}{ccc}
X \times 0 & \longrightarrow & X \times I \\
\downarrow f \times \text{id} & & \downarrow \\
Y \times 0 & \longrightarrow & \text{Cyl}(f)
\end{array}
\]

The **mapping cone** \( \text{Cone}(f) \) is the quotient \( \text{Cyl}(f) / X \times 1 \), i.e., the pushout

\[
\begin{array}{ccc}
X \times 1 & \longrightarrow & \ast \\
\downarrow & & \downarrow \\
\text{Cyl}(f) & \longrightarrow & \text{Cone}(f)
\end{array}
\]

*Note.* If \( X \) and \( Y \) are CG, so are \( \text{Cyl}(f) \) and \( \text{Cone}(f) \), since these are pushouts of CG spaces along inclusions.

2.1. **Proposition.**

1. There is a bijection between (i) continuous maps \( \text{Cyl}(f) \to E \) and (ii) data \((g,H)\) consisting of a map \( g : Y \to E \) and a homotopy \( H_1 \) of maps \( X \to E \) such that \( H_0 = gf \).
2. There is a bijection between (i) continuous maps \( \text{Cone}(f) \to E \) and (ii) data \((g,H,e)\) consisting of: a map \( g : Y \to E \), a homotopy \( H_1 \) of maps \( X \to E \), and a point \( e \in E \), such that \( H_0 = gf \) and \( H_1(X) \subseteq \{e\} \). (If \( X \neq \emptyset \), we can omit \( e \) and simply say that \( H_1 \) is a constant map.)

2.2. **Example.** For an identity map \( \text{id}_X : X \to X \), we have \( \text{Cyl}(\text{id}_X) \approx X \times I \), and \( \text{Cone}(\text{id}_X) \approx \text{Cone}(X) \), i.e., the quotient space \( X / X \times 1 \).

*Remark.* In general, for a subspace \( A \subseteq X \), the space \( X/A \) is defined to be the pushout of \( X \leftarrow A \to \ast \). Note that if \( A = \emptyset \), then \( X/\emptyset = X \cup \{\ast\} \). If \( X \) is CG and \( A \) is a closed subspace, then \( X/A \) is CG. If \( A \) is not closed, then we need to replace \( X/A \) with its CG-ification. Usually we will only form \( X/A \) when \( A \) is a closed subspace, so this will not be an issue.

2.3. **Exercise.** The inclusion \( Y \to \text{Cyl}(f) \) is a homotopy equivalence.

2.4. **Exercise.** There is an evident homeomorphism between \( \text{Cone}(f) \) and the pushout of

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & \ast \\
\downarrow & & \downarrow \\
X & \xrightarrow{j} & \text{Cone}(X)
\end{array}
\]

where \( j \) is the inclusion induced by \( X \approx X \times 0 \to \text{Cone}(X) \).

Note that the pair \((\text{Cyl}(f), X \times 1)\) is a "good pair" in the sense of Hatcher, so that

\[
H_*(\text{Cyl}(f), X \times 1) \approx \tilde{H}_* \text{Cone}(f)
\]

Also, we have a commutative diagram

\[
\begin{array}{ccc}
X \times 1 & \longrightarrow & \text{Cyl}(f) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]
in which the vertical maps are homotopy equivalences. Thus, we get a long exact sequence in homology
\[ \cdots \to H_\ast X \xrightarrow{H_\ast f} H_\ast Y \to \tilde{H}_\ast \text{Cone}(f) \to H_{\ast-1}X \to \cdots. \]

When \( f: A \to X \) is inclusion of a closed subspace, we are interested in comparing the quotient and the mapping cone, via the evident projection
\[ \pi: \text{Cone}(f) \to X/A. \]

We are going to give a condition on \( f \) which ensures that \( \pi \) is a homotopy equivalence; in fact, a homotopy equivalence “rel \( A \)”.

3. Relative homotopy

Given a space \( A \), a “space under \( A \)” is defined to be a map \( i_X: A \to X \) of spaces. A map of spaces under \( A \) is a commutative diagram
\[
\begin{array}{ccc}
A & \xrightarrow{i_X} & X \\
\downarrow & & \downarrow f \\
Y & \xleftarrow{i_Y} & Y
\end{array}
\]

We write \( \text{Top}_A \) for the category of spaces under \( A \).

A homotopy \( H_t \) of maps \( X \to Y \) in \( \text{Top}_A \) is a homotopy \( H: X \times I \to Y \) such that \( H_t(i_X(a)) = i_Y(a) \) for all \( t \in [0,1] \). This is typically known as a homotopy rel \( A \).

3.1. Example. If \( A = * \), then a “space under *” is just a based space \((X,x_0)\). A map of spaces under * is a basepoint preserving map. A homotopy rel * is a homotopy such that the basepoint of \( X \) is sent to the basepoint of \( Y \) “at all times \( t \)”.

We sometimes write \([X,Y]_A\) for the set of “homotopy rel \( A \)” classes of maps in \( \text{Top}_A \).

3.2. Example. Let \( A = \{0,1\} \). Then \( I = [0,1] \) is naturally a space under \( A \). If \( i_X: A \to X \) is another space under \( A \) with \( i_X(n) = x_n \), then
\[ [I,X]_A = \pi_1(X,x_0,x_1), \]

the set of homotopy classes of paths \( x_0 \sim x_1 \).

Given spaces \( X,Y \) under \( A \), let
\[ \underline{\text{Map}}_A(X,Y) \subseteq \text{Map}(X,Y) \]

denote the subspace of maps \( X \to Y \) compatible with the maps from \( A \). (If \( X,Y \) are CG, this is a closed subspace, so as a subspace it is CG.)

3.3. Exercise. Homotopies rel \( A \) are the same as paths in \( \underline{\text{Map}}_A(X,Y) \).

3.4. Example (Based loops). Let \( A = * \), and choose a standard basepoint \( s_0 \) for the circle \( S^1 \). Then for a based space \((X,x_0)\), the based loops are
\[ \Omega X := \underline{\text{Map}}_*(S^1,X) = \{ f: S^1 \to X \mid f(s_0) = x_0 \}. \]

Note that \( \pi_0 \Omega X \approx \pi_1(X,x_0) \).

Here is a really important fact about homotopy rel \( A \). We start with the observation that if we fix a map \( g: A \to B \), then we have a functor
\[ (-) \cup_A B: \text{Top}_A \to \text{Top}_B \]

which takes a space under \( A \) to a space under \( B \), by gluing a copy of \( B \) along \( A \):
\[ (A \to X) \mapsto (B \to X \cup_A B). \]
(Another convenient name for this functor might be $g_\ast$.)

Note: a map $f: X \to Y$ of spaces under $A$ induces a map $f': X' \to Y'$ of spaces under $B$, where $X' = X \cup_A B$ and $Y' = Y \cup_A B$. The idea is that $f'|_X = f$ while $f'|_B$ is the “identity” on $B$.

3.5. **Proposition.** If $H_0, H_1: X \to Y$ are maps which are homotopic rel $A$, then the induced maps $H_0', H_1': X' \to Y'$ are homotopic rel $B$.

**Proof.** Exercise. □

3.6. **Corollary.** If $X, Y$ are spaces under $A$ which are homotopy equivalent rel $A$, then $X' = X \cup_A B$ and $Y' = Y \cup_A B$ are spaces under $B$ which are homotopy equivalent rel $B$.

4. Homotopy extension property

Let $f: A \to X$ be a map. We say that $f$ has the **homotopy extension property (HEP)** if and only if for every commutative diagram of solid arrows

\[
\begin{array}{ccc}
A \times 0 & \longrightarrow & X \times 0 \\
\downarrow f \times \text{id} & & \downarrow h_0 \\
A \times I & \longrightarrow & X \times I \\
\downarrow \text{id} & & \downarrow h \\
& & Z \\
\end{array}
\]

there exists a dotted arrow making the diagram commute.

Equivalently, $f$ has the HEP if for every $g$ there exists a dotted arrow:

\[
\begin{array}{ccc}
\text{Cyl}(f) & \longrightarrow & Z \\
\downarrow j & & \downarrow \text{id} \\
X \times I & \longrightarrow & \\
\end{array}
\]

where $j$ is the “obvious” map (i.e., induced by $f \times \text{id}: A \times I \to X \times I$ and the inclusion $X \times 0 \to X \times I$).

4.1. **Proposition.** A map $f$ has the HEP if and only if there exists $r: X \times I \to \text{Cyl}(f)$ such that $r j = \text{id}_{\text{Cyl}(f)}$.

4.2. **Remark** (Short version). If $f: A \to X$ has the HEP, then both $f$ and $j: \text{Cyl}(f) \to X \times I$ are embeddings. (A map $f: A \to X$ is an embedding if it induces a homeomorphism between $A$ and the subspace $f(A)$ of $X$.) Thus WLOG if $f$ has the HEP we can assume $A \subseteq X$ and $\text{Cyl}(f) = (A \times I) \cup (X \times \{0\}) \subseteq X \times I$ are subspaces.

4.3. **Remark** (Long version). If $f$ has the HEP, then the existence of the retraction $r$ implies that $j$ is injective, and in fact that $j: \text{Cyl}(f) \to X \times I$ an embedding. (A priori, even if $f$ is an embedding, $j$ can fail to be an embedding. It is true that $j$ must be an embedding if $f$ is a closed embedding, which is the usual situation we will see.)

If $f$ has the HEP, then have that $f: A \to X$ is injective, and in fact that $f$ is an embedding. Proof: $A \times \{1\} \to \text{Cyl}(f)$ and $X \times \{1\} \to X \times I$ are embeddings (use the definition of the pushout topology). Because $j: \text{Cyl}(f) \to X \times I$ is an embedding, so is its intersection with the closed subset $X \times \{1\}$. Thus $\text{Cyl}(f) \cap (X \times \{1\}) \to X \times \{1\}$ is an embedding, which is equivalent to the map $f$.

Definitions of HEP you may find elsewhere (such as in Hatcher) will start by assuming $f$ is an embedding (in fact, that $A \subseteq X$ is a subspace). Because of the above remarks, these definitions are actually equivalent. Also, the “retract formulation” of HEP is often stated: there exists a retraction of $X \times I$ onto the subspace $A \times I \cup X \times \{1\}$. It turns out that this condition is also equivalent to
ours: see the Appendix in Hatcher (online revised version, not the printed version, which has an error here).

In the above, I have assumed that $A$ and $X$ are not necessarily CG, and that $\text{Cyl}(f)$ is the usual pushout in Top. In fact, if $A$ and $X$ are CG spaces, then $X \times I$ is CG, and furthermore if $f$ has the HEP then also $\text{Cyl}(f)$ is CG (note that this is not true in general).

Maps with HEP are also sometimes called **Hurewicz cofibrations**, or even just **cofibrations** (though this is potentially confusing, cause there are other kinds of cofibrations). Maps with the HEP which are inclusions of closed subspaces are also called **NDR pairs**.

Examples of maps with HEP.

- The inclusions $A \to A \amalg B \leftarrow B$ into a disjoint union have the HEP. (Exercise.)
- The inclusions $\{0\} \to [0,1] \leftarrow \{1\}$ of endpoints into an interval have the HEP. (Exercise.)
- The inclusions $S^{n-1} \to D^n$ of a unit sphere into a unit disk have the HEP. (Exercise: prove this by exhibiting a retraction $D^n \times I \to \text{Cyl}(S^{n-1} \to D^n)$.)

4.4. **Example.** The inclusions $(0,1) \to \mathbb{R}$ and $A \to I$ where $A = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\}$ do not have the HEP.

The key example of maps with HEP are inclusions into mapping cylinders and mapping cones. We will soon prove that for any map $f: X \to Y$, the inclusion $Y \amalg X \to \text{Cyl}(f)$ has the HEP, and also that $Y \amalg * \to \text{Cone}(f)$ has the HEP.

5. **Lifting properties**

We are going to use the following reinterpretation of the HEP property. Let $e_t: \text{Map}(I, X) \to X$ denote the evaluation map, defined by $\gamma \mapsto \gamma(t)$.

5.1. **Proposition.** A map $f: X \to Y$ has the HEP if and only if for every space $Z$, and every commutative square of the form

$$
\begin{array}{ccc}
X & \overset{u}{\longrightarrow} & \text{Map}(I, Z) \\
\downarrow f & & \downarrow e_0 \\
Y & \overset{v}{\longrightarrow} & Z
\end{array}
$$

there exists a dotted arrow $s$ making both triangles commute.

*Proof.* A map $u: X \to \text{Map}(I, Z)$ corresponds exactly via mapping-space adjunction to a map $\tilde{u}: X \times I \to Z$. Under this correspondence, the identity $e_0 u = v f$ corresponds exactly to the identity $\tilde{u} j_0 = v f$, where $j_0: X \to X \times I$ is the inclusion $j_0(x) = (x,0)$.

In other words, $(u, v)$ exactly corresponds to the data of a map $v = H_0: Y \to Z$ together with a homotopy $\tilde{u} = \overline{\pi}_t$ of maps $X \to Z$.

A map $s: Y \to \text{Map}(I, Z)$ corresponds exactly to a map $\tilde{s}: Y \times I \to Z$. Under this correspondence, the identities $s f = u$ and $e_0 s = v$ correspond to $\tilde{s} (f \times \text{id}_I) = \tilde{u}$, and $\tilde{s} j_0 = v$, where $j_0: Y \to Y \times I$ is the inclusion $j_0(y) = (y,0)$.

In other words, a lift $s$ corresponds exactly to a homotopy $s = H$ of maps $Y \to Z$ extending $\overline{\pi}_t$ and $H_0$. \hfill $\square$

5.2. **Exercise.** A map $f: X \to Y$ has the HEP if and only if an $s$ exists in the statement of the previous theorem in the special case that: $Z = Y \times I$, $u: X \to \text{Map}(I, Y \times I)$ is defined by $u(x)(t) = (f(x),t)$, and $v: Y \to Y \times I$ by $v(y) = (y,0)$.

5.3. **Corollary.** If $f: X \to Y$ has the HEP, and $S$ is any space, then $f \times \text{id}_T: X \to Y$ has the HEP.
Proof. There is a correspondence between the following diagrams:

\[
\begin{array}{ccc}
X \times S & \xrightarrow{u} & \text{Map}(I, Z) \\
\downarrow f \times \text{id} & & \downarrow e_0 \\
Y \times S & \xrightarrow{v} & Z
\end{array} \quad \iff \quad
\begin{array}{ccc}
X & \xrightarrow{\tilde{u}} & \text{Map}(I, \text{Map}(S, Z)) \\
\downarrow f & & \downarrow e_0 \\
Y & \xrightarrow{\tilde{v}} & \text{Map}(S, Z)
\end{array}
\]

where

- \(\tilde{u}(x)(t)(s) = u(x, s)(t)\),
- \(\tilde{v}(y)(s) = v(y, s)\),
- \(\tilde{s}(y)(t)(s) = s(y, s)(t)\).

Verify: that \(e_0 u = v(f \times \text{id})\) iff \(e_0 \tilde{u} = \tilde{v} f\), that \(e_0 s = \tilde{v}\) iff \(e_0 \tilde{s} = \tilde{v}\), and that \(s(f \times \text{id}) = u\) iff \(\tilde{s} f = \tilde{u}\).

A lift \(s\) exists in the left diagram if and only if a lift \(\tilde{s}\) exists in the right diagram. Thus if \(f\) has the HEP, we have that \(f \times \text{id}\) has the HEP.

5.4. Corollary. For any space \(X\), the map \(X \times \{0, 1\} \to X \times I\) has the HEP.

5.5. Proposition. If \(f : X \to Y\) and \(g : Y \to Z\) have the HEP, then \(gf\) has the HEP.

Proof.

5.6. Proposition. Given a pushout square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \text{id} \\
Y & \xrightarrow{Y'} &
\end{array}
\]

if \(f\) has the HEP, then \(f'\) has the HEP.

Proof. Consider

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \xrightarrow{f'} \text{Map}(I, Z) \\
\downarrow & & \downarrow \text{id} \\
Y & \xrightarrow{Y'} \text{Map}(S, Z)
\end{array}
\]

Because the left-hand square is a pushout, to produce a lift \(s'\) in the right-hand square, it suffices to produce a lift \(s\) in the large rectangle.

To summarize, the class \(\text{HEP}\) of maps which have the HEP:

- contains \(\{0, 1\} \to I\) and \(S^{n-1} \to D^n\);
- is closed under coproducts: if \(\{f_i : A_i \to B_i\}\) is a set of maps in \(\text{HEP}\), then \(\bigsqcup f_i\) is in \(\text{HEP}\);
- is closed under composition, i.e., \(gf \in \text{HEP}\) if \(f, g \in \text{HEP}\);
- is closed under cobase change, i.e., if \(f \in \text{HEP}\) and \(f'\) is a pushout of \(f\), then \(f' \in \text{HEP}\);
- is closed under product with any space, i.e., if \(f \in \text{HEP}\) then \(f \times \text{id}_S \in \text{HEP}\) for any space \(S\).

6. Mapping cylinders and the HEP

6.1. Theorem. For any map \(f : X \to Y\), the evident map \(j : Y \amalg X \to \text{Cyl}(f)\) has the HEP, as do the restrictions \(Y \to \text{Cyl}(f)\) and \(X \to \text{Cyl}(f)\).
Proof. Consider

\[
\begin{array}{ccc}
X & \rightarrow & X \times \{0,1\} \\
\downarrow_{f} & & \downarrow \\
Y & \rightarrow & Y \amalg X \\
\downarrow_{j} & & \downarrow \\
& & \text{Cyl}(f)
\end{array}
\]

The left-hand square is a pushout. The outer rectangle is a pushout by definition. Therefore the right-hand square is a pushout, by “pushout pasting”. Thus \(j\) is a pushout if \(\text{id} \times i\), which has the HEP. \(\square\)

6.2. Theorem. For any map \(f: X \rightarrow Y\), the evident map \(j': Y \amalg * \rightarrow \text{Cone}(f)\) has the HEP, as do the restrictions \(Y \rightarrow \text{Cone}(f)\) and \(* \rightarrow \text{Cone}(f)\).

Proof.

\[
\begin{array}{ccc}
X & \rightarrow & X \amalg * \\
\downarrow_{f} & & \downarrow \\
Y & \rightarrow & Y \amalg * \\
\downarrow_{j} & & \downarrow \\
& & \text{Cyl}(f)
\end{array}
\]

Earlier we observed that the outer square is a pushout. Thus the result follows once we know that \(i\) has the HEP, which follows from the pushout diagram

\[
\begin{array}{ccc}
X \times \{0,1\} & \rightarrow & X \times I \\
\downarrow & & \downarrow \\
X \amalg * & \rightarrow & \text{Cone}(X)
\end{array}
\]

\(\square\)

6.3. Corollary. Any inclusion \(X \rightarrow \text{Cone}(X)\) has the HEP. In particular, \(S^{n-1} \rightarrow D^n\) has the HEP.

6.4. Exercise. Show that if \(\{f_i: X_i \rightarrow Y_i\}\) is a set of maps such that each \(f_i \in \text{HEP}\), the the coproduct \(f: \coprod X_i \rightarrow \coprod Y_i\) is in \(\text{HEP}\).

6.5. Exercise. Show that if \(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots\) is a countable sequence of maps such that each \(f_i \in \text{HEP}\), then the induced map \(X_0 \rightarrow \text{colim}_i X_i\) has the HEP.

6.6. Exercise. Show that if \(f: A \rightarrow X\) is a CW inclusion, then \(f\) has the HEP.

7. Homotopy properties of mapping cylinders

We say that \(A \subseteq X\) is a deformation retract of \(A\) if the inclusion map \(j: A \rightarrow X\) is a homotopy equivalence rel \(A\). Explicitly, this means there exists a homotopy \(H_t\) of maps \(X \rightarrow A\) such that \(H_t(a) = a\) for all \(a \in A\), and such that such that \(H_0 = \text{id}_X\) and \(H_1(A) \subseteq A\).

We have noted that for any map \(f: A \rightarrow X\), the inclusion \(X \rightarrow \text{Cyl}(f)\) is a homotopy equivalence, which implies that \(j\): \(\text{Cyl}(f) \rightarrow X \times I\) is a homotopy equivalence. If \(f\) has the HEP we can say something better.

7.1. Proposition. If \(f: A \rightarrow X\) has the HEP, then \(j\): \(\text{Cyl}(f) \rightarrow X \times I\) is a homotopy equivalence rel \(\text{Cyl}(f)\). That is, \(\text{Cyl}(f)\) is a deformation retract of \(X \times I\).
Proof. Recall that since \( f \) has the HEP WLOG we can assume \( A \subseteq X \) and \( \text{Cyl}(f) \subseteq X \times I \) are subspaces, i.e., \( \text{Cyl}(f) = (A \times I) \cup (X \times \{0\}) \subseteq X \times I \). Because \( f \) has the HEP there is a retraction \( r: X \times I \to \text{Cyl}(f) \) so that \( rf = \text{id}_{\text{Cyl}(f)} \), which we can write as
\[
jr = (r', r'') \text{, } r': X \times I \to X \text{, } r'': X \times I \to I,
\]
so that:
- \( r'(a, s) = a \) and \( r''(a, s) = s \) for \( a \in A \) and \( s \in I \),
- \( r'(x, 0) = x \) and \( r''(x, 0) = 0 \) for \( x \in X \), and
- either \( r'(x, s) \in A \) or \( r''(x, s) = 0 \).

(First two are because \( jr = \text{id} \), while the last is because the image of \( r \) is in \( \text{Cyl}(f) \).)

Now define a homotopy \( H_t \) of maps \( X \times I \to X \times I \) by
\[
H_t(x, s) := (r'(x, st), r''(x, s)t + s(1 - t)).
\]

We observe that
\[
H_0(x, s) = (r'(x, 0), s) = (x, s),
\]
\[
H_1(x, s) = (r'(x, s), r''(x, s)) = r(x, s),
\]
\[
H_t(x, 0) = (r'(x, 0), r''(x, 0)t) = (x, 0),
\]
\[
H_t(a, s) = (r'(a, st), r''(x, s)t + s(1 - t)) = (a, st + s(1 - t) + (a, s) \text{ if } a \in A.
\]
Thus \( H_t \) is a homotopy between \( \text{id}_{X \times I} \) and \( jr \) which is rel \( \text{Cyl}(f) \). \( \square \)

7.2. Example. Consider \( A = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\} \) as a (closed) subspace of \( X = [0, 1] \). The inclusion \( f: A \to X \) does not have the HEP. It is not hard to see that \( X \times I \) does not deformation retract onto \( \text{Cyl}(f) \). (Draw picture.)

Here is a consequence.

7.3. Proposition. Let \( f: A \to X \) have the HEP, and let \( M = \text{Cyl}(f) \). Then \( M \) and \( X \) are homotopy equivalent rel \( A \), where \( M \) is viewed as an object of \( \text{Top}_A \) via \( i_M: A = A \times \{1\} \to A \times I \to M \).

Proof. Consider the following three objects of \( \text{Top}_A \).

(1) The inclusion \( A = A \times \{1\} \to \text{Cyl}(f) \).

(2) The inclusion \( A = A \times \{1\} \to X \times I \).

(3) The inclusion \( A \to X \).

Since \( \text{Cyl}(f) \) and \( X \times I \) are homotopy equivalent rel \( \text{Cyl}(f) \), they are homotopy equivalent rel the subspace \( A \times \{1\} \) of \( \text{Cyl}(f) \). Thus (1) and (2) are homotopy equivalent rel \( A \).

It is easy to show that (2) and (3) are homotopy equivalent rel \( A \), since in fact \( X \times \{1\} \) is a deformation retract of \( X \times I \). Thus (1) and (3) are homotopy equivalent rel \( A \), as desired.

Note. We can make this explicit: let \( p: \text{Cyl}(f) \to X \) be the projection \( p(x, s) = x \). Let \( r: X \times I \to \text{Cyl}(f) \) be a choice of retraction. Then
\[
G_t(x) := p(r(x, t)),
\]
\[
H_t(x, s) := r(x, t + s(1 - t)),
\]
define the desired homotopies \( G: \text{id}_X \sim_A pq \) and \( H: \text{id}_{\text{Cyl}(f)} \sim_A qp \), where \( q: X \to \text{Cyl}(f) \) is defined by \( q(x) = r(x, 1) \). \( \square \)

It was already easy to see that \( X \) and \( \text{Cyl}(f) \) are homotopy equivalent, even when \( f \) does not have the HEP. Knowing that the homotopy equivalence can be done rel \( A \) has additional consequences.

7.4. Corollary. Let \( f: A \to X \) have the HEP, and let \( C = \text{Cone}(f) \). Then \( C \) and \( X/A \) are homotopy equivalent rel \( \ast \).
Proof. We have that \( \text{Cone}(f) = \text{Cyl}(f)/A \), where the copy of \( A \) is the inclusion \( i_M : A \to M \) of the previous theorem. Because we are gluing a point \( * \) onto \( A \), and the objects \( X \) and \( \text{Cyl}(f) \) are homotopy equivalent rel \( A \), we get that the glued objects \( \text{Cone}(f) \) and \( X/A \) are homotopy rel \( * \). \( \square \)

The following generalizes Hatcher’s “good pair” observation.

7.5. Corollary. If \( f : A \to X \) has the HEP, then \( H_*(X, A) \cong \overline{H}_* \text{Cone}(f) \cong \overline{H}_*(X/A) \).

Proof. We have a homotopy equivalence \( (\text{Cyl}(f), A) \cong (X, A) \), so \( H_*(X, A) \cong H_*(\text{Cyl}(f), A) \). The pair \( (\text{Cyl}(f), A) \) is a “good pair”, so we have \( H_*(\text{Cyl}(f), A) \cong \overline{H}_* \text{Cone}(f) \). The homotopy equivalence we proved above gives \( \overline{H}_* \text{Cone}(f) \cong \overline{H}_*(X/A) \). \( \square \)

Given maps \( f : A \to X \) and \( g : B \to Y \), the double mapping cylinder \( \text{Cyl}(f, g) \) is the pushout

\[
\begin{array}{ccc}
A \times \{0,1\} & \xrightarrow{f\|g} & X \amalg Y \\
\downarrow & & \downarrow \\
A \times I & \xrightarrow{\cong} & \text{Cyl}(f, g)
\end{array}
\]

7.6. Exercise. There is a homeomorphism \( \text{Cyl}(f, g) \cong \text{Cyl}(f) \cup_A \text{Cyl}(g) \). (Hint: a closed interval can be formed by gluing together two closed intervals.)

Another way to produce the double mapping cylinder is by gluing onto a mapping cylinder: there is a pushout

\[
\begin{array}{ccc}
A & \xrightarrow{g} & Y \\
\downarrow i & & \downarrow \\
\text{Cyl}(f) & \xrightarrow{\cong} & \text{Cyl}(f, g)
\end{array}
\]

where \( i \) is the usual inclusion.

7.7. Proposition. If \( f : A \to X \) and \( g : A \to B \) are maps and \( f \) has the HEP, let \( Y = X \cup_A B \) denote the pushout of \( f \) along \( g \), and \( Y' := \text{Cyl}(f, g) = \text{Cyl}(f) \cup_A B \) denote the double mapping cylinder. Then \( Y \) and \( Y' \) are homotopy equivalent rel \( B \).

Proof. We have \( X \) and \( \text{Cyl}(f) \) homotopy equivalent rel \( A \). Gluing both along \( g : A \to B \) gives spaces homotopy equivalent rel \( B \). \( \square \)

7.8. Exercise. Show that for \( f : A \to X \) and \( g : A \to B \), there is an exact sequence

\[ \to H_*A \to H_*(X \oplus H_*B \to H_*\text{Cyl}(f, g) \to H_{*-1}A \to . \]

Conclude that if \( f \) has the HEP, then there is an exact sequence

\[ \to H_* \to H_*(X \oplus H_*B \to H_*(X \cup_A B) \to H_{*-1}A \to . \]

Finally we have the following.

7.9. Proposition. If \( f : A \to X \) has the HEP, and \( g_i : A \to B \) are homotopic maps for \( i = 0,1 \), then \( Y_i := X \cup_{A,g_i} B \) for \( i = 0,1 \) are homotopy equivalent rel \( B \).

Proof. Let \( G_i \) be the homotopy between \( g_0 \) and \( g_1 \). Let \( Y_i := (X \times I) \cup_{A \times I,G} B \) be the pushout along \( G \). We have obvious inclusions \( Y_0 \to Y_1 \leftarrow Y_1 \)

We show that each of these is a deformation retraction. Since the inclusion of \( B \) is compatible in all 3 spaces, we get homotopy equivalences rel \( B \).
We do the case $i = 0$. We use the fact that $Y_t$ is a pushout of $j$: $\text{Cyl}(f) \to X \times I$ along the obvious map $\text{Cyl}(f) \to Y_0$. This is not too hard to see directly, but here is an argument by pasting. We have

\[
\begin{array}{ccc}
A \times \{0\} & \longrightarrow & A \times I \\
\downarrow f & & \downarrow G \\
X \times \{0\} & \longrightarrow & \text{Cyl}(f) \longrightarrow Y_0 \\
\downarrow j & & \downarrow i_0 \\
X \times I & \longrightarrow & Y_t
\end{array}
\]

the left square and top rectangle are pushouts by definition, so the upper right square is a pushout by pasting. The same argument gives that the bottom square is a pushout.

We know that $j$ is a deformation retraction since $f$ has the HEP. Therefore $i_0$ is a deformation retraction as desired. \hfill \Box

7.10. Corollary. If $f_i : X \to Y$ are homotopic maps, then $\text{Cone}(f_0)$ and $\text{Cone}(f_1)$ are homotopy equivalent rel $Y$.

Proof. We have $\text{Cone}(f_i) \approx \text{Cone}(X) \cup_{X, f_i} Y$. Since $X \to \text{Cone}(X)$ has the HEP, the previous proposition gives that $\text{Cone}(f_0)$ and $\text{Cone}(f_1)$ are homotopy equivalent rel $Y$. \hfill \Box

Here is one more principle.

7.11. Proposition. Let $i_X : A \to X$ and $i_Y : A \to Y$ be spaces under $A$, and suppose both $i_X$ and $i_Y$ have the HEP, and let $f : X \to Y$ be a map such that $fi_X = i_Y$. If $f$ is a homotopy equivalence, then $f$ is a homotopy equivalence rel $A$.

This says that, for a map $f$ in $\text{Top}_A$ between objects with the HEP, to determine if $f$ is a homotopy equivalence rel $A$, it suffices to know that it is a plain old homotopy equivalence. The HEP condition is necessary. (For instance, given a subspace $T \subseteq I$, consider $A = X = (T \times I) \cup (I \times \{0\}) \to Y = I \times I$. Because $A = X$ is the mapping cylinder of $T \to I$, we know that $X \to Y$ is a homotopy equivalence. But it can fail to be a deformation retraction, e.g., if $T = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\}$.)

For instance, this gives the following.

7.12. Corollary. If $f : X \to Y$ is a basepoint preserving map between non-degenerately based spaces, and if $f$ is a homotopy equivalence, then $f$ is a homotopy equivalence rel the basepoint.

To prove this, we need the following lemma. (The argument is from May’s book.)

7.13. Lemma. Let $i : A \to X$ be a map with the HEP, and let $f : X \to X$ be a map under $A$ which is homotopic to $i_d X$ (but perhaps not homotopic rel $A$). Then there exists a left homotopy inverse to $f$ rel $A$, i.e., a map $g : X \to X$ under $A$ and a homotopy $gf \sim_A i_d X$.

Proof. Let $H_t : f \sim \text{id}_X$ be a homotopy. Apply the HEP to $(H_t | A, \text{id}_X)$ to get a homotopy $K_t : \text{id}_X \sim g$ such that $K_t i = H_t$. Note that this works because $H_0 i = fi = i = \text{id}_X i$. (It’s a little counterintuitive, because $\text{id}_X$ is at the 1-end of $H$, but at the 0-end of $K$.)

The map $g : = K_1$ satisfies $gi = K_0 i = H_0 i = fi = i$, so $g$ is a map under $A$.

We will construct a homotopy $gf \sim_A \text{id}_X$, by applying the HEP for $A \times I \to X \times I$. We have the following diagram:

\[
\begin{array}{ccc}
A \times \{0\} & \longrightarrow & A \times I \\
\downarrow i \times \text{id} & & \downarrow L \\
X \times I \times \{0\} & \longrightarrow & X
\end{array}
\]
where $J$ and $L$ are homotopies defined by

$$J(x, s, 0) := \begin{cases} K(f(x), 1 - 2s) & \text{if } s \leq 1/2, \\ H(x, 2s - 1) & \text{if } s \geq 1/2, \end{cases}$$

and

$$L(a, s, t) = \begin{cases} K(i(a), 1 - 2s(1 - t)) & \text{if } s \leq 1/2, \\ H(i(a), 1 - 2(1 - s)(1 - t)) & \text{if } s \geq 1/2. \end{cases}$$

Note that $J$: $gf \sim \mathrm{id}_X$, and that $L_{st} = i$ if either $s \in \{0, 1\}$ or $t = 1$. By the HEP there is a common extension $M: X \times I \times I \to X$. If we traverse along the edges $(0, 0) \sim (0, 1) \sim (1, 1) \sim (1, 0)$ of $I \times I$ the map $M$ restricts to a homotopy $J': gf \sim \mathrm{id}_X$, except in this case the homotopy is rel $A$.

**Proof of Proposition.** We first prove a weaker statement: any $f: X \to Y$ in $\text{Top}_A$ which is a homotopy equivalence (but not rel $A$) admits a “left homotopy inverse rel $A$”, i.e., there exists $g: Y \to X$ such that $gi_Y = i_X$ and $gf \sim_A \mathrm{id}_X$.

Given this the proposition follows: assuming given $f$ as in the hypotheses, the weak version of the proposition gives a $g$ with $gf \sim_A \mathrm{id}_X$. Since $f$ is itself a homotopy equivalence, then so is $g$, and since $g$ is a map in $\text{Top}_A$ we can apply the weak version version again, to get $f': X \to Y$ such that $f'g \sim_A \mathrm{id}_Y$. Then $f \sim_A f'g \sim_A f'$ whence $fg \sim_A f'f \sim_A \mathrm{id}$ as desired.

Now we prove the weaker statement.

**Step 1.** We find a left homotopy inverse $g': Y \to X$ to $f$ (not rel $A$) which satisfies $g'i_Y = i_X$. Pick any $\bar{g}: Y \to X$ which is homotopy inverse to $f$, but perhaps not rel $A$, and probably with $g'i_Y \neq i_X$. Since $\bar{g}f \sim \mathrm{id}_X$, by restricting this homotopy to $A$ we obtain a homotopy $H_t: \bar{g} = \bar{g}fi_X \sim i_X$ of maps $A \to X$. Using the HEP of $i_X$ applied to $(H_t, \bar{g})$, we obtain a new map $g': Y \to X$ and such that $g'i_Y = i_X$ and homotopy $\bar{g} \sim g'$ extending $H_t$, so in particular $g'$ is a homotopy inverse of $f$ and is a map in $\text{Top}_A$.

**Step 2.** By construction the composite $g'f: X \to X$ is a map under $A$ which is homotopic (not rel $A$) to $\mathrm{id}_X$. By the lemma, $g'f$ admits a left homotopy inverse $v$ rel $A$, i.e., $v\bar{g} \sim_A \mathrm{id}_X$. Thus $(vg')f = v(g'f) \sim_A \mathrm{id}_X$, i.e., $g := vg': Y \to X$ is a left homotopy inverse of $f$ rel $A$, as desired.

7.14. **Corollary.** For any contractible space $A$, the inclusion $i: A \to \text{Cone}(A)$ is a deformation retraction.

**Proof.** Since $\text{Cone}(A)$ is always contractible, if $A$ is also contractible then the map $i$ is a homotopy equivalence. Both $\mathrm{id}_A$ and $i$ have the HEP, so by the previous proposition $i$ is a homotopy equivalence rel $A$.

Combining this with what we have done gives the following.

7.15. **Proposition.** If $f: A \to X$ has the HEP and $A$ is contractible (=homotopy equivalent to $*$), then $X \to X/A$ is a homotopy equivalence.

**Proof.** First note that since $A$ is contractible, $A$ is a deformation retract of $\text{Cone}(A)$ (exercise). That is, the inclusion $A \to \text{Cone}(A)$ is a homotopy equivalence rel $A$. Therefore, by gluing with $A \to X$, we get that

$$X = X \cup_A A \to \text{Cone}(f) = X \cup_A \text{Cone}(A)$$

is a homotopy equivalence.

On the other hand, since $f$ has the HEP we know that $\text{Cone}(f) \to X/A$ is a homotopy equivalence. Thus the composite map $X \to X/A$ (which is just the projection) is a homotopy equivalence.

The moral of all this is that gluing constructions along maps with the HEP are “well-behaved”.

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8. Interlude: Hopf invariant

Recall that if \( f_0, f_1 : A \to X \) are homotopic maps, then there is a homotopy equivalence \( \text{Cone}(f_0) \approx \text{Cone}(f_1) \) rel \( X \).

Let's consider the case of \( A = S^{n-1} \). Then for \( f : S^{n-1} \to X \), we have

\[
\text{Cone}(f) = X \cup_{f,S^{n-1}} \text{Cone}(S^{n-1}) = X \cup_{f,S^{n-1}} D^n,
\]

i.e., \( Y = \text{Cone}(f) \) is the space obtained by attaching an \( n \)-dimensional cell along its boundary along the map \( f \).

Note: it is typical to use the notation

\[
\eta \in C^n(f) = \partial Y 
\]

for the cone. The idea is that \( Y \) as a set is a disjoint union of \( X \) and \( e^n = \text{Int}D^n \), somehow glued together by \( f \).

The question is, what can we say about how the homotopy type of \( \text{Cone}(f) \) depends on the map \( f \). We of course have the positive result: if \( f_0 \sim f_1 \) then \( \text{Cone}(f_0) \approx_X \text{Cone}(f_1) \). We are going to consider a negative result: given two maps, we will show that \( \text{Cone}(f_0) \not\approx_X \text{Cone}(f_1) \), and therefore that \( f_0 \) and \( f_1 \) cannot be homotopic.

We consider the case of maps \( S^3 \to S^2 \). Here are two such maps:

- The constant map \( c : S^3 \to \ast \to S^2 \). We have

\[
\text{Cone}(c) \approx S^2 \cup_{c,S^3} D^4 \approx S^2 \cup_{\ast} D^4 / S^3 \approx S^2 \lor S^4.
\]

Here we use the notation \( X \lor Y \) (“wedge”) for the one-point union of two based spaces, where we identify the common base point.

- The Hopf map \( \eta : S^3 \to S^2 \). This is constructed as follows. Let \( S^3 \subset \mathbb{C}^2 \) be the length 1 vectors. The subgroup \( S^1 \subset \mathbb{C}^\times \) of length 1 complex scalars acts on \( S^3 \) by \( \lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2) \). The quotient \( S^1 \backslash S^3 \) of this action is the projective space \( \mathbb{C}P^1 \): a point in \( \mathbb{C}P^1 \) corresponds exactly to a 1-dimensional subspace in \( \mathbb{C}^2 \).

There is a well-known homeomorphism \( S^2 \approx \mathbb{C}P^1 \). To see this, recall that we can describe \( S^2 \approx \mathbb{C} \cup \{ \infty \} \) (i.e., \( S^2 \) is the 1-point compactification of the plane), and the homeomorphism we want extends the map \( \mathbb{C} \to S^3 \) by \( z \mapsto (z,1)/\|z,1\| \).

The map \( \eta \) is the attaching map for the second cell in \( \mathbb{C}P^2 \). Thus

\[
\text{Cone}(\eta) \approx S^2 \cup_{\eta} e^4 \approx \mathbb{C}P^2.
\]

8.1. Proposition. \( \eta \not\sim c \).

Proof. We can compute the homology/cohomology of either mapping cone. In each case,

\[
H^* \text{Cone}(c) \approx H^* \text{Cone}(\eta) \approx (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, 0, \ldots).
\]

However, cohomology has a cup product. If we write \( x \in H^2 \) and \( y \in H^4 \) for generators, then we know that

- In \( \text{Cone}(c) \) we have \( x^2 = 0 \). (Proof: the inclusion \( j : S^2 \to \text{Cone}(c) \) admits a retraction \( r : \text{Cone}(c) \to S^2 \). We have that \( x = r^* \overline{x} \) where \( \overline{x} \in H^2 S^2 \) is a generator, so \( x^2 = r^* (x^2) = r^2(0) = 0 \).

- In \( \text{Cone}(\eta) \) we have \( x^2 = \pm y \) (depending on which generator we chose in \( H^4 \)). This is just the calculation of the cup product structure on \( \mathbb{C}P^n \), which is itself a consequence of Poincare duality.

□
9. Homotopy groups

For a based space \((X, x_0)\), the set \(\pi_n(X, x_0)\) is defined to be the set of homotopy classes of maps of pairs
\[ (I^n, \partial I^n) \to (X, x_0). \]
We can think \(X\) as a space under \(\partial I^n\) via \(\partial I^n \to \{x_0\} \to X\), in which case these are homotopy classes rel \(\partial I^n\).

Or we can reinterpret as homotopy classes of based maps
\[ (S^n, s_0) \approx (I^n / \partial I^n, \ast) \to (X, x_0). \]
For \(n \geq 1\), \(\pi_n(X, x_0)\) admits a group structure. Pick any coordinate \(k = 1, \ldots, n\), and define
\[ (\alpha \ast_k \beta)(t_1, \ldots, t_n) := \begin{cases} 
\alpha(t_1, \ldots, 2t_k, \ldots, t_n) & \text{if } t_k \leq 1/2, \\
\beta(t_1, \ldots, 2t_k - 1, \ldots, t_n) & \text{if } t_k \geq 1/2.
\end{cases} \]
Exercise: the operation \(\ast_k\) descends to based homotopy classes, and defines a group structure on \(\pi_n(X, x_0)\). In every case the identity element is the class of the constant map.

9.1. Lemma. For any \(i \neq j\), we have \((\alpha \ast_i \beta) \ast_j (\gamma \ast_i \delta) \sim (\alpha \ast_j \gamma) \ast_i (\beta \ast_j \delta)\). That is, \(\ast_i\) and \(\ast_j\) “commute”.

9.2. Corollary (Eckmann-Hilton). If \(n \geq 2\), then \(\ast_i\) and \(\ast_j\) agree on homotopy classes for all \(i, j\), and are commutative.

Proof. If we have a set \(X\) with two binary operations \(\ast_1\) and \(\ast_2\), such that:

1. there exists \(e \in X\) which is an identity element for both operations, i.e., \(x \ast_1 e = e \ast_1 x = x = x \ast_2 e = e \ast_2 x\), and
2. the two operations commute, i.e., \((a \ast_1 b) \ast_2 (c \ast_1 d) = (a \ast_2 c) \ast_1 (b \ast_2 d)\),
then
\[ a \ast_1 b = (a \ast_2 e) \ast_1 (e \ast_2 b) = (a \ast_1 c) \ast_2 (e \ast_1 b) = a \ast_2 b \]
and
\[ a \ast_1 b = (e \ast_2 a) \ast_1 (b \ast_2 e) = (e \ast_1 b) \ast_2 (a \ast_1 e) = b \ast_2 a. \]
It makes more sense if you write \(\ast_1\) horizontally and \(\ast_2\) vertically. \(\square\)

Thus, all choices give the same commutative group law on \(\pi_n(X, x_0)\) for \(n \geq 2\). The same idea proves the following.

An H-space is \((X, e, \mu)\) a based space \((X, e)\) with a map \(\mu: X \to X \to X\), such that \(e\) is an identity for \(\mu\), i.e., \(\mu(\epsilon, -) = \text{id}_X = \mu(-, e)\). (Warning: some people will also put an associativity-up-to-homotopy condition here; I would call that an “associative H-space”.)

9.3. Proposition. If \((X, e)\) admits an H-space structure, then \(\pi_1(X, e)\) is commutative.

We note another way to describe the group law. Given a pair of maps \(\alpha, \beta: I^n \to X\) sending \(\partial I^n\) to the base point, the group law \(\alpha \ast \beta = \alpha \ast_1 \beta\) we gave above is a composite:
\[ I^n = [0, 1] \times I^{n-1} \overset{m}{\to} [0, 2] \times I^{n-1} \overset{\alpha \# \beta}{\to} X \]
where
\[ m(t_1, t_2, \ldots, t_n) = (2t_1, t_2, \ldots, t_n) \]
and
\[ (\alpha \# \beta)(t_1, t_2, \ldots, t_n) = \begin{cases} 
\alpha(t_1, t_2, \ldots, t_n) & \text{if } t_1 \leq 1, \\
\beta(t_1 - 1, t_2, \ldots, t_n) & \text{if } t_1 \geq 1.
\end{cases} \]
Let $\infty := \{0, 1, 2\} \times I^{n-1} \cup (I \times \partial I^{n-1})$, a subspace of $[0, 2] \times I^{n-1}$. We get induced maps on quotients

$$I^n/\partial I^n \to ([0, 2] \times I^{n-1})/\infty \xrightarrow{\alpha \# \beta} X$$

which we can reinterpret as

$$S^n \xrightarrow{m} S^n \vee S^n \xrightarrow{(\alpha, \beta)} X.$$ 

Thus, the group law in $\pi_n$ is given by: $\alpha * \beta = (\alpha, \beta)m$. The fact that $m$ is associative (and commutative if $n \geq 2$) amounts to the fact that $m$ is “coassociative” up to homotopy (and “cocommutative” up to homotopy).

10. Hopf invariant

Consider maps $f : S^{4k-1} \to S^{2k}$, where $k \geq 1$. (E.g., $S^3 \to S^2$, $S^7 \to S^4$, etc.) We will assume that such maps preserve a choice of basepoint.

The Hopf invariant is a function

$$H : \pi_{4k-1} S^{2k} \to \mathbb{Z}$$

defined as follows. Given $f$, we obtain the sequence

$$S^{2k} \xrightarrow{j} \text{Cone}(f) \xrightarrow{p} S^{4k},$$

where $p : \text{Cone}(f) \to \text{Cone}(f)/j(S^{2k}) \approx \text{Cone}(S^{2k})/S^{2k} \approx S^{4k}$. (Important note: the identification $\text{Cone}(f)/j(S^{2k}) \approx \text{Cone}(S^{2k})/S^{2k}$ is canonical.)

In general, we write $S(X) := \text{Cone}(X)/X$ for the unreduced suspension of $X$.

Because $j$ has the HEP, we have a long exact sequence in reduced cohomology. The non-zero parts of this sequence are:

$$0 \leftarrow \prod^{4k} \text{Cone}(f) \leftarrow \prod^{4k} S^{4k} \leftarrow 0$$

and

$$0 \leftarrow \prod^{2k} S^{2k} \leftarrow \prod^{2k} \text{Cone}(f) \leftarrow 0.$$ 

Designate generators $\overline{x} \in H^{2k} S^{2k}$ and $\overline{y} \in H^{4k} S^{4k}$. (This requires that we have fixed orientations for both spheres.) These determine generators $x \in H^{2k} \text{Cone}(f)$, $y \in H^{4k} \text{Cone}(f)$ characterized by $j^*x = \overline{x}$, $p^*\overline{y} = y$. Now the Hopf invariant is to be the unique integer $H(f)$ such that

$$x^2 = H(f)y.$$

10.1. Proposition. $H(f)$ only depends on $f$ up to homotopy.

Proof. Let $F : S^{4k-1} \to S^{2k}$ be a homotopy between $f_0$ and $f_1$. We have a commutative diagram

$$\begin{array}{ccc}
S^{2k} & \xrightarrow{j_0} & \text{Cone}(f_0) & \xrightarrow{p_0} & S^{4k} \\
\approx & & \approx & & \\
S^{2k} & \xrightarrow{J} & \text{Cone}(F) & \xrightarrow{P} & S(S^{4k-1} \times I) \\
\approx & & \approx & & \\
S^{2k} & \xrightarrow{j_1} & \text{Cone}(f_1) & \xrightarrow{p_1} & S^{4k}
\end{array}$$

Each of the vertical maps is a homotopy equivalence (in fact, a deformation retraction). Taking cohomology shows that the long exact sequences for $f_0$ and $f_1$ are isomorphic, and that under this isomorphism the generators $\overline{x}$ and $\overline{y}$ correspond to each other. \qed

Now we have the following.
10.2. **Proposition.** The function \( H: \pi_{4k-1}S^{2k} \to \mathbb{Z} \) is a homomorphism. I.e., \( H(f + g) = H(f) + H(g) \).

**Proof.** Consider \((f, g): S^{4k-1} \vee S^{4k-1} \to S^{2k}\). This has mapping cone with cell structure
\[
\text{Cone}((f, g)) = S^{2k} \cup_{(f,g)} (e_1 \cup e_2^k).
\]

We have a commutative diagram of cofiber sequences
\[
\begin{array}{ccc}
S^{2k} & \xrightarrow{\sim} & \text{Cone}(f) \xrightarrow{i_1} S^{4k} \\
\downarrow & & \downarrow i_1 \\
S^{2k} & \xrightarrow{\sim} & \text{Cone}((f, g)) \xrightarrow{i_1} S^{4k} \vee S^{4k} \\
\downarrow & & \downarrow i_1 \\
S^{2k} & \xrightarrow{\sim} & \text{Cone}(g) \xrightarrow{i_2} S^{4k}
\end{array}
\]
where the vertical maps are the "obvious" inclusions. These exist because \text{Cone} is actually a functor:
\[
\text{Cone}: \text{Fun}([1], \text{Top}) \to \text{Top}.
\]

Here \([1]\) is the category with shape \((0 \to 1)\). A functor \([1] \to \text{Top}\) is just a map \(f: A \to B\), while a map \(f \Rightarrow f'\) of functors is a commutative square:
\[
\begin{array}{ccc}
A & \xrightarrow{gA} & A' \\
f & \downarrow & f' \\
B & \xrightarrow{gB} & B'
\end{array} \implies \text{Cone}(f) \to \text{Cone}(f').
\]

The naturality of this construction is what makes the diagram commute.

We can trace all these maps on cohomology classes:
\[
\begin{array}{cccc}
x & \xleftarrow{i_1} & y, & 0, \\
& \uparrow & & \uparrow \\
& \downarrow & & \downarrow \\
& x & \xleftarrow{i_2} & y_1, \ y_2, \ \overline{y}_1, \ \overline{y}_2 \\
& \uparrow & & \uparrow \\
& \uparrow & & \uparrow \\
& x & \xleftarrow{i_1} & 0, \ y, \ 0, \ \overline{y}
\end{array}
\]

In \( H^* \text{Cone}((f, g)) \) we have \( x^2 = a_1 y_1 + a_2 y_2 \) for some \( a_1, a_2 \in \mathbb{Z} \). Applying \( i_1^* \) gives \( x^2 = a_1 y_1 + a_2 0 \), so \( a_1 = H(f) \). Likewise, \( a_2 = H(g) \). We also have a diagram
\[
\begin{array}{ccc}
S^{2k} & \xrightarrow{\sim} & \text{Cone}(f + g) \xrightarrow{\tilde{m}} S^{4k} \\
\downarrow & & \downarrow \tilde{m} \\
S^{2k} & \xrightarrow{\sim} & \text{Cone}((f, g)) \xrightarrow{m} S^{4k} \vee S^{4k}
\end{array}
\]
where \( f + g = (f, g)m \), where \( m: S^{4k-1} \to S^{4k-1} \vee S^{4k-1} \), and the vertical maps are induced by \( m \). The key observation is that \( S(m)^* \) sends both \( \overline{y}_1 \) and \( \overline{y}_2 \) to \( \overline{y} \). Thus \( \tilde{m}^* \) sends the equation \( x^2 = H(f)y_1 + H(g)y_2 \) to \( x^2 = H(f)y + H(g)y = (H(f) + H(g))y \). Thus \( H(f + g) = H(f) + H(g) \) as desired.

We note the following famous theorem.
10.3. Theorem (Adams). There exists \( f : S^{4k-1} \to S^{2k} \) with \( H(f) = 1 \) if and only if \( 2k = 2, 4, 8 \).

Thus, in these dimensions the corresponding Hopf map generates a free summand.

11. **Whitehead products**

Here is an easier theorem.

11.1. Proposition. For every \( k \geq 1 \), there exists \( f \in \pi_{4k-1}S^{2k} \) such that \( H(f) = 2 \).

I will construct this map using a general construction called the **Whitehead product**. This will be an operation \( [-,-] : \pi_m \times \pi_n X \to \pi_{m+n-1}X \), written as a bracket. It is defined by precomposition with a map \( w_{m,n} : S^m \vee S^n \to S^{m+n-1} \), i.e., \( [f,g] := (f,g)w_{m,n} \).

The map in question is the attaching map for \( S^m \times S^n \) as a CW-complex. We regard spheres as based spaces, with basepoint called \( * \). We have a subspace \( V := (S^m \times *) \cup (*) \times S^n \approx S^m \vee S^n \subseteq S^m \times S^n \).

The complement of \( V \) is \( (S^m \setminus *) \times (S^n \setminus *) \approx \mathbb{R}^m \times \mathbb{R}^n \) is an open \((m+n)\)-cell. In fact, we can form the product by attaching a closed cell to \( V \) along its boundary:

\[
S^m \times S^n \approx (S^m \vee S^n) \cup_{w_{m,n}} e^{m+n}.
\]

Familiar example: \( m = n = 1 \).

You actually have to prove that such a map \( w_{m,n} \) exists. One way to do it is to use the identification \( S^m \approx I^m/\partial I^m \). If we do this, then the product as the form

\[
S^m \times S^n \approx (I^m/\partial I^m) \times (I^n/\partial I^n).
\]

In general, we can analyze \( X/A \times Y/B \) as a pushout:

\[
\begin{array}{ccc}
X \times B \cup_{A \times B} A \times Y & \longrightarrow & X/A \vee Y/B \\
\downarrow & & \downarrow \\
X \times Y & \longrightarrow & X/A \times Y/B
\end{array}
\]

where the map on the top is induced by the obvious projections

\[
X \times B \to X \to X/A \quad \text{and} \quad A \times Y \to Y \to Y/B,
\]

both of which send \( A \times B \) to the basepoint. In our case this gives

\[
\begin{array}{ccc}
I^m \times \partial I^n \cup \partial I^n \times I^m & \longrightarrow & S^m \vee S^n \\
\downarrow & & \downarrow \\
I^m \times I^n & \longrightarrow & S^m \times S^n
\end{array}
\]

In fact, \( I^m \times \partial I^n \cup \partial I^n \times I^n = \partial I^{m+n} \approx S^{m+n-1} \). The map on the top is thus the definition of \( w_{m,n} \).

11.2. Example. If \( m = n = 1 \), then \( [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1} \), the commutator in \( \pi_1 \), hence the notation.

11.3. Example. If \( m = 1 \) and \( n \geq 2 \), so \( m + n - 1 = n \), we can write

\[
[\alpha, \beta] = \tau_n(\beta) - \beta,
\]

where \( \tau_n : \pi_n X \to \pi_n X \) describes the action of \( \pi_1 \) on \( \pi_n \).
Here are some more properties (where I assume a certain sign convention in the definition of the Whitehead product, which is tricky to describe.)

- For $\alpha \in \pi_m$, $\beta \in \pi_n$, $\gamma \in \pi_p$ with $m, n, p \geq 2$, we have $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$.
- If $\alpha \in \pi_m$ and $\beta \in \pi_n$ with $m, n \geq 2$, then $[\beta, \alpha] = -(-1)^{(m-1)(n-1)}[\alpha, \beta]$.
- If $\alpha \in \pi_m$, $\beta \in \pi_n$, $\gamma \in \pi_p$ with $m, n, p \geq 2$, then
  
  $[[\alpha, \beta], \gamma] + (-1)^{m(p+n)}[[\beta, \gamma], \alpha] + (-1)^{p(m+n)}[[\gamma, \alpha], \beta].$

In other words, the Whitehead product makes homotopy groups into a kind of graded Lie algebra.

More precisely, if we set $A_k := \pi_{k+1}$, then $A_\ast$ admits a graded Lie bracket $[-, -] : A_m \times A_n \rightarrow A_{m+n}$ (possibly after making some different sign choices).

11.4. Remark. The decomposition $\partial I^{m+n} = I^m \times \partial I^n \cup \partial I^m \times \partial I^n \partial I^m \times I^n$ can be written

$S^{m+n-1} \approx D^m \times S^{n-1} \cup S^{m-1} \times S^{n-1} S^{m-1} \times D^n$.

For instance, if $m = n = 2$ we get

$S^3 \approx D^2 \times S^1 \cup S^1 \times S^1 \times D^2$,

i.e., the 3-sphere is obtained by gluing together two solid tori together along their boundary tori. It is possible to visualize this using $S^3 \approx \mathbb{R}^3 \cup \{\infty\}$.

Let's use this to describe the Hopf map. Because $S^1 \subset D^2 \subset \mathbb{C}$, we have a multiplication $\mu : D^2 \times D^2 \rightarrow D^2$. We get

$$
\begin{array}{ccc}
D^2 \times S^1 & \xrightarrow{\mu} & S^1 \times S^1 \\
\downarrow & & \downarrow \\
D^2 & \xrightarrow{\mu} & S^1 \\
\end{array}
$$

Taking colimits in the rows gives the Hopf map $\eta : S^1 \rightarrow S^2$. All Hopf maps arise from this Hopf construction: $D^1 \subset \mathbb{R}$, $D^2 \subset \mathbb{C}$, $D^4 \subset \mathbb{H}$, $D^8 \subset \mathbb{O}$ give $2\ell, \eta, \nu, \sigma$.

We now compute the Hopf invariant of $[\iota_n, \iota_n] \in \pi_{2n-1}S^n$ for even $n$.

11.5. Proposition. For every even $n \geq 2$, we have $H([\iota_n, \iota_n]) = 2$.

Proof. Write $t = \iota_n$. By definition $[\iota_n, \iota_n] = (\iota_n, \iota_n) \circ w$ where $w$ is the universal Whitehead product. By functoriality of the cone construction we have a commutative diagram

$$
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{w} & S^n \cup S^n \\
\downarrow & \downarrow (\iota, \iota) & \downarrow \\
S^{2n-1} & \xrightarrow{[\iota_n, \iota_n]} & S^n \\
\downarrow & \downarrow & \downarrow \\
S^n & \rightarrow & \text{Cone}([\iota_n, \iota_n]) \\
\end{array}
$$

because $S^n \times S^n = \text{Cone}(w)$. We have $H^n(S^n \cup S^n) = \mathbb{Z}\{x_1, x_2\}$ so that $j_k(x_i) = \delta_{ik}x$, where $H^nS^n = \mathbb{Z}\{x\}$ and $j_1, j_2 : S^n \rightarrow S^n \cup S^n$ are the two inclusion maps. Then we calculate that

$$(\iota, \iota)^\ast(x) = x_1 + x_2$$

and this also describes the effect of $H^n \text{Cone}([\iota_n, \iota_n]) \rightarrow H^n(S^n \times S^n)$ The square of the class $x_1 + x_2 \in H^nS^n \times S^n$ is $(x_1 + x_2)^2 = 2x_1x_2 \in H^{2n}(S^n \times S^n)$, and since $x_1x_2 \in H^{2n}S^{2n}$ is the generator, we see that

$H([\iota_n, \iota_n]) = 2$. 

□
**Question:** Given two based maps \( f : S^p \to X \) and \( g : S^q \to X \), does there exist a map \( S^p \times S^q \to X \) extending them?

\[
\begin{array}{c}
S^p \vee S^q \\
\downarrow \\
S^p \times S^q
\end{array}
\xrightarrow{(f,g)}
\begin{array}{c}
X
\end{array}
\]

**Answer:** In general, NO. There is no general reason why you would expect to be able to do this.

Better answer: an extension exists if and only if \([f,g] = 0 \) in \( \pi_{p+q-1} X \). Such an identity is the same as the existence of a null homotopy of \((f,g) \circ w_{p,q}\), which is the same as a map from the pushout:

\[
\begin{array}{c}
S^{p+q-1} \\
\downarrow \\
D^{p+q}
\end{array}
\xrightarrow{w_{p,q}}
\begin{array}{c}
S^p \vee S^q \\
\downarrow \\
S^p \times S^q
\end{array}
\xrightarrow{(f,g)}
\begin{array}{c}
X
\end{array}
\]

That is: The Whitehead product \([f,g] \) is precisely the obstruction to extending a map from a wedge of spheres to the product.

**11.6. Proposition.** If \( X \) admits the structure of an \( H \)-space, then all Whitehead products in \( \pi_* X \) are trivial.

**Proof.** If \( \mu : X \times X \to X \) is the \( H \)-space product (with identity at the basepoint), any \((f,g) : S^p \vee S^q \to X \) extends to

\[
\begin{array}{c}
S^p \times S^q \\
\downarrow \\
X \times X
\end{array}
\xrightarrow{f \times g}
\begin{array}{c}
X \xrightarrow{\mu} X.
\end{array}
\]

\[\Box\]

**11.7. Corollary.** Even dimensional spheres cannot be \( H \)-spaces.

**Proof.** We constructed a non-trivial Whitehead product in \( \pi_* \) of an even sphere. \[\Box\]

Note: this is a somewhat complicated way of proving this. We computed \( H([i,i]) = 2 \) by a calculation involving the cup product structure on \( H^*(S^n \times S^n) \). You can actually directly prove the non-existence of an \( H \)-space structure on \( S^n \) using this cup product structure.

**12. Cofibration sequence**

Fix a map \( f : X \to Y \) of based spaces. I am usually going to suppose that the basepoint inclusions \( * \to X \) and \( * \to Y \) have the HEP. (This is sometimes called a “non-degenerate basepoint”.)

We get a sequence of maps \( X \xrightarrow{f} Y \xrightarrow{\Delta} \text{Cone}(f) \). Unfortunately, the second map isn’t a based map. We can fix this by forming the **reduced cone**:

\[
\text{Cone}(f) := \text{Cone}(f)/\text{Cone}(*).
\]

You can also think of this as \( \overline{\text{Cone}(f)} = Y \cup_X \overline{\text{Cone}(X)} \), where \( \overline{\text{Cone}(X)} = \text{Cone}(X)/\text{Cone}(*_0) \).

**12.1. Proposition.** If \((X,x_0)\) has non-degenerate basepoint, then (i) \( \text{Cone}(*_0) \to \text{Cone}(X) \) has the HEP, whence \( \text{Cone}(X) \to \text{Cone}(X) \) is a homotopy equivalence, and (ii) \( X \to \text{Cone}(X) \) has the HEP.
Proof. There is a diagram
\[
\begin{array}{c}
\{x_0\} \times I \cup X \times \{1\} \rightarrow X \times \{0\} \cup \{x_0\} \times I \cup X \times \{1\} \rightarrow X \times I \\
\downarrow \\
\{x_0\} \times I \rightarrow X \times \{0\} \cup \{x_0\} \times I \rightarrow \text{Cone}(X) \\
\downarrow \\
X \times \{0\} \rightarrow \text{Cone}(X)
\end{array}
\]
in which every square is a pushout, and the maps indicated by “\(\rightarrow\)” have the HEP.

12.2. Corollary. If \(f: X \to Y\) is a map of based spaces with non-degenerate basepoints, then \(g: Y \to \text{Cone}(f)\) has the HEP, and \(\text{Cone}(f) \to \text{Cone}(f)\) is a homotopy equivalence.

Proof. \(g: Y \to \text{Cone}(f)\) is pushout of \(X \to \text{Cone}(X)\), which has the HEP as we have just shown, and therefore \(g\) has the HEP.

We have a diagram of pushout squares
\[
\begin{array}{c}
X \times \{1\} \rightarrow X \times \{0\} \cup \{x_0\} \times I \cup X \times \{1\} \rightarrow X \times I \\
\downarrow \\
* \rightarrow X \times \{0\} \cup \text{Cone}(\{x_0\}) \rightarrow \text{Cone}(X) \\
\downarrow \\
\text{Cone}(\{x_0\}) \rightarrow \text{Cone}(f)
\end{array}
\]
in which the indicated arrow has the HEP since \((X, x_0)\) has non-degenerate basepoint.

12.3. Theorem. Let \(f: X \to Y\) be a based map of spaces with non-degenerate base point. For any based space \(W\), the sequence
\[
[\text{Cone}(f), W]_* \xrightarrow{g_*} [Y, W]_* \xrightarrow{f_*} [X, W]_*
\]
is exact, in the sense that \(\alpha \in [Y, W]_*\) is in the image of \(g_*\) if and only if \(f^*(\alpha) = 0\) (i.e., \(\alpha f\) is homotopic to the constant map).

Proof. This is essentially a triviality.

Note that “exactness” only tells you about the preimage of 0 under \(f^*\), not the preimage of any other element of \([X, W]_*\).

13. Puppe sequence

We want to turn this sequence into a long exact sequence.

We note that the reduced cone construction actually defines a functor
\[
K: \text{Fun}([1], \text{Top}_*) \to \text{Fun}([1], \text{Top}_*),
\]
which on objects sends \((f: X \to Y)\) to \((g: Y \to \overline{\text{Cone}}(f))\). Note that the source of \(Kf\) is the target of \(f\), and this is compatible with the functorial structure. Thus, given a map \(f\) of based spaces we obtain a natural sequence
\[
f \xrightarrow{f} Kf \xrightarrow{K^2f} K^3f \xrightarrow{\ldots}
\]
of maps of based spaces, called the Puppe sequence or cofibration sequence. We may write this as
\[
X \xrightarrow{f} Y \xrightarrow{g} \overline{\text{Cone}}(f) \xrightarrow{h} \overline{\text{Cone}}(g) \xrightarrow{f'} \overline{\text{Cone}}(h) \xrightarrow{g'} \overline{\text{Cone}}(f') \xrightarrow{h'} \overline{\text{Cone}}(g') \xrightarrow{f''} \overline{\text{Cone}}(h') \xrightarrow{g''} \ldots.
\]
If the original spaces have non-degenerate basepoints, then every map in the sequence (except perhaps \( f \)) has the HEP, and every object has a non-degenerate basepoint.

Here is the key observation: the sequence repeats after three steps, up to homotopy. Consider the tautological quotient map

\[
\pi : \text{Cone}(g) \to \text{Cone}(f)/Y.
\]

Since \( \text{Cone}(g) \approx Y \cup_X \text{Cone}(X) \), we have

\[
\text{Cone}(f)/Y \approx (Y \cup_X \text{Cone}(X))/Y \approx \text{Cone}(X)/X \approx \Sigma X.
\]

the unreduced suspension of \( X \). Because we assume \( g \) has the HEP, we know that \( \pi \) is a homotopy equivalence. We get a diagram

\[
\begin{array}{cccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} \text{Cone}(f) \\
& \searrow & & \swarrow \text{Cone}(g) \\
& & Y/X & \text{Cone}(f)/Y \\
& & & \sim \\
& & & \text{Cone}(f)/Y \quad \text{Cone}(g)/Y \\
& & & \downarrow \downarrow \\
& & & \Sigma X \\
& & & \Sigma Y \\
\end{array}
\]

where the indicated maps are based homotopy equivalences. (If \( f \) has the HEP then \( \text{Cone}(f) \to Y/X \) is also a homotopy equivalence.)

Thus, shifted by 3 terms the objects of the sequence “look like” (i.e., are based homotopy equivalent) to the suspensions of the original objects. Shifting by 6 terms gives 2-fold suspensions, etc.

Unfortunately, the situation with maps is a little more complicated. For any \( X \), the reduced suspension

\[
\Sigma X = (X \times I)/(X \times \{0,1\}) \cup (\{x_0\} \times I)
\]

admits an involution \( \nu : \Sigma X \to \Sigma X \), defined using the map \( X \times I \to X \times I \) given by \( (x,t) \mapsto (x,1-t) \) and passing to the quotient. Note that \( \nu \nu = \text{id} \), and that \( \nu \) is a natural isomorphism \( \Sigma \to \Sigma \).

13.1. Proposition. The square

\[
\begin{array}{ccc}
\text{Cone}(g) & \xrightarrow{f'} & \text{Cone}(h) \\
\pi \downarrow & & \pi \downarrow \\
\text{Cone}(f)/Y & \xrightarrow{\sim} & \text{Cone}(g)/Y \\
& & \nu \circ (\Sigma f) \quad \nu \circ (\Sigma f) \\
\Sigma X & \xrightarrow{\nu \circ (\Sigma f)} & \Sigma Y \\
\end{array}
\]

commutes up to homotopy.

I gave several ways to think about this; I might put some of that in here. You can apparently prove this by writing down an explicit homotopy, as in G. Whitehead, *Elements of homotopy theory*, Ch. III, Lemma 6.15.

Note: people usually write “\(-\Sigma f\)” for \( \nu \circ (\Sigma f) \).

Combined with the short exact sequence we gave above, we get
13.2. **Corollary.** If \( f : X \to Y \) is a map between non-degenerately based spaces, and \( W \) any based space, then we have a long exact sequence of the form

\[
[X,W]_* \xrightarrow{f_*} [Y,W]_* \xleftarrow{\Sigma X,W} [\Sigma Y,W]_* \xleftarrow{- (\Sigma g)^*} [\Sigma Z,W]_* \xleftarrow{\Sigma Z,W} \cdots
\]

where \( Z = \text{Cone}(f) \).

The sets \( [\Sigma^k T,W]_* \) have a natural group structure when \( k \geq 1 \), which is abelian if \( k \geq 2 \). The maps \( (- \Sigma f)^* \) are anti-homomorphisms. For higher suspensions, they are just homomorphisms.

(Note: the group structure is defined using the suspension coordinate.)

If \( W \) is itself a grouplike commutative H-space, there is another group structure on \( [\Sigma^k T,W]_* \), which commutes with the one coming from the suspension coordinate. Thus in this case we just get an exact sequence of abelian groups.

I briefly described the notion of an \( \Omega \)-spectrum, and its relation to generalized cohomology theories. In doing so I stressed the necessity of working with weak equivalence rather than homotopy equivalence.

14. **Weak equivalence**

A map \( f : X \to Y \) of spaces is a **weak equivalence** if for each \( n \geq 0 \) and each choice \( x_0 \in X \) of basepoint, the induced map \( f_* : \pi_n(X,x_0) \to \pi_n(Y,f(x_0)) \) is a bijection.

For \( n = 0 \), \( \pi_0 \) is a based set. (We don’t usually care about the basepoint in this case.)

Recall that if \( x_0 \) and \( x_1 \) are connected by a path \( \gamma \), then we get a homomorphism \( c_{\gamma} : \pi_n(X,x_0) \xrightarrow{\sim} \pi_n(X,x_1) \) defined by “conjugation” by \( \gamma \). This conjugation commutes with \( f_* \), i.e., \( c_{f \gamma} f_* = f_* c_{\gamma} \).

This means that to check that \( f \) is a weak equivalence, we only need to check at one basepoint in each path component of \( X \).

A map of based spaces is a **weak equivalence** if it is so as a map of unbased spaces. Thus, even for based spaces, to check weak equivalence we need to look at all path components, not just the basepoint component. (Obviously if the spaces are path connected this is not a problem.)

The key theorem about weak equivalence is the following.

14.1. **Theorem** (Whitehead). If \( f : X \to Y \) is a weak equivalence and \( K \) is a CW-complex, then \( f_* : [K,X] \to [K,Y] \) is a bijection.

The same statement remains true with based spaces.

15. **HELP**

Consider maps \( i : A \to B \) and \( f : X \to Y \) in \( \text{Top}_* \). We say that \( i \) has the **homotopy extension and lifting property** (HELP) wrt \( f \) if for every commuting solid diagram of the form

\[
\begin{array}{ccc}
A \times \{1\} & \xrightarrow{u} & B \times \{1\} & \xrightarrow{\bar{u}} & X \\
\downarrow & & \downarrow & & \downarrow f \\
(A \times I) \cup_{A \times \{0\}} (B \times \{0\}) & \xrightarrow{(h,v)} & B \times I & \xrightarrow{\bar{h}} & Y \\
\end{array}
\]

there exist dotted arrows \( \bar{v} \) and \( \bar{h} \) making the diagram commute.
That is, given a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{h} & & \downarrow{f} \\
B & \xrightarrow{v} & Y
\end{array}
\]

which commutes up to a homotopy \( h \), there exists an extension \( \bar{u} \) of \( u \) to \( B \) (so \( \bar{u}i = u \)), and a homotopy \( \bar{h} : v \sim f\bar{u} \) of maps \( B \to X \), which extends the homotopy \( h : vi \sim fu \) of maps \( A \to Y \), so that \( \bar{u} \) can be regarded as a lift of \( v \) up to homotopy.

15.1. **Remark.** Consider the special case when \( A = \emptyset \). Then \( i : \emptyset \to B \) has the HELP with respect to \( f \) if and only if every \( B \to Y \) can be lifted along \( f \) up to homotopy, i.e., if \( f_* : [B,X] \to [B,Y] \) is surjective.

15.2. **Remark.** Consider the special case of \( i : K \times \{0,1\} \to K \times I \). If \( i \) has the HELP wrt \( f \), then for any two maps \( u_0, u_1 : K \to X \), if \( f u_0 \sim f u_1 \), then \( u_0 \sim u_1 \), i.e., \( f_* : [K,X] \to [K,Y] \) is injective.

I’ll write HELP(\( f \)) for the class of all maps \( i \) which have the HELP with respect to \( f \). We are going to prove the following theorem, which implies the Whitehead theorem I gave earlier.

15.3. **Theorem.** Let \( f : X \to Y \) be a map. TFAE.

1. \( f \) is a weak equivalence.
2. Every CW-inclusion \( K \subseteq L \) is in HELP(\( f \)).
3. For all \( n \geq 0 \), the inclusion \( S^{n-1} \to D^n \) is in HELP(\( f \)).

To see this, we first reformulate HELP in terms of an honest lifting diagram.

Given \( f \), consider the space

\[
\text{Path}(f) := \text{Map}(\{1\},X) \times_{\text{Map}(\{1\},Y)} \text{Map}(I,Y)
\]

called the **path space** of \( f \). A point in Path(\( f \)) is a pair \((x, \gamma)\) consisting of a point \( x \in X \) and a path \( \gamma \) in \( Y \) whose endpoint is \( f(x) \).

There is an evident inclusion

\[
i : X \to \text{Path}(f), \quad x \mapsto (x, f(x)),
\]

using the constant path at \( f(x) \in Y \).

15.4. **Proposition.** The map \( i \) is a deformation retraction.

**Proof.** Consider the homotopy \( H_t \) of maps \( \text{Path}(f) \to \text{Path}(f) \) defined by

\[
H_t(x, \gamma) = (x, (s \mapsto \gamma(1-t(1-s))))
\]

\[\square\]

There is an evaluation map

\[
p_{\text{fib}}(f) = p : \text{Path}(f) \to Y, \quad (x, \gamma) \mapsto \gamma(0),
\]

called the **path fibration** associated to \( f \).

We define the following relation on morphisms on spaces: given \( i : A \to B \) ad \( p : U \to V \), say \( "i \sqcup p" \) if for every pair \((u,v)\) of maps \( u : A \to U \) and \( v : B \to V \) such that \( pu = vi \), there exists \( s : B \to U \) such that \( si = u, ps = v \).
15.5. Example. A map \( i \) has the HEP iff \( i \vartriangleright e_0 \) for every space \( X \), where \( e_0 : \text{Map}(I, X) \rightarrow \text{Map}(\{0\}, X) \) is the evaluation map.

15.6. Proposition. We have that \( i \in \text{HELP}(f) \) if and only if \( i \vartriangleright \text{pfib}(f) \).

Proof. This is a straightforward exercise. □

As a consequence, we discover that \( \text{HELP}(f) \) has closure properties similar to \( \text{HEP}(f) \). Thus,

- the composite of two maps in \( \text{HELP}(f) \) is in \( \text{HELP}(f) \),
- the coproduct of any collection of elements in \( \text{HELP}(f) \) is in \( \text{HELP}(f) \),
- if \( i \in \text{HELP}(f) \), then any pushout \( i' \) of \( i \) is in \( \text{HELP}(f) \),
- if \( X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \) is a sequence such that each \( X_k \rightarrow X_{k+1} \) is in \( \text{HELP}(f) \), then \( X_0 \rightarrow X_\infty = \text{colim} X_k \) is in \( \text{HELP}(f) \).

This gives us part of our theorem.

15.7. Proposition. Every CW-inclusion is in \( \text{HELP}(f) \) if and only if every \( i_n : S^{n-1} \rightarrow D^n \) is in \( \text{HELP}(f) \) for all \( n \geq 0 \).

To get the other part, we need to relate the condition \( i_n \in \text{HELP}(f) \) to the effect of \( f \) on homotopy groups. To do this, fix \( f : X \rightarrow Y \), and suppose we pick a basepoint \( x_0 \in X \), hence a basepoint \( y_0 = f(x_0) \in Y \). We are going to define a notion

\[ \pi_n(f) \]

of homotopy groups of the map \( f \) of based spaces (which will only be a group if \( n \geq 1 \)).

In most textbooks, you will find the notion

\[ \pi_n(f) \]

of homotopy groups of a pair, where \( x_0 \in X \subseteq Y \). We are generalizing this to maps which are not merely inclusions of subspaces, but to all maps. This is kind of nonstandard, but we can do it, so we will. It is especially handy to do so for this particular application.

We work in the category

\[ \text{Top}_*^{[1]} = \text{Fun}([1], \text{Top}_*), \]

whose objects are functors \( U : [1] \rightarrow \text{Top}_* \), i.e., are maps \( u : U_0 \rightarrow X_1 \) of based spaces, and whose morphisms \( \phi : U \rightarrow V \) are commutative squares

\[
\begin{array}{ccc}
U_0 & \xrightarrow{u} & U_1 \\
\phi_0 \downarrow & & \downarrow \phi_1 \\
V_0 & \xrightarrow{v} & V_1
\end{array}
\]

of based spaces. (The problem with \( \text{Top}_*^{[1]} \) is that the notation is annoying.)

An example of an object in \( \text{Top}_*^{[1]} \) is the map \( i_n : S^{n-1} \rightarrow D^n \), where we use a basepoint \( s_0 \in S^{n-1} \).

There is a notion of homotopy of maps \( \Psi : U \rightarrow V \) in \( \text{Top}_*^{[1]} \), which is homotopies

\[ \Phi_0 : U_0 \times I \rightarrow V_0, \quad \Phi_1 : U_1 \times I \rightarrow V_1 \]

rel basepoints such that \( v \circ \Phi_0 = \Phi_1 \circ (u \times \text{id}) \).

Another way to think of this: there is a subspace

\[ \text{Map}_{\text{Top}_*^{[1]}}(U, V) \subseteq \text{Map}_*(U_0, V_0) \times \text{Map}_*(U_1, V_1) \]

of points \( (f_0, f_1) \) such that \( v f_0 = f_1 u \). A homotopy is just a path in this space.

Yet another way to think about this: given any object \( U = (u : U_0 \rightarrow U_1) \in \text{Top}_*^{[1]} \), we can define \( U \times I \) to be \( u \times \text{id} : U_0 \times I \rightarrow U_1 \times I \). Then a homotopy of maps \( U \rightarrow V \) is just a single map \( U \times I \rightarrow V \).
For $n \geq 1$, define $$\pi_n(f) := [(i_n:S^{n-1} \rightarrow D^n, (f:X \rightarrow Y)]_*,$$
both $i_n$ and $f$ thought of as objects of Top$^{[1]}$. I might write $\pi_n(f;x_0)$ if I want to make the choice of basepoint clear.

When $n = 0$, $\pi_n(f:x_0)$ doesn’t exist: $S^{-1} = \emptyset$ has no basepoint! For each $n \geq 1$, it is a based set, where the basepoint is the constant map to the basepoints of $X$ and $Y$. The based set $\pi_n(f;x_0)$ admits a natural group structure when $n \geq 2$, which is abelian if $n \geq 3$. To see this we use a particular presentation of the map $i_n$. Consider the spaces $$I^n \supset \partial I^n \supset J^{n-1} := (\partial I^{n-1} \times I) \cup (I^{n-1} \times \{0\}).$$
Collapsing the subspace $J^{n-1}$ gives a based arrow $$(J^{n-1}/J^{n-1}) \in \partial I^n/J^{n-1} \rightarrow I^n/J^{n-1}$$
which is seen to be isomorphic to $i_n$. Thus a map $i_n \rightarrow f$ in Top$^{[1]}$ corresponds exactly to a map $\partial I^n \rightarrow I^n \rightarrow f$ such that $f(J^{n-1}) = \{x_0\}$. We obtain a group law by “concatenation” along any of the first $n-1$ coordinates.

15.8. Proposition. Let $f: X \rightarrow Y$ be an arrow, and $n \geq 1$. The following are equivalent.

Th 28 Sep

(1) $(i_n:S^{n-1} \rightarrow D^n) \in \text{HELP}(f)$.
(2) For all $x_0 \in X$, we have $\pi_n(f;x_0) = \{\ast\}$.

Proof. First consider any $i: A \rightarrow B$, and let’s restate “$i \in \text{HELP}(f)$” in the language of Top$^{[1]}$. By definition, $i \in \text{HELP}(f)$ means that a dotted arrow exists in any diagram of the form

\[
\begin{array}{ccc}
(A \times \{1\}) & \xrightarrow{i} & (B \times \{1\}) \\
\downarrow & & \downarrow \text{u} \\
((A \times I) \cup_{A \times \{0\}} (B \times \{0\})) & \rightarrow & B \times I
\end{array}
\]

where $j$ is the obvious map and $u$ is anything. This diagram is in Top$^{[1]}$. If we have a basepoint $a_0 \in A$, this determines a basepoint in all the other spaces, and the diagram is in Top$^{[1]}$.

Now let $i = i_n:S^{n-1} \rightarrow D^n$. The map $j$ is represented by the square

\[
\begin{array}{ccc}
S^{n-1} \times \{1\} & \xrightarrow{j} & D^n \times \{1\} \\
\downarrow & & \downarrow \\
(S^{n-1} \times I) \cup_{S^{n-1} \times \{0\}} (D^n \times \{0\}) & \rightarrow & D^n \times I
\end{array}
\]

which is the usual picture of: an open can, and a lid, both mapping into a solid can. Now lets think about the condition $\pi_n(f;x_0) = \{\ast\}$, i.e., that every $(S^{n-1} \rightarrow D^n) \Rightarrow (X \rightarrow Y)$ is homotopic to a constant map (by a homotopy rel basepoints). We know that if we are talking about a map $A \rightarrow X$ of based spaces, it is null-homotopic (rel basepoint) if and only if it extends to $A \subseteq \overline{\text{Cone}(A)} \rightarrow X$. This is because $\overline{\text{Cone}(A)}$ is the quotient $A \times I/(A \times \{1\}) \cup (\{a_0\} \times I)$.

The same is true in Top$^{[1]}$: a map $(A \xrightarrow{i} B) \Rightarrow (X \xrightarrow{j} Y)$ is null-homotopic if and only if there exists a lift in

\[
\begin{array}{ccc}
(A \xrightarrow{i} B) & \xrightarrow{u} & (X \xrightarrow{j} Y) \\
\downarrow & & \downarrow \\
(\overline{\text{Cone}(A)} \rightarrow \overline{\text{Cone}(B)})
\end{array}
\]
We observe that

\[ (A \rightarrow B) \times I / (((A \rightarrow B) \times \{1\}) \cup \{(a_0) \rightarrow \{b_0\} \times I}) \]

in Top\[^{[1]}\].

Now if \( i = i_n \), we see that \( (S^{n-1} \rightarrow D^n) \Rightarrow (\text{Cone}(S^{n-1}) \rightarrow \text{Cone}(D^n)) \) is homeomorphic to the one involved in the HELP.

The previous proposition did not handle the case of \( i_0 \in \text{HELP}(f) \). Instead we have the following.

15.9. **Proposition.** \( (i_0: \emptyset \rightarrow D^0) \in \text{HELP}(f) \) if and only if \( f_\ast: \pi_0X \rightarrow \pi_0Y \) is surjective.

We are going to produce an exact sequence of based sets:

\[ \cdots \rightarrow \pi_n(X; x_0) \xrightarrow{f_\ast} \pi_n(Y; f(x_0)) \rightarrow \pi_{n-1}(f; x_0) \rightarrow \pi_{n-1}(X; x_0) \xrightarrow{f_\ast} \pi_{n-1}(Y; x_0) \rightarrow \cdots \]

\[ \cdots \rightarrow \pi_2(f; x_0) \rightarrow \pi_1(X; x_0) \xrightarrow{f_\ast} \pi_1(Y; f(x_0)) \rightarrow \pi_1(f; x_0) \rightarrow \pi_0(X; x_0) \xrightarrow{f_\ast} \pi_0(Y; f(x_0)). \]

The maps in the sequence will be maps of groups whenever that makes sense.

Assuming this for a moment, let’s finish the proof of the theorem:

\[ f: X \rightarrow Y \text{ is a weak equivalence iff } (i_n: S^{n-1} \rightarrow D^n) \in \text{HELP}(f) \text{ for all } n \geq 0. \]

In light of what we just proved, this is:

\[ f: X \rightarrow Y \text{ is a weak equivalence iff } (i) \pi_0f \text{ is surjective, and (ii) } \pi_n(f; x_0) = \{\ast\} \text{ for all } n \geq 1, \text{ } x_0 \in X. \]

*End of proof of theorem.* This is just a standard argument involving the long exact sequence. It is worthwhile to think carefully about how this works for the small values of \( n \), e.g., \( n = 0, 1 \), since \( \pi_n(f; x_0) \) is either undefined or not a group in this case.

Now let's define the long exact sequence of relative homotopy groups (called the long exact sequence of a pair when \( f: X \rightarrow Y \) is inclusion of a subspace). We start with the following infinite sequence of maps in Top\[^{[1]}\].

\[
\begin{align*}
A_{n-1} &= (\ast \rightarrow S^{n-1}) \\
B_{n-1} &= (S^{n-1} \rightarrow S^{n-1}) \\
C_{n-1} &= (S^{n-1} \rightarrow D^n) \\
A_n &= (\ast \rightarrow S^n) \\
B_n &= (S^n \rightarrow S^n)
\end{align*}
\]

We observe that

\[
[A_{n-1}, f]_* \approx \pi_{n-1}(Y; y_0), \quad [B_{n-1}, f]_* \approx \pi_{n-1}(X; x_0), \quad [C_{n-1}, f]_* \approx \pi_n(f; x_0),
\]

and also that the map \([B_{n-1}, f]_* \rightarrow [A_{n-1}, f]_*\) coincides with \( f_\ast: \pi_{n-1}(X; x_0) \rightarrow \pi_{n-1}(Y; y_0)\).

This gives us the sequence we want. Now we need to show it is exact.

Given a map \( F: (A_0 \rightarrow A_1) \Rightarrow (B_0 \rightarrow B_1) \) of objects in Top\[^{[1]}\], we can define its reduced mapping cone:

\[ \text{Cone}\[^{[1]}\](F): \text{Cone}(F_0; A_0 \rightarrow B_0) \rightarrow \text{Cone}(F_1; A_1 \rightarrow B_1). \]

This is just the functoriality of the cone construction. It can also be described in terms of colimits in Top\[^{[1]}\]:

\[ ((A \times I) \cup_{AX\{0\}} (B \times \{0\})) / ((A \times \{1\}) \cup \{(a_0) \times I\}). \]
Easy exercise: for any $W \in \text{Top}^*[\mathbb{I}]$, we get an exact sequence of sets
\[
\text{Cone}^*[\mathbb{I}](F), W) \rightarrow [B, W] \rightarrow [A, W].
\]

Now we make the observation that the sequence
\[
A_{n-1} \xrightarrow{\alpha_{n-1}} B_{n-1} \xrightarrow{\beta_{n-1}} C_{n-1} \xrightarrow{\gamma_{n-1}} A_n \xrightarrow{\alpha_n} B_n \rightarrow \cdots
\]
in $\text{Top}^*[\mathbb{I}]$ is isomorphic to the cofiber sequence of $\alpha$:
\[
A_{n-1} \xrightarrow{\alpha_{n-1}} B_{n-1} \xrightarrow{j} \text{Cone}^*[\mathbb{I}](\alpha_{n-1}) \xrightarrow{\delta} \Sigma A_n \rightarrow \Sigma B_n \rightarrow \cdots.
\]
This immediately gives exactness at $B$: the sequence $\pi_n(f; x_0) \rightarrow \pi_{n-1}(X; x_0) \rightarrow \pi_{n-1}(Y; y_0)$ is exact. For exactness at the other two positions, we need to know that the other places in the sequence are the same as the relevant mapping cones.

In the case of $\text{Top}^*$, we used that if $f: A \rightarrow X$ is a map with the HEP, then $\text{Cone}(f) \rightarrow X/A$ is a based homotopy equivalence. (I have suggested that you also need non-degenerate basepoints, but it turns out you don’t.) The proof can be given by explicit constructions:

- Choose a retraction $r: X \times I \rightarrow \text{Cyl}(f)$ of $j: \text{Cyl}(f) \rightarrow X \times I$ given by the HEP.
- Let $p: \text{Cyl}(f) \rightarrow X$ be given by $p((a, s)) = a$, $p((x, 0)) = x$.
- Let $q: X \rightarrow \text{Cyl}(f)$ be defined by $q(x) = r(x, 1)$.
- Let $G_t(x) = p(r(x, t))$, defining a homotopy $\text{id}_x \sim_A pq$ of maps $X \rightarrow X$.
- Let $H_t(x) = r(x, t + s(1 - t))$, defining a homotopy $\text{id}_{\text{Cyl}(f)} \sim_{A \times \{1\}} qp$ of maps $\text{Cyl}(f) \rightarrow \text{Cyl}(f)$.
- Clearly $p$ and $q$ preserve the basepoints $a_0 \in A \subset X$ and $(a_0, 1) \in \text{Cyl}(f)$. Furthermore, $G_t$ is rel $a_0$.
- The homotopy $H$ satisfies $H_t(\{a_0\} \times I) \subseteq \{a_0\} \times I$.

Therefore, passing to the quotients $X/A$ and $\text{Cone}(f) = \text{Cyl}(f)/((A \times \{1\}) \cup \{(a_0) \times I\})$ gives the desired homotopy equivalence.

Say that a map $F: A \Rightarrow X$ in $\text{Top}^*[\mathbb{I}]$ has the HEP if the evident map $A \times I \cup X \times \{0\} \rightarrow X \times I$ admits a retraction. This is the same as saying that we have retractions $r_\epsilon$ of $j_\epsilon: A_\epsilon \times I \cup X_\epsilon \times \{0\} \rightarrow X_\epsilon \times I$ for $\epsilon = 0, 1$ which commute with the maps induced by the $a: A_0 \rightarrow A_1$ and $x: X_0 \rightarrow X_1$. (Draw the digrams.)

The argument I gave for $\text{Top}^*$ formally works in $\text{Top}^*[\mathbb{I}]$: once you have your compatible retractions $r_\epsilon$, the rest of the construction gives you once you want, since every step after that point is explicit and functorial. We have essentially proved the following.

15.10. **Proposition.** If $F: A \Rightarrow B$ in $\text{Top}^*[\mathbb{I}]$ has the HEP, then $\text{Cone}^*[\mathbb{I}](F) \rightarrow B/A$ is a homotopy equivalence in $\text{Top}^*[\mathbb{I}]$.

Thus, to show that
\[
\text{Cone}^*[\mathbb{I}](\beta_{n-1}) \rightarrow C_{n-1}/B_{n-1} \approx \Sigma A_{n-1}, \quad \text{Cone}^*[\mathbb{I}](\gamma_{n-1}) \rightarrow \text{Cone}^*[\mathbb{I}](\beta_{n-1})/C_{n-1} \approx \Sigma B_{n-1}
\]
are homotopy equivalences in $\text{Top}^*[\mathbb{I}]$, we need to show that
\[
\beta_{n-1}: B_{n-1} \rightarrow C_{n-1}, \quad \gamma_{n-1}: C_{n-1} \rightarrow \text{Cone}(\beta_{n-1})
\]
have the HEP. Recall that $\beta_{n-1}$ is
\[
(S^{n-1} \rightarrow S^{n-1}) \Rightarrow (S^{n-1} \rightarrow D^n),
\]
while $\gamma_{n-1}$ is the incusion into the mapping cone, which looks like
\[
(S^{n-1} \rightarrow D^n_+) \Rightarrow (D^n \rightarrow D^{n+1}).
\]
I’ll do this using the following criterion, which is easily checked in the two examples.
15.11. **Proposition.** Let $F : (A_0 \xrightarrow{a} A_1) \Rightarrow (B_0 \xrightarrow{b} B_1)$ be a map in $\text{Top}^{[1]}$. This map has the HEP if and only if the maps

$$A_0 \xrightarrow{F_0} B_0, \quad A_1 \cup A_0 \xrightarrow{(F_1, b)} B_1$$

have the HEP in $\text{Top}$.

Proving this uses the following description of maps in $\text{Top}^{[1]}$. A map $F : B \rightarrow U$ consists of two pieces of data:

$$F_0 : B_0 \rightarrow U_0, \quad F_1 : B_1 \rightarrow U_1,$$

together with a constraint:

$$uF_0 = F_1 b.$$

The idea is that we can package this information into a 2-step process.

1. A map

$$B_0 \xrightarrow{F_0} U_0$$

2. An extension $F_1$ in

$$\begin{array}{ccc}
B_0 & \xrightarrow{uF_0} & U_1 \\
\downarrow b & & \downarrow F_1 \\
B_1 & & \\
\end{array}$$

Note that the second step is dependent on the first.

**Proof of proposition.** Having the HEP amounts to a lifting property

$$\begin{array}{ccc}
(A_0 \xrightarrow{a} A_1) & \xrightarrow{U} & (\text{Map}(I, W_0) \rightarrow \text{Map}(I, W_1)) \\
\downarrow F & & \downarrow F \\
(B_0 \xrightarrow{b} B_1) & \xrightarrow{V} & (\text{Map}(\{0\}, W_0) \rightarrow \text{Map}(\{0\}, W_1))
\end{array}$$

Finding $s$ amount to successively solving the lifting problems

$$\begin{array}{ccc}
A_0 & \xrightarrow{F_0} & \text{Map}(I, W_0) \\
\downarrow F_0 & & \downarrow s_0 \\
B_0 & \xrightarrow{s_0} & \text{Map}(\{0\}, W_0)
\end{array} \quad \quad \begin{array}{ccc}
A_1 \cup A_0 & \xrightarrow{(u_1, s_0)} & \text{Map}(I, W_1) \\
\downarrow (F_1, b) & & \downarrow s_1 \\
B_1 & \xrightarrow{s_1} & \text{Map}(\{0\}, W_1)
\end{array}$$

We have now proved what we needed to get the LES of relative homotopy theory. To summarize what we have proved, we have shown that for a map $f : X \rightarrow Y$ of spaces, TFAE:

1. $f_* : \pi_n(X; x_0) \rightarrow \pi_n(Y; f(x_0))$ is a bijection for all $x_0 \in X$, $n \geq 0$. (**Weak equivalence.**)
2. $\pi_n(f; x_0) \approx \{\ast\}$ for all $x_0 \in X$, $n \geq 1$, and $\pi_0 X \rightarrow \pi_0 Y$ is surjective.
3. Every CW inclusion $K \hookrightarrow L$ is in $\text{HELP}(f)$.

The final property implies that if $f$ is a weak equivalence, then $[K, X] \rightarrow [K, Y]$ is a bijection whenever $K$ is a CW-complex.

As a consequence, we get Whitehead’s theorem: a map between CW-complexes is a homotopy equivalence if and only if it is a weak equivalence.
16. Relative homotopy and homotopy fibers

Now we observe that relative homotopy groups \( \pi_n(f; x_0) \) of a map \( f: X \to Y \) are actually the homotopy groups of a space.

Given a map \( g: U \to V \), we write \( \text{Fib}(g; v_0) := g^{-1}(v_0) \) for the fiber. For a general map \( f: X \to Y \) and \( y_0 \in Y \) let

\[
\text{hFib}(f; y_0) := \text{Fib}(pfib(f); y_0) = \{ (x, \gamma) \in X \times \text{Map}(I, Y) \mid \gamma(1) = f(x), \gamma(0) = y_0 \},
\]

the \textbf{homotopy fiber} of \( f \), defined to be the actual fiber of the path fibration associated to \( f \).

16.1. Proposition. \textit{We have a natural isomorphism} \( \pi_n(f; x_0) \approx \pi_{n-1}(\text{hFib}(f; f(x_0), x_0)) \).

16.2. Example. If \( f: \{x_0\} \to X \), then \( \text{hFib}(f; x_0) \approx \Omega X \), the base loop space. Thus we have isomorphisms

\[
\pi_n(X, x_0) \xrightarrow{\approx} \pi_n(f, x_0) \approx \pi_{n-1}(\Omega X)
\]

for \( n \geq 1 \).

\textit{Proof}. I'll actually prove something much stronger: for a based space \( T \), there is a homeomorphism

\[
\text{Map}_* (T, \text{hFib}(f, f(x_0))) \approx \text{Map}_* ((T \to \overline{\text{Cone}(T)}), (X \xrightarrow{f} Y)).
\]

Setting \( T = S^{n-1} \) and looking at path components gives the result.

Recall that \( \text{hFib}(f; y_0) = \text{fiber of Map}(\{0\}, X) \times_{\text{Map}(\{0\}, Y)} \text{Map}(I, Y) \to \text{Map}(\{1\}, Y) \) over \( y_0 \). A map \( T \to \text{hFib}(f; y_0) \) is precisely a diagram

\[
\begin{array}{ccc}
T \times \{0\} & \longrightarrow & X \\
\downarrow & & \downarrow f \\
T \times I & \longrightarrow & Y
\end{array}
\]

such that \( v(T \times \{1\}) = \{y_0\} \); i.e., \( v \) factors through \( \text{Cone}(T) \) sending the vertex to \( y_0 \).

When \( f \) is a based map, i.e., \( y_0 = f(x_0) \), then \( \text{hFib}(f; f(x_0)) \) has a natural basepoint \( (x_0, y_0) \). A based map \( T \to \text{hFib}(f; f(x_0)) \) exactly the same as a diagram

\[
\begin{array}{ccc}
T & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\overline{\text{Cone}(T)} & \longrightarrow & Y
\end{array}
\]

in based spaces. \hfill \square

The path fibration of \( f \) has the form \( \pi: P \to Y \), with \( P = \text{Path}(f) \) homotopy equivalent to \( X \) and with fiber \( F = \text{hFib}(f) \). Thus we can reinterpret the relative LES:

\[
\cdots \to \pi_n X \to \pi_n Y \to \pi_n f \to \pi_{n-1} X \to \pi_{n-1} Y \to \cdots
\]

as the LES:

\[
\cdots \to \pi_n P \to \pi_n Y \to \pi_{n-1} F \to \pi_{n-1} X \to \pi_{n-1} Y \to \cdots.
\]

17. Serre fibrations and the long exact sequence of a Serre fibration

A map \( f: X \to Y \) \textbf{is a Serre fibration} if for every \( n \geq 1 \), and every diagram of the form

\[
\begin{array}{ccc}
I^{n-1} \times \{0\} & \longrightarrow & X \\
\downarrow & & \downarrow f \\
I^{n-1} \times I & \longrightarrow & Y
\end{array}
\]
a lift exists. Equivalently, if \((I^{n-1} \times \{0\} \to I^{n-1} \times I) \sqcup f\) for all \(n\).

17.1. Example. Every covering map has this property. In fact, in a covering map, such lifts are always unique.

17.2. Example. Any projection \(\pi: F \times Y \to Y\) is a Serre fibration.

17.3. Example. Any fiber bundle is a Serre fibration.

17.4. Example. The evaluation map \(\text{Map}(I, X) \to \text{Map}(\{\emptyset\}, X)\) is a Serre fibration.

17.5. Example. The path fibration of any map is a Serre fibration. In fact, the pullback of a Serre fibration along any map is a Serre fibration, because of the lifting property.

Note that if \(p: E \to B\) is a Serre fibration, then the class \(\mathcal{A}\) of maps \(j\) such that \(j \sqcup p\) is closed under composition, pushout, etc. For instance, the inclusion

\[
(S^{n-1} \times \{0\}) \cup (\{s_0\} \times I) \to S^{n-1} \times I
\]

(\(n\) end of cylinder + base point line segment) is in this class, because there is a pushout square of the form

\[
(D^n \times \{0\}) \cup (S^{n-1} \times I) \to D^n \times I
\]

and the map along the top is homeomorphism to \(I^n \times \{0\} \to I^n \times I\).

More generally, for any CW-inclusion \(j: K \to L\), the induced inclusion

\[
\text{Cyl}(j) = (L \times \{0\}) \cup (K \times I) \to L \times I
\]

lifts against any Serre fibration. (Proof: induction on the cells of \(L \setminus K\): for each \(n\)-cell you glue in a \((D^n \times \{0\}) \cup (S^{n-1} \times I) \to D^n \times I\) on the mapping cylinder.) This is called the **covering homotopy extension property** (CHEP).

17.6. Lemma (Covering homotopy extension property). For any Serre fibration \(p: E \to B\) and any CW inclusion \(j: K \to L\), and any diagram

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & \text{Map}(L, E) \\
\downarrow & & \downarrow \\
I & \longrightarrow & \text{Map}(j, p)
\end{array}
\]

a lift exists.

If \(p\) and \(j\) are based maps, the same result holds with \(\text{Map}\) replaced with \(\text{Map}_{\ast}\).

In other words, given a “1-parameter family of lifting problems” relating \(j\) and \(p\) such that a lift exists at \(t = 0\), then there exists a continuous family of lifts at every \(t\) (if \(j\) is CW and \(p\) a Serre fibration).

Proof. This is the usual adjunction argument. Note that the vertical map on the right sends \(s: L \to E\) to the map \((K \xrightarrow{j} L) \Rightarrow (E \xrightarrow{p} B)\) defined by \(sj: K \to E\) and \(ps: L \to B\).

If \(j\) and \(p\) are based maps and we consider \(u: \{0\} \to \text{Map}_{\ast}(L, E) \subseteq \text{Map}(L, E)\) and \(v: I \to \text{Map}_{\ast}(j, p) \subseteq \text{Map}(j, p)\), we see that any lift \(s: I \to \text{Map}(L, E)\) automatically lands in \(\text{Map}_{\ast}(L, E)\), since \(v\) already tells us \(\{k_0\} \times I \to K \times I \to E\) goes to the basepoint of \(E\).

17.7. Proposition. Let \(p: E \to B\) be a Serre fibration, let \(b_0 \in B\) and \(F = \text{Fib}(p, b_0)\), and choose \(e_0 \in F \subseteq E\). Then the map \((F \xrightarrow{\pi} \{b_0\}) \Rightarrow (E \xrightarrow{p} B)\) in \(\text{Top}^{[1]}\) induces isomorphisms

\[
\pi_n(\pi, e_0) \xrightarrow{\sim} \pi_n(p, e_0).
\]
Proof. The function $\pi_n(F \to \ast, e_0) \to \pi_n(E \to B, e_0)$ is given by

$$
\begin{array}{ccc}
S^{n-1} & \rightarrow & F \\
\downarrow i_n & & \downarrow p \\
D^n & \rightarrow & \{b_0\} \\
\end{array}
$$

To show that it is surjective, we need to show that any based map $\alpha: (S^{n-1} \to D^n) \Rightarrow (E \to B)$ can be lifted to $(F \to \{b_0\})$ up to homotopy.

First choose $H_t$, a deformation retraction of $D^n$ to its basepoint $s_0$, so $H_0 = \text{id}$, $H_1 = \{s_0\}$, and $H_t(s_0) = s_0$. Then let $K$ be the composite

$$
D^n \times I \xrightarrow{H} D^n \xrightarrow{\alpha_1} B.
$$

Note that $K(D^n \times \{1\}) = \{b_0\}$. Consider

$$
\begin{array}{ccc}
(S^{n-1} \times \{0\}) \cup (\{s_0\} \times I) & \xrightarrow{(\alpha_0, \pi_0)} & E \\
\downarrow H & & \downarrow p \\
S^{n-1} \times I & \rightarrow & D^n \times I \\
\end{array}
$$

Because $p$ is a Serre fibration a lift $H$ exists. In terms of CHEP, we have a path $I \rightarrow \text{Map}_x(\{s_0\} \to S^{n-1}, p)$ together with a lift at the 0 end, and thus by CHEP have a lift along the whole path.

Therefore $(H, K)$ gives a homotopy of based maps $(S^{n-1} \to D^n) \Rightarrow (E \to B)$, which at the 0-end is the original $\alpha$, and at the 1 end factors through $(F \to \{b_0\})$. This gives the surjectivity.

A similar but slightly more complex argument gives injectivity.

The relative LES gives bijections $\pi_n(\pi; x_0) \to \pi_{n-1}(F; x_0)$, and thus we get a LES

$$
\cdots \rightarrow \pi_n(X; x_0) \rightarrow \pi_n(Y; f(x_0)) \rightarrow \pi_{n-1}(F; x_0) \rightarrow \cdots.
$$

Examples.

If $f: X \to Y$ is a covering map, then $\pi_k F \approx 0$ when $k \geq 1$. Thus we get an exact sequence of based sets

$$
1 \rightarrow \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0)) \rightarrow \pi_0(F, x_0) \rightarrow \pi_0(X, x_0) = \pi_0(Y, f(x_0)),
$$

and isomorphisms $\pi_k(X, x_0) \approx \pi_k(Y, f(x_0))$ if $k \geq 2$.

For instance, this computes $\pi_\ast S^1$, using the universal cover $\mathbb{R} \to S^1$.

For every $n$, there is a circle fibration

$$
S^1 \to S^{2n+1} \to \mathbb{C}P^n.
$$

Passage to the direct limit gives $S^1 \to S^{\infty} \to \mathbb{C}P^{\infty}$. So $\mathbb{C}P^{\infty} \approx K(\mathbb{Z}, 2)$.

For $n = 1$ this is the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. Thus we can read off:

$$
\pi_2 S^2 \approx \pi_1 S^1 \approx \mathbb{Z},
$$

$$
\pi_3 S^2 \approx \pi_3 S^3 \approx \mathbb{Z},
$$

$$
\pi_k S^2 \approx \pi_k S^3 \text{ if } k \geq 3.
$$

Similar arguments apply to $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$. Note that $S^3 \to S^7$ is null homotopic, so we actually get short exact sequences

$$
0 \rightarrow \pi_k S^7 \rightarrow \pi_k S^4 \rightarrow \pi_{k-1} S^3 \rightarrow 0.
$$

We can say a little more.
17.8. **Proposition.** We have $\pi_k S^4 \approx \pi_k S^7 \oplus \pi_{k-1} S^3$.

More generally.

17.9. **Proposition.** If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration with the inclusion $i$ null-homotopic, then $\pi_k B \approx \pi_k E \oplus \pi_{k-1} F$.

**Proof.** Choose a null homotopy of $i$, so $H_t$ is a homotopy of maps $F \to E$ such that $H_0 = i$ and $H_1 =$ constant map at basepoint. I'm going to construct a function $\pi_{k-1} F \to \pi_k B$ which is a section of $\pi_k B \to \pi_{k-1} F$.

Recall that $\pi_{k-1} F \approx \pi_k (F \to \{b_0\})$. Define a function $\pi_k (F \to *) \to \pi_k B$ as follows. Given an element in the domain represented by $(f, g): (S^{k-1} \to D^k) \Rightarrow (E \to B)$, we get a map

$$(g, p \circ H_t \circ f): S^k \approx D^k \cup_{S^k} \text{Cone}(S^k) \to B.$$ 

You can show this is a group homomorphism for $k \geq 2$ (because there is a free coordinate). Now check that

18. **Connectivity**

A map $f: X \to Y$ is $n$-**connected** if (i) $\pi_0 X \to \pi_0 Y$ is surjective, and (ii) $\pi_k (f; x_0) \approx \{\ast\}$ for all $k \leq n$ and all $x_0 \in X$. By the relative LES, this means exactly that

1. $\pi_k (X; x_0) \to \pi_k (Y; f(x_0))$ is a bijection if $k \leq n$, and
2. $\pi_n (X; x_0) \to \pi_n (Y; f(x_0))$ is a surjection.

This is also called an $n$-**equivalence**.

The above definition makes sense for all $n \geq 0$.

**Warning.** There are a lot of $\pm 1$ issues in this convention. Some people might prefer to call this $(n - 1)$-connected, for instance, though in classical algebraic topology the convention I am using is pretty standard.

A space $X$ is said to be $n$-**connected** (for $n \geq 0$) if it is non-empty, and if $\pi_k (X; x_0) \approx \{\ast\}$ for all $k \leq n$ and all $x_0 \in X$.

We may also say that $X$ is $(-1)$-connected if it is non-empty.

**Warning.** $X$ is $n$-connected if and only if the canonical projection $X \to *$ is an $(n + 1)$-connected map.

Alternately, if $X$ is non-empty, then $X$ is $n$-connected if and only if $\{x_0\} \to X$ is an $n$-connected map for every $x_0$.

18.1. **Proposition.** $f: X \to Y$ is $n$-connected if and only if $i_k \in \text{HELP}(f)$ for all $k \leq n$.

18.2. **Proposition.** Let $K$ be a finite -dimensional CW-complex. If $f: X \to Y$ is $n$-connected, then $[K, X] \to [K, Y]$ is a bijection if $\dim K < n$, and a surjection if $\dim K = n$.

**Proof.** Just like the Whitehead theorem: apply HELP to $\emptyset \to K$ for surjectivity, or to $K \times \partial I \to K \times I$ for injectivity.

Note that we can reformulate connectivity in terms of homotopy fibers.

18.3. **Proposition.** A map $f: X \to Y$ is $n$-connected if and only if for all $y_0 \in Y$ the homotopy fiber $\text{hFib}(f; y_0)$ is $(n - 1)$-connected as a space.
19. Cellular approximation

19.1. Proposition. $\pi_k S^n = 0$ if $k < n$.

This is a consequence of the following.

19.2. Lemma. Let $j: X \to Y = X \cup e^n$. Then for any $k < n$, a map $f: I^k \to Y$ is homotopic rel $C := f^{-1}(X)$ to a map $f'$ such that $f'(I^k) \subseteq X$.

Proof. This is proved as Hatcher 4.10; refer to this for full details. The idea is to first find a homotopy rel $C$ to a map $g: I^k \to Y \setminus \{p\}$, where $p \in e^n$ is a point in the interior of the new cell. Then use the fact that $Y \setminus \{p\}$ deformation retracts to $X$ (because $D^n \setminus \{0\}$ deformation retracts to $S^{n-1}$) to get the desired $f'$ and homotopy.

To produce $g$, identify the open $n$-cell $e^n$ with $\mathbb{R}^n$. Choose closed balls $B_1 \subseteq B_2$ of radius 1 and 2 around the origin. Using compactness of $f^{-1}(B_2)$ there’s an $\epsilon > 0$ such that $|x - y| < \epsilon$ implies $\text{len}f(x) - f(y) < 1/2$ for $x, y \in f^{-1}(B_2)$. Choosing a sufficiently small mesh, we can subdivide $I^k$ into small simplices, and then find unions of small simplices $K_1, K_2$ such that $f^{-1}(B_1) \subseteq K_1 \subseteq K_2 \subseteq f^{-1}(B_2)$. Now you can deform $f$ to $g$, so that $f|I^k \setminus K_2 = f'|I^k \setminus K_2$, and so $g|K_1$ is piecewise linear (e.g., it agrees with $f$ on the vertices of the subdivision of $K_1$). Because points outside of $K_1$ are outside $B_1$, their images under $f$ are far from the origin, and thus likewise for $f'$. Therefore there is a point $p \in B_1$ not in the image of $g$, since the image of the PL-map is a finite union of simplices of dimension less than $n$.

19.3. Corollary. If $Y = X \cup_f e^{n+1}$ for some $f: S^n \to X$, then $j: X \to Y$ is $n$-connected.

Proof. Given $(S^{k-1} \to D^k) \Rightarrow (X \to Y)$ such that $k < n + 1$, by the lemma this is homotopic (rel basepoint, and in fact rel $S^{k-1}$), to a map which factors through $(D^k \to D^k) \Rightarrow (X \to Y)$, and $(D^k \to D^k)$ is homotopy equivalent to the terminal object in $\text{Top}^{[1]}$. Thus $\pi_k(j) = 0$ for $k \leq n$.

Thus, if $j: X \to Y = X \cup_f e^{n+1}$, the relative long exact sequence has the form

$$\cdots \to \pi_{n+1}(j) \to \pi_n X \to \pi_n Y \to 0 \to \pi_{n-1} X \to \pi_{n-1} Y \to 0 \to \cdots$$

So the “first” change in homotopy groups is in dimension $n$, where $\pi_n Y \approx \pi_n X/K$. Clearly $[f] \in K$. In general it is not the case that $K$ is generated as a group by $[f]$.

In fact, we have the following.

19.4. Proposition. If $j: A \to X$ is a CW-inclusion such that all cells in $X \setminus A$ have dimension > $n$, then $j$ is $n$-connected.

19.5. Theorem (Cellular approximation). Any map $f: X \to Y$ of CW-complexes is homotopic to a cellular map ($f(X_k) \subseteq Y_k$ for all $k$). If $f$ is already cellular on a subcomplex $A$, then the homotopy can be chosen to be rel $A$.

19.6. Theorem (Whitehead). For every space $X$ there exists a CW-complex $K$ and a weak equivalence $K \to X$.  

Proof. Let’s do $n = 0$. The map $f: X \to Y$ is 0-connected if and only if $\pi_0 X \to \pi_0 Y$ is surjective, i.e., if for every $y_0 \in Y$ there exists a point $x \in X$ and a path $\gamma: f(x) \sim y_0$. This data is exactly a point in $\text{hFib}(f; y_0)$, so $f$ is 0-connected iff all homotopy fibers are non-empty, i.e., $(-1)$-connected.

To handle the general case, we need to note that if $y_0, y_1 \in Y$ are connected by a path, then $\text{hFib}(f; y_0)$ and $\text{hFib}(f; y_1)$ are weakly equivalent. □
More generally, for any map \( f : X \to Y \) there exists a commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{j} & L \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that the vertical maps are weak equivalences and \( j \) is a CW-inclusion.

20. \( n \)-cartesian squares

Let us consider a map \((g, h) : (X \xrightarrow{f} Y) \Rightarrow (X' \xrightarrow{f'} Y')\) in \( \text{Top} \). This is the same as a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{h} & Y'
\end{array}
\]

Note that giving such a map is exactly the same as giving a map

\((f, f') : (X \xrightarrow{g} X') \Rightarrow (Y \xrightarrow{h} Y')\)

in \( \text{Top} \).

Say that a map \((g, h)\) in \( \text{Top} \) as above is \( n \)-connected (for some \( n \geq 1 \)) if

1. for all \( x_0 \in X \) the induced map \( \pi_k(f; x_0) \to \pi_k(f'; g(x_0)) \) is a bijection when \( k < n \) and a surjection if \( k = n \), and
2. complicated condition.

This looks just like one characterization for \( n \)-connected map of spaces, except instead of a map in \( \text{Top} \) we have a map in \( \text{Top} \).

You might expect that we could reformulate this in terms of a notion of “relative-relative homotopy groups” \( \pi_*(g, h) \) of the map \((X \to X') \to (Y \to Y')\), which would fit into a LES also involving \( \pi_*(f) \) and \( \pi_*(f') \). You would be correct, but I won’t do this.

You can also reformulate this in terms of homotopy fibers.

20.1. Proposition. \((g, h) : (X \xrightarrow{f} Y) \Rightarrow (X' \xrightarrow{f'} Y')\) is \( n \)-connected if and only if for all \( y_0 \in Y \), the induced map \( \text{hFib}(f, y_0) \to \text{hFib}(f', h(y_0)) \) on homotopy fibers is \((n - 1)\)-connected.

Which you can then reinterpret in terms of homotopy fibers of homotopy fibers.

20.2. Proposition. \((g, h) : (X \xrightarrow{f} Y) \Rightarrow (X' \xrightarrow{f'} Y')\) is \( n \)-connected if and only if for all \( y_0 \in Y \) and all \( z_0 \in \text{hFib}(f', h(y_0)) \), the homotopy fiber of \( \text{hFib}(f, y_0) \to \text{hFib}(f', h(y_0)) \) over \( z_0 \) is \((n - 2)\)-connected as a space.

Let’s think about this homotopy fiber of homotopy fibers.

20.3. Corollary. The map \((g, h) : (X \xrightarrow{f} Y) \Rightarrow (X' \xrightarrow{f'} Y')\) is \( n \)-connected if and only if \((f, f') : (X \xrightarrow{g} X') \Rightarrow (Y \xrightarrow{h} Y')\) is \( n \)-connected.

So \( n \)-connectedness of a map of maps is really just a property of the commutative square of maps, sometimes called \( n \)-cartesian.

Here’s another more invariant way to say what an \( n \)-cartesian square is. Given a commutative
square

\[
\begin{array}{c}
X \xrightarrow{i_1} X_1 \\
\downarrow \quad \downarrow f_0 \\
X_0 \xrightarrow{f_1} X_{01}
\end{array}
\]

form the path fibrations for \(f_0\) and \(f_1\), and take the pullback:

\[
\begin{array}{c}
X \\
\downarrow \quad \downarrow \quad \downarrow j_0 \\
\quad P \\
\downarrow \quad p_0 \\
P_0 \quad \xrightarrow{p_1} X_{01}
\end{array}
\]

so that

\[
P_0 = \text{Map}(\{0\}, X_0) \times_{\text{Map}(\{0\}, X_{01})} \text{Map}(\{0, \frac{1}{2}\}, X_{01}), \quad (x_0, \gamma) \mapsto \gamma(1/2): P_0 \xrightarrow{p_0} X_{01},
\]

and

\[
P_1 = \text{Map}(\{1\}, X_1) \times_{\text{Map}(\{1\}, X_{01})} \text{Map}(\{\frac{1}{2}, 1\}, X_{01}), \quad (x_1, \delta) \mapsto \delta(1/2): P_1 \xrightarrow{p_0} X_{01}.
\]

This means that

\[
P \approx \text{Map}(\{0\}, X_0) \times_{\text{Map}(\{0\}, X_{01})} \text{Map}(\{0, 1\}, X_{01}) \times_{\text{Map}(\{1\}, X_{01})} \text{Map}(\{1\}, X_1),
\]

and \(j: X \to P\) is the map \(x \mapsto (j_0(x), p_0j_1(x) = p_1j_0(x), j_1(x))\).

Then the square is \(n\)-cartesian if and only if \(j\) is \((n - 1)\)-connected.

When \(j\) is a weak equivalence, we say the original square is a **homotopy pullback**.

21. **Homotopy excision**

21.1. **Theorem** (Homotopy excision). A pushout square

\[
\begin{array}{c}
X \xrightarrow{g} X' \\
\downarrow f \quad \downarrow f' \\
Y \xrightarrow{h} Y'
\end{array}
\]

such that either \(f\) or \(g\) has the HEP, and such that \(f\) is \(m\)-connected and \(g\) is \(n\)-connected, and at least one of \(m, n\) is positive, is \((m + n)\)-Cartesian, i.e., \(\pi_k(f) \to \pi_k(f')\) is bijective for \(k < n\) and surjective for \(k = n\).

Here is an explicit example. Suppose \(X\) is a 1-connected space, and we have a pushout square

\[
\begin{array}{c}
S^n \xrightarrow{f} X \\
\downarrow i \\
D^{n+1} \xrightarrow{j} Y = X \cup_f e^{n+1}
\end{array}
\]

The map \(i\) is \(n\)-connected by inspection. Assuming \(n \geq 2\), then \(f\) is 1-connected. Then the square is \((n + 1)\)-cartesian, i.e., the induced maps

\[
\pi_k(S^n \xrightarrow{i} D^{n+1}) \to \pi_k(X \xrightarrow{j} Y)
\]
are surjective for \( k = n + 1 \) and bijective for \( k \leq n \). Putting this into the relative LES:
\[
\cdots \to \pi_{n+1}(j) \to \pi_n X \to \pi_n Y \to \pi_n(j) \to \pi_{n-1} X \to \pi_{n-1} Y \to \pi_{n-1}(j) \to \cdots
\]
gives a LES
\[
\pi_{n+1}(j) \to \pi_n X \to \pi_n Y \to \pi_n(j) \to \pi_{n-1} X \to \pi_{n-1} Y \to \pi_{n-1}(j) \to \cdots.
\]
(The sequence does not extend to the left because \( \pi_{n+1}(i) \to \pi_{n+1}(j) \) is merely a surjection.) On the other hand, the relative LES also gives \( \pi_k(i) \approx \pi_k S^n \), and so we get an exact sequence
\[
\pi_{n}(S^n) \to \pi_n X \to \pi_n Y \to \pi_{n-1} X \to \pi_{n-1} Y \to \pi_{n-1}(j) \to \cdots.
\]
Thus we learn that \( \pi_k X \sim \pi_k Y \) is a bijection if \( k < n \) (which we already knew by cellular approximation), and also that the kernel of \( j_\ast \): \( \pi_k X \to \pi_k Y \) is the subgroup generated by \( f \).

In other words, if we attach an \( n \)-cell to a simply connected space along a map \( f: S^n \to X \), then \( \pi_{n-1} Y \approx \pi_{n-1}X/(f) \).

21.2. Example. Let \( f: X \to Y \) be a map with the HEP, so that \( Y/X \approx \text{Cone}(f) \). That is we have pushout squares:

\[
\begin{array}{ccc}
A & \longrightarrow & \text{Cone}(A) \\
\downarrow f & & \downarrow \sim \\
X & \longrightarrow & \text{Cone}(f) \\
\end{array}
\]  
\( \sim \longrightarrow \) \( X/A \)

If the space \( A \) is \((m-1)\)-connected (so \( A \to * \) is \( m \)-connected) and \( f \) is an \( n \)-connected map, then the square on the left (and hence on the right) is \((m+n)\)-cartesian, so that
\[
\pi_k(A \xrightarrow{f} X) \to \pi_k(\text{Cone}(A) \to \text{Cone}(f)) = \pi_k(X/A)
\]
is an isomorphism for \( k < m + n \) and surjective for \( k = m + n \). Thus, we get a long exact sequence relating \( \pi_A, \pi_X, \pi_A X/A \) in a range of dimensions:
\[
\pi_{m+n-1} A \to \pi_{m+n-1} X \to \pi_{m+n-1} (X/A) \to \pi_{m+n-2} A \to \cdots.
\]
(The next term to the left would be \( \pi_{m+n}(f) \), but this only surjects to \( \pi_{m+n} X/A \).)

We can apply this to \( Y = X \cup_f e^{n+1} \), so that \( Y/X = S^{n+1} \). The map \( f \) is \( n \)-connected. Assume \( Y \) is \((m-1)\)-connected. Then we get an exact sequence
\[
\pi_{m+n-1} X \to \pi_{m+n-1} Y \to \pi_{m+n-1} S^{n+1} \to \pi_{m+n-2} X \to \cdots.
\]

If \( Y \) is at least simply connected, then \( m \geq 2 \), so we get
\[
\pi_{n+1} X \to \pi_{n+1} Y \to \pi_{n+1} S^{n+1} \to \pi_n X \to \pi_n Y.
\]

21.3. Example. Consider

\[
\begin{array}{ccc}
S^{p+q-1} & \longrightarrow & S^p \vee S^q \\
\downarrow & & \downarrow \\
D^{p+q} & \longrightarrow & S^p \times S^q
\end{array}
\]

If \( p, q \geq 2 \), then \( S^p \vee S^q \) is simply connected, then using the fact that \( \pi_k(X \times Y) \approx \pi_k X \times \pi_k Y \), we have
\[
\pi_k(S^p \vee S^q) \approx \pi_k S^p \times \pi_k S^q, \quad k < p + q - 1,
\]
and an exact sequence
\[
\mathbb{Z} \overset{[p,q]}{\longrightarrow} \pi_{p+q-1}(S^p \vee S^q) \to \pi_{p+q-1} S^p \times \pi_{p+q-1} S^q \to 0.
\]

For instance, when \( p = q = 2 \) this proves that
\[
\pi_3(S^2 \vee S^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},
\]
generated by \( i_1 \circ \eta, i_2 \circ \eta, w = [i_1, i_2] \), where \( \eta: S^3 \to S^2 \) is the Hopf map and \( i_k: S^2 \to S^2 \vee S^2 \) are the inclusions.

21.4. Exercise. Suppose \( f = a(i_1 \circ \eta) + bw + c(i_2 \circ \eta) \) where \( a, b, c \in \mathbb{Z} \), and form

\[ X_f := (S^2 \vee S^2) \cup f e^4. \]

Compute \( H^*(X_f) \) with its cup product structure. (Hint: use naturality of the mapping cone construction with respect to various choices of maps \( p: S^2 \vee S^2 \to S^2 \):

\[
\begin{array}{ccc}
S^3 & \xrightarrow{f} & S^2 \vee S^2 \\
\downarrow{p} & & \downarrow{p} \\
S^3 & \xrightarrow{pf} & S^2 \\
\end{array}
\]

together with calculation of the Hopf invariant of \( pf \).

21.5. Example. If \( X \) is \((m - 1)\)-connected and \( Y \) is \((n - 1)\)-connected, then

\[
\pi_k(X \vee Y) \to \pi_k X \times \pi_k Y
\]

is iso if \( k < m + n - 1 \) and surjective if \( k = m + n - 1 \). Use the pushout square

\[
\begin{array}{c}
X \vee Y \xrightarrow{f} \text{Cone}(X) \vee Y \\
\downarrow \quad \downarrow \\
X \vee \text{Cone}(Y) \xrightarrow{g} \text{Cone}(X) \vee \text{Cone}(Y)
\end{array}
\]

Alternate proof: model \( X \) and \( Y \) as CW complexes, and think about \( X \wedge Y \).

21.6. Proposition. Suppose \( f: X \to Y \) has the HEP and \( X \) is simply connected. If \( Y/X \) is weakly contractible, then \( f \) is a weak equivalence.

More generally, if \( Y/X \) is \( d \)-connected, then \( f \) is \( d \)-connected.

Proof. Use the square

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \text{Cone}(X) \xrightarrow{\sim} \ast \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{\sim} & \text{Cone}(f) \xrightarrow{\sim} Y/X
\end{array}
\]

Suppose \( X \) is simply connected, so \( i \) is \( n \)-connected with \( n \geq 2 \). Let \( m = \text{connectivity of } f \), i.e., \( \pi_k(f) = 0 \) if \( k \leq m \).

By homotopy excision the square is \( m + n \geq m + 2 \) cartesian, i.e., \( \pi_k(f) \to \pi_k(g) \) is an iso if \( k < m + 2 \), in particular for \( k = m + 1 \). But if \( \text{Cone}(f) \approx Y/X \) is contractible, \( \pi_k(g) = 0 \), so we have learned:

\[
\text{if } f \text{ is } m \text{-connected, then } \pi_{m+1}(f) = 0 \text{ so } f \text{ is } m+1 \text{-connected}.
\]

Thus we have shown that if \( X \) is simply connected and \( Y/X \) contractible, then \( f \) is a weak equivalence. \( \square \)

This fails if \( X \) is not simply connected. Take \( A = S^1 \vee S^2 \), so that \( \pi_2 A \approx \mathbb{Z}[t, t^{-1}] \). (Note: \( \pi_2 = A \) is in general not a ring, but it is a module over the group ring \( \mathbb{Z}[\pi_1 A] \). In this case, it is a free module on one generator over \( \pi_1 A = \mathbb{Z}[t, t^{-1}] \).)

Let \( g: S^2 \to A \) represent the element \( 2t + 1 \), and form \( Y = (S^1 \vee S^2) \cup_g e^3 \). We also have an inclusion \( f: X = S^1 \to Y \).

Claim. \( Y/X \) is contractible, but \( f \) is not a homotopy equivalence.
Proof. Note that $Y/X \approx S^2 \cup e^3$, where $\overline{g}$ is the composite $S^2 \xrightarrow{g} S^1 \vee S^2 \xrightarrow{\nu} S^2$ of $g$ with the pinch map. The observation is the pinch map $p$ induces

$$p_* : \pi_2(S^1 \vee S^2) \to \pi_2 S^2, \quad f(t) \mapsto f(1) : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}.$$  

Thus $\overline{g} = pg \sim \text{id}_{S^2}$. Therefore $S^2 \cup_{\overline{g}} e^3 \approx S^2 \cup_{\text{id}} e^3 = \text{Cone}(S^2)$, so $Y/X$ is contractible.

On the other hand, we can compute that $\pi_2 Y \approx \mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}$, using a universal cover $\tilde{Y}$ for $Y$. To see this, first think about the universal cover $\tilde{A} \to A$. The attaching map $g : S^2 \to A$ can be lifted infinitely many ways to $\tilde{A}$, e.g., we can form

$$\tilde{g} = (\tilde{g}_i) : \prod_{\mathbb{Z}} S^2 \to \tilde{A}$$

where each component of $\tilde{g}$ is a different lift of $g$. Under the isomorphism $\pi_2 \tilde{A} \approx \pi_2 A \approx \mathbb{Z}[t, t^{-1}]$, the lifts $\tilde{g}_i$ correspond to the classes $t^i(2t - 1)$.

Then form $\tilde{Y}$ as a pushout, and note that each term of the square comes with a projection to the pushout defining $Y$:

$$\begin{array}{ccc}
\prod S^2 & \xrightarrow{\tilde{g}} & \tilde{A} \\
\downarrow & & \downarrow \\
\prod D^3 & \longrightarrow & \tilde{Y} \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
S^2 & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
D^3 & \longrightarrow & Y \\
\end{array}$$

The map $\tilde{Y} \to Y$ is also a covering map, whose fibers can be identified with $\mathbb{Z}$. (In general, given a diagram

$$\begin{array}{ccc}
E' & \xleftarrow{E} & E'' \\
\downarrow & & \downarrow \\
B' & \xleftarrow{B} & B'' \\
\end{array}$$

where the vertical maps are covering maps and the squares are pullbacks, the induced map $E' \cup_E E'' \to B' \cup_B B''$ is also a covering map, with the "same fibers" as all the other ones.)

Because $\tilde{A}$ is simply connected we see that $\tilde{Y}$ also is, so $\tilde{Y} \to Y$ is also a universal cover. In any case,

$$\pi_2 Y \approx \pi_2 \tilde{Y} \approx \pi_2 \tilde{A}/(\tilde{g}_i)_{i \in \mathbb{Z}} \approx \mathbb{Z}[t, t^{-1}]/(t^i(2t - 1))_{i \in \mathbb{Z}} \approx \mathbb{Z}[t, t^{-1}]/(2t - 1) \approx \mathbb{Z}[\frac{1}{2}].$$

Thus, we have produced a CW-inclusion $X \to Y$ such that $Y/X$ is contractible but $\pi_2 X \not\approx \pi_2 Y$.

22. HUREWICZ THEOREM

We can use homotopy excision to prove the Hurewicz theorem. The Hurewicz map is a group homomorphism

$$\pi_n(X; x_0) \to \tilde{H}_n X, \quad [f] \mapsto f_*[S^n].$$

where $[S^n] \in \tilde{H}_n S^n$ is the fundamental class. It is natural with respect to maps of based spaces.

22.1. Theorem. If $X$ is $n - 1$-connected ($n - 1 \geq 0$), then the Hurewicz map induces isomorphisms

$$\pi_k(X; x_0)^{ab} \to \tilde{H}_k X$$

for $k \leq n$.

There is also a relative Hurewicz map

$$\pi_n(X \to Y, x_0) \to H_n(Y, X)$$
define for inclusions: send \( f: (S^{n-1} \to D^n) \Rightarrow (X \to Y) \) to \( f_*[D^n, S^{n-1}] \in H_n(Y, X) \), where \([D^n, S^{n-1}] \in H_n(D^n, S^{n-1}) \approx \mathbb{Z}\) is the generator. This is also natural, is a group homomorphism if \( n \geq 2 \), and gives a map of long exact sequences.

22.2. Lemma. Let \( X = \bigsqcup_{k=1}^n S^n \) be a finite wedge of \( n \)-spheres with \( n \geq 2 \). Then the Hurewicz theorem holds for \( X \), i.e.,

\[
\pi_n X \approx \tilde{H}_n X \approx \mathbb{Z}^k.
\]

Proof. Let \( Y = \prod_{k=1}^n S^n \). The space \( Y \) has a CW-structure with cells in dimensions \( d \) for \( d = 0, \ldots, k \), and \( Y_n = X \). Thus \( X \to Y \) is \((2n-1)\)-connected. Note that \( 2n-1 \geq n+1 \) since \( n \geq 2 \).

Homotopy excision thus gives an exact sequence

\[
\pi_{n+1}(X \to Y) \to \pi_n(\bigsqcup S^n) \to \pi_n \prod S^n \to \pi_n(X \to Y),
\]

where the two end terms are 0, so \( \pi_n(\bigsqcup S^n) \approx \prod \pi_n S^n \approx \bigoplus \mathbb{Z} \). Using naturality of the Hurewicz map and the Künneth theorem (or just the fact that \( \tilde{H}_{n/n+1}(Y, X) = 0 \)) gives the result. \( \square \)

Recall that if a Hausdorff space \( X \) is a directed colimit of closed subsets \( X_\alpha \), then any map \( K \to X \) from a compact Hausdorff space factors through some \( X_\alpha \subset X \). This implies \( \pi_n X \approx \text{colim}_\alpha \pi_n X_\alpha \). Since also \( \tilde{H}_n X \approx \text{colim}_\alpha \tilde{H}_n X_\alpha \), and an infinite wedge of spheres can be regarded as a union of all the finite wedges, and thus we can extend the previous lemma to infinite wedges of spheres.

Proof. Both homotopy groups and homology are weak homotopy invariant. E.g., if \( f: X \to Y \) is a weak equivalence, then \( f_*: H_* X \to H_* Y \) is an isomorphism. Since the Hurewicz map is natural, we can WLOG assume that \( X \) is a CW-complex, by CW-approximation.

Furthermore, if \( X \) is \( n-1 \)-connected, then we can find a CW-complex with \( X_{n-1} = * \), so we will do so. This immediately gives \( \tilde{H}_n X \approx 0 \) if \( k < n \).

By the lemma and subsequent comments, the Hurewicz map is iso for the space \( X_n \). Now let \( f: X_n \to X_{n+1} \) be the inclusion. We have a map of exact sequences

\[
\begin{array}{c}
\pi_{n+1}(f) \to \pi_n X_n \to \pi_n X_{n+1} \to \pi_n(f) \\
H_{n+1}(X_{n+1}, X_n) \to H_n X_n \to H_n X_{n+1} \to H_n(X_{n+1}, X_n)
\end{array}
\]

Because \( f \) is \( n \)-connected and \( X_n \to * \) is \( n \)-connected, we see that \( \pi_k(f) \approx \pi_k(X_{n+1}/X_n) \) for \( k \leq 2n-1 \), which includes \( k = n+1 \) since we assume \( n \geq 2 \). We read off that the Hurewicz map \( \pi_n X_{n+1} \to H_n X_{n+1} \).

We use the same argument to go from \( X_{n+1} \) to \( X \), using that the inclusion \( X_{n+1} \to X \) is \((n+1)\)-connected and that \( \pi_n(X/X_{n+1}) \approx 0 \) by cellular approximation. \( \square \)

22.3. Corollary. If \( X \) is a simply connected space with \( \tilde{H}_* X \approx 0 \), then \( X \to * \) is a weak equivalence.

From this we get a beautiful result.

22.4. Theorem (Homology Whitehead theorem). Let \( f: X \to Y \) be a map between simply connected spaces such that \( f_*: H_* X \to H_* Y \) is an isomorphism. Then \( f \) is a weak equivalence.

22.5. Corollary. Let \( f: X \to Y \) be a map of simply connected CW-complexes. Then \( f_*: H_* X \to H_* Y \) iso implies \( f \) is a homotopy equivalence.

Proof. WLOG we can replace \( f \) with a CW-inclusion. Because \( X \) is simply connected, we know that \( f \) is a weak equivalence iff \( Y/X \) is contractible, and that \( f \) is an \( H_* \)-equivalence iff \( H_*(Y, X) \approx H_* Y/X \approx 0 \). \( \square \)
It is hard (often effectively impossible) to compute homotopy groups, but usually possible to compute homology groups. Thus this gives a very effective criterion for being a weak equivalence.

**Warning.** This is a criterion for a map, not for comparing two spaces. It is easy to construct pairs of spaces $X$ and $Y$ which are not weakly equivalent but are such that $H_* X \approx H_* Y$.

What if the spaces aren’t simply connected? Then you have the following.

22.6. **Theorem.** Let $f: X \to Y$ be a map between nice path connected spaces (where nice = a universal cover exists). Then if

1. $f_*: \pi_1 X \to \pi_1 Y$ is an isomorphism, and
2. the induced map $\tilde{f}: \tilde{X} \to \tilde{Y}$ between universal covers induces an isomorphism in homology,

then $f$ is a weak equivalence.

**Proof.** There is a commutative square

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

we know $f$ induces iso on $\pi_1$ by hypothesis. The vertical maps induce isos on $\pi_k$ for $k \geq 2$, so it suffices to show that $\tilde{f}$ is a weak equivalence. This follows from the hypotheses and the previous theorem. □

### 23. Relative remarks

It is convenient to have the notion of a **relative cell complex**: a map $f: X \to Y$ obtained by iteratively attaching cells. (There is no condition on $X$.) That is, $Y = \bigcup Y_n$, where $Y_0 = X$ and each $Y_n$ is obtained as a pushout from $Y_{n-1}$ along a map $\coprod S^{n_i-1} \to \coprod D^{n_i}$.

Every relative CW pair is a relative cell complex, but not conversely: in a relative cell complex you are not required to attach cells in order of dimension, and $X$ is not even required to be a CW complex anyway. This provides a little more flexibility, at little cost.

Every relative cell complex has the HEP.

The arguments we gave earlier actually prove the following.

23.1. **Proposition.** If $f$ is a relative cell complex and $g$ a weak equivalence, then $f \in \text{HELP}(g)$.

We can prove relative versions of the Whitehead theorems.

23.2. **Corollary.** Let $A \to B$ be a relative cell complex, and let $g: X \to Y$ be a map in $\text{Top}_A$ which is a weak equivalence of underlying spaces. Then $[B, X]_A \to [B, Y]_A$ is a bijection.

23.3. **Corollary.** If $A \to X$ and $A \to Y$ are relative cell complexes, and $f: X \to Y$ is a map in $\text{Top}_A$ which is a weak equivalence, then $f$ is a homotopy equivalence rel $A$.

23.4. **Corollary.** Let $A \to X$ be any map. Then there exists a relative cell complex $A \to B$ and a map $B \to X$ in $\text{Top}_A$ which is a weak equivalence.

23.5. **Corollary.** If $A \to X$ is any $n$-connected map, then there exists a factorization $A \to B \to X$ such that $B \to X$ is a weak equivalence, and $A \to B$ is a relative cell complex constructed by only attaching along maps $S^{k-1} \to D^k$ with $k \geq n + 1$. 


24. Truncation

A space $X$ is $n$-truncated if
\[ \pi_k(X; x_0) \approx \{\star\} \quad \text{for all } k > n, x_0 \in X. \]
Equivalently, $X$ is $n$-truncated if $[D^{k+1}, X] \to [S^k, X]$ is a bijection for all $k > n$.

24.1. **Example.** A space is $(-1)$-truncated if $\pi_k(X; x_0) \approx \{\star\}$ for all $k \geq 0$ and all choices of basepoint. A $(-1)$-truncated space is either: the empty set, or weakly contractible.

24.2. **Example.** A space is 0-connected iff it is weakly equivalent to a discrete space.

24.3. **Example.** The circle is 1-truncated, as is $RP^\infty$, while $CP^\infty$ is 2-truncated.

24.4. **Proposition.** If $f: A \to B$ is an $n + 1$-connected relative cell complex, and $X$ is $n$-truncated, then any $A \to X$ extends to a map $B \to X$.

*Proof.* By remarks above, we can find a relative cell complex $g: A \to C$ such that cells are added only in dimensions $\geq n + 2$, together with a weak equivalence $C \to B$ in $\text{Top}_A$, which is therefore a homotopy equivalence rel $A$. Therefore $[B, X]_A \to [C, X]_A$ is a bijection, so $[B, X]_A \neq \emptyset$ if and only if $[C, X]_A \neq \emptyset$.

If we write $C = \bigcup C_k$ with $C_0 = A$, then to prove the result it suffices to show we can extend any $C_k \to X$ to $C_{k+1}$. This is clear by the definition of truncated.

An $n$-truncation of a space $X$ is a map $f: X \to Y$ such that:

1. $f$ is $(n + 1)$-connected, and
2. $Y$ is $n$-truncated.

Explicitly, this means:

- for all $x_0 \in X$ and $k \leq n$, $\pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is a bijection, and
- for all $y_0 \in Y$ and $k > n$, $\pi_k(Y, y_0) \approx \{\star\}$.

24.5. **Example.** The map $S^2 = \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is an example of an $n$-truncation. This is because the fiber sequence $S^1 \to S^\infty \to \mathbb{C}P^\infty$ computes $\pi_*\mathbb{C}P^\infty$.

24.6. **Proposition.** For every space $X$, there always exists a relative cell complex $X \to Y$ which is an $n$-truncation.

*Proof.* Form $Y_{n+2} := X \cup \bigcup \alpha e^{n+2}$ by attaching along a collection $\{S^{n+1} \to X\}$, such that these elements generate the $\pi_{n+1}(X, x_0)$s. (If $X$ is not path connected, you need to attach cells for each path component separately.)

Iterate, and set $Y = \bigcup Y_k$. That this is a truncation is immediate from the construction and the fact that $X \to X \cup e^k$ is $k - 1$ connected.

24.7. **Proposition.** Given two relative cell complexes $f: X \to Y$ and $f': X \to Y'$ which are $n$-truncations, there exists a homotopy equivalence $Y \approx Y'$ rel $X$.

*Proof.* Since $f$ is $(n + 1)$-connected, we can extend $X \to Y'$ along $Y$, obtaining $g: Y \to Y'$ in $\text{Top}_X$. This is clearly a weak equivalence, so is a homotopy equivalence rel $X$.

I’ll write $X \to X_{\leq n}$ for any choice of $n$-truncation of $X$.

25. Eilenberg MacLane spaces

Let $n \geq 0$, and let $G$ be an abelian group (or just a group when $n = 1$, or just a based set when $n = 0$). An **Eilenberg-Mac Lane space** of type $K(G, n)$ is a based space $X$ together with an isomorphism $\pi_n X \cong G$, such that $\pi_k X \approx 0$ for all $k \neq n$. 
An Eilenberg-MacLane space of every type can be constructed as a CW complex, by truncation. First build \( X = \bigvee S^n \cup \bigcup e^{n+1} \) so that \( \pi_n X \approx G \) (using a presentation of \( G \) by generators and relations). Then \( X \leq n \) is a \( K(G, n) \). Note that we can build this \( K(G, n) \) as a CW-complex.

Write \( \Omega X := \text{Map}_*(S^1, X) \). It is clear that \( [K, \text{Map}(S^1, X)]_* \approx [K \wedge S^1, X]_* \).

Therefore taking \( K = S^n \) we obtain an isomorphism \( \pi_n(\Omega X, x_0) \approx \pi_{n+1}(X, x_0) \).

25.1. \textbf{Proposition.} If \( X \) is a \( K(G, n) \), then \( \Omega X = \text{Map}_*(S^1, X) \) is a \( K(G, n-1) \).

We are going to prove the following.

25.2. \textbf{Proposition.} Let \( G \) be an abelian group. For \( X \) a based CW-complex, we have that \( [X, K(G, n)]_* \approx \tilde{H}^n(X, G) \).

For \( X \) an unbased CW-complex, we have that \( [X, K(G, n)] \approx H^n(X, G) \).

The second statement follows from the first in the following way. Let \( X = X/\emptyset = X \amalg \{*\} \).

Then it is easy to see that there are natural isomorphisms \( [X_+, Y]_* \approx [X, Y] \), \( \tilde{H}^n(X_+, G) \approx H^n(X; G) \).

26. \textbf{Spectra and generalized cohomology}

An \( \Omega \)-\textit{spectrum} is data consisting of based spaces \( E_n \) and weak equivalences \( \epsilon_n: E_n \to \Omega E_{n+1} \), for all \( n \in \mathbb{Z} \).

Note: if we only give this data for \( n \geq 0 \) (say), we can fill in negative values by setting \( E_{-n} = \Omega^n E_0 \).

Let \( E = \{E_n, \epsilon_n\} \) be an \( \Omega \)-spectrum. For based spaces \( X \) which have the homotopy type of CW-complexes we define \( \tilde{h}^n(X, E) := [X, E]_* \).

There is also a natural map \( \sigma: h^n(X, E) \to \tilde{h}^{n+1}(X, E) \), defined by \( [X, E]_* \to [X, \Omega E_{n+1}]_* \approx [\Sigma X, E_{n+1}]_* \).

These functors satisfy the following properties:

1. \( \tilde{h}^n(-, E) \) takes homotopic maps to identical maps.
2. If \( X \xrightarrow{f} Y \to Y/X \) where \( f \) has the HEP, then there is an exact sequence \( h^n(Y/X, E) \to h^n(Y, E) \to h^n(X, E) \).
3. For any cell complex \( X \), \( \sigma: h^n(X, E) \to h^{n+1}(\Sigma X, E) \) is an isomorphism.
4. If \( X \approx \bigvee X_\alpha \), then \( \tilde{h}^n(X, E) \to \prod \tilde{h}^n(X_\alpha, E) \)

induced by the inclusions \( X_\alpha \to X \) is an isomorphism.

The proofs are straightforward: (1) and (4) are obvious, (2) is the Puppe sequence, (3) is the loop isomorphism.

Data \( (\tilde{h}^*, \sigma) \) satisfying these axioms is called a generalized cohomology theory.

We need the following, which is actually a special case of the fact that EM spaces represent cohomology (on spaces which are highly connected).
26.1. **Lemma.** Let $X$ and $Y$ be based spaces such that (i) $* \to X$ is a relative cell complex which is $(n - 1)$-connected, and (ii) $Y$ is a $K(G,n)$. Then for every homomorphism $\phi: \pi_n(X) \to G$ there exists a based map $f: X \to Y$ such that $f_\ast = \phi$.

**Proof.** Assume $n \geq 2$. WLOG we can replace $X$ with a homotopy equivalent $X'$ with $X'_{n-1} = \{\ast \}$. Then $X_{n+1} = \bigvee_a S^n \cup \bigcup_\beta e^{n+1}$, and there is an exact sequence of groups

$$\pi_{n+1}(X_{n+1}/X_n) \approx \pi_{n+1}(X_n \to X_{n+1}) \to \pi_n X_n \to \pi_n X_{n+1} \to 0$$

which describes a presentation of $\pi_n X_{n+1} \approx \pi_n X$ as a quotient of a map of free abelian groups.

Because $X_n = \bigvee_a S^n$, there exists a map $f_n: X_n \to Y$ inducing the composite $\pi_n X_n \to \pi_n X \overset{\phi}{\to} \pi_n Y = G$ in homotopy. Each attaching map $\beta$ for $X_{n+1}$ gives

$$\begin{array}{ccc}
S^n & \to & X_n \\
\downarrow & & \downarrow \\
D^{n+1} & \to & X_{n+1}
\end{array}$$

where $S^n \overset{\alpha}{\to} X_n \to Y$ is null homotopic since the image of $[\alpha] \in \pi_n S^n = \pi_{n+1}(S^n \to D^{n+1}) \to \pi_{n+1}(X_n \to X_{n+1}) \to \pi_n X_n \to G$ is trivial. Thus we can extend $f_n$ to $f_{n+1}: X_{n+1} \to Y$, and by construction the induced map on $\pi_\ast$ is as given. Because $Y$ is $n$-truncated, we can then extend to higher skeleta of $X$.

When $n = 1$, the proof is similar except that $\pi_1 X_2$ is presented as a group by generators and relations according to the van Kampen theorem. When $n = 0$ it is straightforward. □

26.2. **Corollary.** Let $K_n$ be a $K(G,n)$ which is a based cell complex. Then there exists a based weak equivalence $K_n \to \Omega K_{n+1}$.

This corollary construct an example of an **Eilenberg MacLane spectrum**, which I’ll denote by $HG$.

Next, we are going to construct natural transformations

$$\hat{h}^n(X, HG) \to \tilde{H}^n(X, G)$$

(on based cell complexes $X$).

Let $K_n$ be a $K(G,n)$ which is a based cell complex. Let’s note that we regard the isomorphism $\pi_n K_n \overset{\sim}{\to} G$ as part of the data of $K_n$.

Because $K_n$ is $(n - 1)$-connected, the universal coefficient theorem

$$0 \to \text{Ext}^1(\tilde{H}_{n-1} K_n, G) \to \tilde{H}^n(K_n, G) \to \text{Hom}(\tilde{H}_n K_n, G) \to 0$$

has $\text{Ext}^1 = 0$. Combined with the Hurewicz theorem, we get

$$\tilde{H}^n(K_n, G) \overset{\sim}{\to} \text{Hom}(\tilde{H}_n K_n, G) \overset{\sim}{\to} \text{Hom}(\pi_n K_n^{ab}, G) \overset{\sim}{\to} \text{Hom}(G, G).$$

(Note: if $n \geq 1$, then $(\pi_n K_n)^{ab} = \pi_n K_n = G$. If $n = 0$, then $(\pi_0 K_0)^{ab} = \mathbb{Z}[G]$, so in this case $\text{Hom}(\pi_0 K_0^{ab}, G) \overset{\sim}{\to} \text{Hom}_{\text{Set}_*}(G, G)$.)

We define

$$\lambda_X: [X, K_n]_\ast = \hat{h}^n(X, G) \to \tilde{H}^n(X, G)$$

by sending $f: X \to K_n$ to $f^*\langle \iota_n \rangle \in \tilde{H}^n(X, G)$. It is straightforward to show that this is natural in $X$.

The **tautological class** $\iota_n \in \tilde{H}^n(K_n, G)$ is the unique class corresponding to $\text{id}: G \to G$.

26.3. **Lemma.** The classes $\iota_n$ and $\iota_{n+1}$ correspond to each other with respect to the maps

$$\tilde{H}^n(K_n, G) \overset{\sigma}{\to} \tilde{H}^{n+1}(\Sigma K_n, G) \overset{\epsilon_n}{\leftarrow} \tilde{H}^{n+1}(K_{n+1}, G),$$

i.e., $\epsilon_n^* (\iota_{n+1}) = \sigma (\iota_n)$. 

Proof. The construction of the tautological classes used the universal coefficient theorem and the Hurewicz theorem. The proof of the lemma is an application of the fact that (i) the maps
\[ \tilde{H}^n(X, G) \to \text{Hom}(\tilde{H}_nX, G) \to \text{Hom}((\pi_nX)^{ab}, G) \]
are natural in \(X\), and (ii) both the Hurewicz map and the coefficient maps are compatible with suspension:
\[
\begin{array}{ccc}
\tilde{H}^n(X, G) & \xrightarrow{\sim} & \text{Hom}(\tilde{H}_nX, G) \\
\downarrow & & \downarrow \\
\tilde{H}^{n+1}(\Sigma X, G) & \xrightarrow{\sim} & \text{Hom}(\tilde{H}_{n+1}\Sigma X, G)
\end{array}
\]
Hence, the lemma gives the result.

26.4. Proposition. The diagram
\[
\begin{array}{ccc}
\tilde{h}^n(X, HG) & \xrightarrow{\sigma} & \tilde{h}^{n+1}(\Sigma X, HG) \\
\lambda_X \downarrow & & \lambda_{\Sigma X} \downarrow \\
\tilde{H}^n(X, G) & \xrightarrow{\sigma} & \tilde{H}^{n+1}(\Sigma X, G)
\end{array}
\]
commutes.

Proof. Given \(f \in [X, K_n]\), we have \(\sigma(f) = \epsilon_n(\Sigma f) \in [\Sigma X, K_{n+1}]\), and thus
\[ \lambda_{\Sigma X}(\sigma(f)) = (\Sigma f)^* \epsilon_n(\iota_{n+1}) \]
while \(\lambda_X(f) = f^*(\iota_n)\) and thus
\[ \sigma(\lambda_X(f)) = \sigma(f^*(\iota_n)) = (\Sigma f)^*(\sigma(\iota_n)) \]
by naturality of homology suspension. The lemma gives the result.

Thus, the \(\lambda\) give a natural transformation of (generalized) cohomology theories \(\tilde{h}^*(-, HG) \to \tilde{H}^*(-, G)\). (It is important that the transformation commute with all the structure, which mean the suspension isomorphisms.)

26.5. Proposition. Let \(\lambda: E^* \to F^*\) be a natural transformation of cohomology theories. If \(\lambda_{S^0}: \tilde{E}^*(S^0) \to \tilde{F}^*(S^0)\) is an isomorphism, then \(\lambda_X\) is an isomorphism for all cell complexes \(X\).

Proof. If \(X\) is a based cell complex, we get the result for each \(X_n\) by an induction on cells.

To pass to \(X\), let \(V = \bigvee X_n\), and note that \(X\) is homotopy equivalent to the reduced mapping torus of two maps \(V \xrightarrow{\sim} V\).

The mapping torus of \(f, g: A \to B\) is the space obtained as a pushout:
\[
\begin{array}{ccc}
A \times \{0, 1\} & \xrightarrow{(f, g)} & B \\
\downarrow & & \downarrow \\
A \times [0, 1] & \to & T
\end{array}
\]
If \(f = g = \text{id}_A\) then \(T = A \times S^1\). When the spaces and maps are based we can also form the reduced mapping torus \(\overline{T} = T/\langle (a_0) \times [0, 1]\rangle\).

Let \(V_n = \bigvee_{k \leq n} X_k\). Then we can form the mapping torus \(T_n\) of \(V_{n-1} \xrightarrow{\sim} V_n\). Claim: this is homotopy equivalent to \(X_n\). Then since \(V = \bigcup V_n\) we get a weak equivalence \(V \to X\).
27. POSTNIKOV TOWERS

Recall that for any space $X$ we can always construct a “truncation tower”

$$
\begin{array}{ccc}
X & \xrightarrow{f_2} & X_{\geq 1} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
X_{\leq n} & \xrightarrow{k} & K(G, n + 2)
\end{array}
$$

so that each map $f_k$ is a $k$-truncation. Therefore each $X_{\geq n+1} \to X_{\geq n}$ is a map which is an isomorphism on $\pi_k$ for all $k$ except $k = n + 1$, where $\pi_{n+1}X_{\geq n} = 0$.

We would like to be able to construct $X_{\geq n+1}$ from $X_{\geq n}$, by choosing a map $k: X_{\geq n} \to K(G, n + 2)$, $G = \pi_{n+2}X$, and taking the homotopy fiber. That is, want to form a diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & EK(G, n + 2) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f_n} & X_{\leq n} & \xrightarrow{k} & K(G, n + 2)
\end{array}
$$

where $EK(G, n+2) = \text{Map}(I, K(G, n+2)) \times_{\text{Map}(\{0\}, K(G,n+2))} \{\ast\} = \{\gamma: [0,1] \to K(G, n + 2) \mid \gamma(0) = \ast\}$ is the path space of the inclusion $\{\ast\} \to K(G, n + 2)$. Then $Y$ is defined as the pullback of the square, i.e., $Y = \text{hFib}(K, \ast)$. Finally, we would like to produce a lift $g$ of $f_n$, so that $g$ is an $n+1$-truncation.

There is an obstruction to doing this in general, which can happen in the case when $\pi_1X$ is non-trivial.

27.1. Example. Let $X = S^1 \vee S^2$. Then $X_{\leq 1} \approx K(Z, 1) \approx S^1$. Since $\pi_2X = G = Z[t^\pm]$, we need an appropriate map $k: S^1 \to K(G, 3)$. In fact, $[S^1, K(G, 3)] \approx H^3(S^1, G) \approx 0$, so any such map is null-homotopic. This implies that the pullback $Y$ along $k$ will be weakly equivalent to a product $X_{\leq 1} \times K(G, 2)$. (Proof needed.)

But if $Y \approx S^1 \times K(G, 2)$, then we know not only that $\pi_2Y = G$, but also that $\pi_1Y$ acts trivially on $\pi_2Y$. This is of course not the case for $X$, so there can be no map $X \to Y$ which induces isomorphisms on $\pi_1$ and $\pi_2$, since it would have to be compatible with this action.

When $\pi_1X = \{0\}$ we can proceed. Given an $n$-truncation $f: X \to X_{\geq n}$, form the mapping cone as the pushout

$$
\begin{array}{ccc}
X & \xrightarrow{f} & \text{Cone}(X) \\
\downarrow & & \downarrow \\
X_{\leq n} & \xrightarrow{f} & \text{Cone}(f)
\end{array}
$$

We can apply homotopy excision to this square: since $X$ is simply connected the map $X \to \text{Cone}(X)$ is 2-connected, while $f$ is $(n + 1)$-connected. Thus $\pi_k(X \xrightarrow{f} X_{\leq n}) \to \pi_k(\text{Cone}(X) \to \text{Cone}(f))$ is an isomorphism if $k < 2 + (n + 1) = n + 3$, and in particular is iso in $n + 2$. Since $\text{Cone}(X)$ is contractible we have $\pi_k(\text{Cone}(X) \to \text{Cone}(f)) \xrightarrow{i} \text{Cone}(f)$, and therefore we conclude that: $\text{Cone}(f)$ is $(n + 1)$-connected with $\pi_{n+2}\text{Cone}(f) \approx G$.

27.2. Remark. If $X$ is merely 1-connected, then we would only know that $G = \pi_{n+2}(X \xrightarrow{f} X_{\leq n}) \to \pi_{n+2}\text{Cone}(f)$ is surjective. In fact it turns out that in this case $\pi_{n+2}\text{Cone}(f) \approx G/N$, where $N$ is
the normal subgroup generated by \( x - \text{conj}_a(x) \) for \( x \in G \) and \( a \in \pi_1 X \), i.e., \( G/N \) is the maximal quotient of \( G \) on which \( \pi_1 \) acts trivially. Thus in general it is possible to carry out what we want to do exactly when \( \pi_1 X \) acts trivially on \( G = \pi_{n+1} X \).

We now obtain a map \( \tilde{k} \colon \text{Cone}(f) \to K(G, n + 2) \) to an Eilenberg MacLane space which is an isomorphism in \( \pi_{n+2} \): for instance, we can take \( \tilde{k} \) to be the \((n + 2)\)-truncation of \( \text{Cone}(f) \). We define \( k := \tilde{k}j = : X_{\leq n} \to K(G, n + 2) \).

The restriction of \( \tilde{k} \) to \( \text{Cone}(X) \subseteq \text{Cone}(f) \) is null-homotopic (because \( \text{Cone}(X) \) is contractible). By choosing such a homotopy we obtain a lift of this restriction to a map \( \tilde{k} \colon \text{Cone}(X) \to EK(G, n + 2) \) (a map \( A \to EK(G, n + 2) \) is the same thing as a map \( H \colon A \times [0, 1] \to K(G, n + 2) \) such that \( H_1(A) = * \).

Thus we have a commutative diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \text{Cone}(X) & \xrightarrow{\tilde{k}} & EK(G, n + 2) \\
\downarrow & & \downarrow & & \downarrow \\
X_{\leq n} & \xrightarrow{j} & \text{Cone}(f) & \xrightarrow{k} & K(G, n + 2)
\end{array}
\]

Taking relative homotopy of the vertical arrows gives

\[
\pi_{n+2}(X_{\leq n} \Rightarrow X) \to \pi_{n+2}(\text{Cone}(X) \Rightarrow \text{Cone}(f)) \to \pi_{n+2}(EK(G, n + 2) \Rightarrow K(G, n + 2))
\]

and both these maps are isomorphisms: the first by homotopy excision as we have already seen, and the second because it is isomorphic to the map \( \pi_{n+2} \text{Cone}(f) \Rightarrow \pi_{n+2}K(G, n + 2) \) which we chose to be an isomorphism.

We can also factor the composite \( (X \xrightarrow{f} X_{\leq n}) \Rightarrow (EK(G, n + 2) \Rightarrow K(G, n + 2)) \) through the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y & \xrightarrow{k} & EK(G, n + 2) \\
\downarrow & & \downarrow & & \downarrow \\
X_{\leq n} & \xrightarrow{k} & X_{\leq n} & \xrightarrow{k} & K(G, n + 2)
\end{array}
\]

where the right-hand square is the pullback. Taking relative homotopy gives maps

\[
\pi_{n+2}(X_{\leq n} \Rightarrow X) \to \pi_{n+2}(Y \Rightarrow X_{\leq n}) \to \pi_{n+2}(EK(G, n + 2) \Rightarrow K(G, n + 2)),
\]

the second of which is an isomorphism since the square is a pullback of a Serre fibration. Thus the first map is an isomorphism as well. From this we conclude using the LES of relative homotopy that \( \pi_{n+1} X \Rightarrow \pi_{n+1} Y \) is an isomorphism. It is straightforward to check that \( \pi_k X \Rightarrow \pi_k Y \) is an isomorphism \( k \leq n \) (since \( \pi_k Y \Rightarrow \pi_k X_{\leq n} \Leftarrow \pi_k X \) are isomorphisms in these dimensions), so \( g \) is \((n + 2)\)-connected, and therefore is an \((n + 1)\)-truncation.

Example: \( (S^2)_{\leq 3} \Rightarrow K(Z, 2) \Rightarrow K(Z, 4) \). Because we know \( K(Z, 2) = \mathbb{C}P^{\infty} \) we can show that \( k \) is the map representing the cup square.

28. Freudenthal suspension

28.1. Theorem. If \( X \) is an \((n - 1)\)-connected space (with non-degenerate basepoint), there are isomorphisms \( \pi_q X \Rightarrow \pi_{q+1}X \) for \( q < 2n - 1 \), and a surjection for \( q = 2n - 1 \).

For instance, \( \pi_3 S^2 \Rightarrow \pi_4 S^3 \) is surjective, while \( \pi_4 S^3 \Rightarrow \pi_5 S^4 \Rightarrow \pi_6 S^5 \).

What is the kernel?
28.2. Proposition. If \( f, g \in \pi_* X \), then their Whitehead product \([f,g]\) is in the kernel of the suspension map.

Proof. By naturality it suffices to consider the universal Whitehead product \( S^{p+q-1} \xrightarrow{w} S^p \vee S^q \).

I claim that \( \Sigma w \sim 0 \). The mapping cone of \( \Sigma w \) is \( \Sigma(S^p \times S^q) \). The claim amounts to showing that \( \Sigma(S^p \vee S^q) \to \Sigma(S^p \times S^q) \) admits a retraction up to homotopy. This follows from a more general observation. \( \Box \)

28.3. Proposition. Let \( X, Y \) be based CW-complexes, and let \( j : X \vee Y \to X \times Y \) be the evident inclusion. Then the map \( \Sigma(j) : \Sigma(X \vee Y) \to \Sigma(X \times Y) \) admits a retraction up to basepoint preserving homotopy.

Proof. Let \( i_X : X \to X \vee Y, \quad i_Y : Y \to X \vee Y \) be the evident inclusions. Then the composites \( j_X : ji_X : X \to X \times Y \) and \( j_Y : ji_Y : Y \to X \vee Y \) are “slice” inclusions. Let \( \pi_X : X \times Y \to X, \quad \pi_Y : X \times Y \to Y \) be projections. Thus

\[
\begin{align*}
\pi_X j_X &= \text{id}_X, & \pi_Y j_X &= *, \\
\pi_X j_Y &= *, & \pi_Y j_Y &= \text{id}_Y.
\end{align*}
\]

Now let \( G \) be an \( H \)-space. I will use these maps to show that

\[
[X \times Y, G]_* \xrightarrow{j^*} [X \vee Y, G]_* \cong [X, G]_* \times [Y, G]_*
\]

is surjective. Given \( f : X \vee Y \to G \), consider the maps

\[
fi_X \pi_X : X \times Y \to G, \quad fi_Y \pi_Y : X \times Y \to G,
\]

and define \( g := \mu(fi_X \pi_X, fi_Y \pi_Y) : X \times Y \to G \) using the \( H \)-space structure \( \mu : G \times G \to G \). I claim that \( jg \sim f \). To show this we compute the restrictions of these maps along \( i_X, i_Y \):

\[
\begin{align*}
(fi_X \pi_X)j_X &= fi_X, & (fi_Y \pi_Y)j_X &= *, \\
(fi_X \pi_X)j_Y &= *, & (fi_Y \pi_Y)j_Y &= fi_Y.
\end{align*}
\]

The claim follows.

Now let \( G = \Omega \Sigma(X \vee Y) \), and let \( f : X \vee Y \to \Omega \Sigma(X \vee Y) \) be the suspension map, which is adjoint to \( \text{id}_X \Sigma(X \vee Y) \). The above argument gives \( g : X \times Y \to \Omega \Sigma(X \vee Y) \) such that \( jg \sim f \), and thus the adjoint \( \tilde{g} : \Sigma(X \times Y) \to \Sigma(X \vee Y) \) satisfies \( \tilde{g}(\Sigma j) \sim \text{id} \). \( \Box \)

In our example, we know that \( [\iota_2, \iota_2] = 2\eta \) in \( \pi_3 S^2 \) (using the Hopf invariant and the isomorphism \( \pi_3 S^2 \cong \mathbb{Z}_2 \)). Thus \( \pi_4 S^3 \) is either \( \mathbb{Z}/2 \) or 0.

How do we resolve this? Here is one method: show that there is a CW-complex approximation to \( \Omega \Sigma S^2 = \Omega S^3 \) with the form

\[
S^2 \cup_{[\iota, \iota]} e^4 \cup e^6 \cup e^8 \cup \cdots \cong \Omega S^3.
\]

This approximation has cells only in even degrees, and the first attaching map is \( [\iota, \iota] \in \pi_3 S^2 \). Homotopy excision applies to give us \( \pi_4 S^3 \cong \pi_3 \Omega S^3 \cong \pi_3 S^2/(\iota, \iota) \cong \mathbb{Z}/2 \), since \( H(\iota, \iota) = 2 \).

We'll construct such a CW-structure by first computing the homology of \( \Omega S^3 \). The first observation is that for a simply connected space you can always construct a CW-approximation which “closely approximates” the homology groups. This is clearest in the case when the homology groups are all free groups.
28.4. Proposition. Suppose $X$ is a simply connected space with $H_*X$ a free abelian group in every dimension. Then there is a CW-approximation $K \to X$ in which the cells of $K$ are in bijective correspondence with a choice of basis for $H_*X$. That is, $H_nX \approx H_n(K_n, K_{n-1})$.

Proof. We inductively construct the $n$-skeleton $K_n$ of $K$ together with a map $f : K_n \to X$ which is an $H_*$-isomorphisms through dimension $n$.

Start with $K_0 = K_1 = \{\ast\}$.

If $n = 2$, the Hurewicz theorem says $\pi_2X \approx H_2X$, so just take $K_2 = \bigvee_\alpha S^2$ corresponding to a basis of $H_2X$.

Suppose $f_n : K_n \to X$ exists as claimed. Then Cone$(f_n)$ is $n$-connected, since it is simply connected (because $K_n$ and $X$ are) and $\tilde{H}_k$Cone$(f_n) = 0$ if $k \leq n$. From this we can deduce that $f_n$ is an $n$-connected map. (We did this argument before: $\pi_k(K_n \to X) \to \pi_k$Cone$(f_n)$ is an isomorphism in the range $k < 2 + \text{conn}(f_n)$, so if Cone$(f_n)$ is $d$-connected then one inductively shows that $f_n$ is also $d$-connected.)

Since $f_n$ is $n$-connected and $K_n$ is simply connected, homotopy excision and the Hurewicz theorem give us isomorphisms

$$\pi_{n+1}(K_n \to X) \xrightarrow{\sim} \pi_{n+1}$\text{Cone}(f_n) \xrightarrow{\sim} H_{n+1}$\text{Cone}(f_n) \approx H_{n+1}X.$$

Choose a basis $\{\alpha\}$ for $H_{n+1}X$. Then for each basis element we can choose a map $S^{n+1} \to$ Cone$(f_n)$ representing it, and then lift it to a map $j_\alpha : (S^n \to D^{n+1}) \Rightarrow (K_n \to X)$. These $j_\alpha$ give us attaching maps to construct $K_{n+1}$ together with a map

$$K_{n+1} := K_n \cup \bigcup_\alpha e^{n+1} \to X$$

which is an isomorphism in homology through dimension $n + 1$. \hfill \Box

29. Homology of Loop-Suspension

Fix a based space $X$, which will soon be assumed to be connected. The space $\Omega \Sigma X$ is an associative H-space via the loop composition map. Therefore

$$A_* := H_*\Omega \Sigma X$$

obtains the structure of a graded associative algebra. We note also the suspension map $\sigma : X \to \Omega \Sigma X$, which induces a map of abelian groups

$$V_* = H_*X \to A_* = H_*\Omega \Sigma X.$$ 

I will also write $\tilde{V}_* = \tilde{H}_*X$ and $\tilde{A}_* = \tilde{H}_*\Omega \Sigma X$. If $X$ is connected, then so is $\Omega \Sigma X$ (by homotopy excision), so $\tilde{V}_*$ and $\tilde{A}_*$ are just the positive degree parts of $V_*$ and $A_*$.

Given any graded abelian group $M_*$, we can form the tensor algebra

$$T(M_*) := \bigoplus_{n \geq 0} M_*^{\otimes n} \approx \mathbb{Z} \oplus M_* \oplus M_* \otimes M_* \oplus M_* \otimes M_* \otimes M_* \oplus \cdots.$$ 

It is the free graded associative algebra on $M_*$. In particular, the suspension map extends to a map

$$T(\tilde{H}_*X) \to H_*\Omega \Sigma X$$

of graded associative algebras, so that $v_1 \otimes \cdots \otimes v_n \mapsto \sigma(v_1) \cdots \sigma(v_n)$.

29.1. Theorem (Bott-Samelson). If $X$ is a based connected space and $\tilde{H}_*X$ is torsion free, then

$$T(\tilde{H}_*X) \to H_*\Omega \Sigma X$$

is an isomorphism of algebras.

This gives us the homology of $\Omega S^3 = \Omega \Sigma S^2$. 
29.2. Proposition. As a ring, $H_*\Omega S^3 \approx \mathbb{Z}[x]$ where $x \in H_2\Omega S^3$ is the image of the generator of $H_2 S^2$ under the suspension map $\sigma: S^2 \to \Omega S^3$.

Another natural piece of structure on any space $T$ is the diagonal map $\Delta: T \to T \times T$. If $H_*T$ is torsion free, then the Künneth theorem says that the Künneth map

$$H_* T \otimes H_* T \to H_*(T \times T)$$

is an isomorphism, and thus we get a comultiplication

$$\Delta_*: H_* T \to H_*(T \times T) \cong H_* T \otimes H_* T$$

which is necessarily counital, coassociative, and cocommutative. If $T$ is an H-space, then the diagonal map is automatically a map of H-spaces (because “diagonal” is a natural transformation $\mathrm{Id} \to \mathrm{Id} \times \mathrm{Id}$ of functors $\mathrm{Top}_* \to \mathrm{Top}_*$, so it “commutes” with any continuous map). Therefore for $T = \Omega \Sigma S^2$ we see that

$$\Delta_*(x) = x \otimes 1 + 1 \otimes x$$

for dimension reasons, and because it commutes with the Pontryagin product we have

$$\Delta_*(x^n) = (x \otimes 1 + 1 \otimes x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}.$$ 

Because $H_*\Omega S^3$ are projective, we have $H^n\Omega S^3 \cong \text{Hom}(H_3\Omega S^3, \mathbb{Z})$, and $\Delta_*$ turns into the cup-product in cohomology. If we let $\gamma_n \in H^{2n}\Omega S^3$ be the element dual to $x^n \in H_{2n}\Omega S^3$ under the Kronecker pairing, we read off that

$$\gamma_i \cup \gamma_j = \binom{i+j}{i} \gamma_{i+j},$$

which implies

$$\gamma_1^n = (n!)\gamma_n,$$

i.e., $H^*\Omega S^3$ is a divided power algebra on the generator $\gamma_1$. In particular, $\gamma_1^2 = 2\gamma_2$ in $H^4\Omega S^3$, so the first attaching map in $\Omega S^3 \approx S^2 \cup e^4 \cup \cdots$ has Hopf invariant 2, and so is homotopic to $[i, i]$.

Now we prove the Bott-Samelson theorem. Consider the pushout square that defines suspension:

$$\begin{array}{ccc}
X & \rightarrow & \ast \\
\downarrow & & \downarrow \\
\ast & \rightarrow & \text{Cone}(X) \quad q \rightarrow \Sigma X
\end{array}$$

Note that $\text{Cone}(X)$ is contractible, so $\ast \rightarrow \text{Cone}(X)$ is a homotopy equivalence. Let

$$P(\Sigma X) = \text{Map}(I, \Sigma X) \times_{\text{Map}(\{1\}, \Sigma X)} \ast = \{ \gamma: [0, 1] \to \sigma X \mid \gamma(1) = * \},$$

the path space associated to $\ast \rightarrow \Sigma X$. Note that $P(\Sigma X)$ is contractible, and is equipped with an evaluation map $\epsilon: P(\Sigma X) \to \Sigma X$ sending $\gamma \mapsto \gamma(0)$.

Let’s form the pullback of $\epsilon: P(\Sigma X) \to \Sigma X$ along each map in the above diagram. We obtain a diagram

$$\begin{array}{ccc}
X \times \Omega \Sigma X & \xrightarrow{\pi} & \Omega \Sigma X \\
\downarrow & & \downarrow \\
\Omega \Sigma X & \xrightarrow{j} & E & \xrightarrow{q} & P(\Sigma X)
\end{array}$$

where $\pi$ is the obvious projection, and

$$E = \text{Cone}(X) \times_{\Sigma X} P(\Sigma X) = \{ (y, \gamma) \in \text{Cone}(X) \times \text{Map}(I, \Sigma X) \mid q(0) = \pi(y), \gamma(1) = * \}$$

where $q: \text{Cone}(X) \to \Sigma X$ is the quotient map.
We make three claims:

1. The inclusion $j$ is a weak equivalence.
2. The diagram

$$
\begin{array}{ccc}
X \times \Omega \Sigma X & \rightarrow & E \\
\downarrow & & \downarrow \\
\Omega \Sigma X & \xrightarrow{j} & \rightarrow
\end{array}
$$

commutes up to homotopy, where the diagonal map is $\alpha(x, \gamma) \mapsto \mu(\sigma(x), \gamma)$, where $\mu : \Omega \Sigma X \rightarrow \Omega \Sigma X$ is loop composition.

3. The maps in the above square fit into a Mayer-Vietoris-type long exact sequence in homology:

$$
\cdots \rightarrow H_n(X \times \Omega \Sigma X) \rightarrow H_n(\Omega \Sigma X) \times H_n E \rightarrow H_n P(\Sigma X) \rightarrow H_{n-1}(X \times \Omega \Sigma X) \rightarrow \cdots
$$

Given these claims, if we take homology we get a commutative diagram

$$
\begin{array}{ccc}
V_s \otimes A_s & \xrightarrow{\pi_s} & A_s \\
\downarrow \mu_s & & \downarrow \\
A_s & \xrightarrow{\sim} & \mathbb{Z}
\end{array}
$$

where I have written $\mu_s$ for the composite $V_s \otimes A_s \rightarrow A_s \otimes A_s \rightarrow A_s$ of the suspension map with the multiplication. The upper left is by the coefficient theorem and the fact that $H_\ast X$ is torsion free. By (3) we see that the kernels of the horizontal maps must be isomorphic, i.e., we get an isomorphism

$$
\mu : \tilde{V}_s \otimes A_s \xrightarrow{\sim} \tilde{A}_s.
$$

Now, we compare this map with the analogous one for the tensor algebra:

$$
\begin{array}{ccc}
\tilde{V}_s \otimes T(\tilde{V}_s) & \xrightarrow{\sim} & \tilde{T}(\tilde{V}_s) \\
\downarrow \phi & & \downarrow \\
\tilde{V}_s \otimes A_s & \xrightarrow{\sim} & \tilde{A}_s
\end{array}
$$

We see that the vertical map is an isomorphism in each degree, by induction on degree. It is clear in degree 0, while if degree $d > 0$ we have

$$
\begin{array}{ccc}
V_d \otimes \mathbb{Z} & \oplus & V_{d-1} \otimes T(\tilde{V}_s)_{d-1} & \oplus & \cdots & \oplus & V_1 \otimes T(\tilde{V}_s)_{d-1} \\
\downarrow \text{id} \otimes \phi & & \downarrow \phi & & \cdots & \downarrow \text{id} \otimes \phi \\
V_d \otimes \mathbb{Z} & \oplus & V_{d-1} \otimes A_1 & \oplus & \cdots & \oplus & V_1 \otimes A_{d-1} \\
& & \xrightarrow{\sim} & & \xrightarrow{\sim} & & \tilde{A}_s
\end{array}
$$

and the map on the left is an isomorphism by the inductive hypothesis.

30. Spectra and homology theories

Given a based space $X$, define

$$
\pi^{\text{st}}_n(X) := \text{colim}_{k \to \infty} \pi_{n+k} \Sigma^k X,
$$

where the direct limit is over the sequence

$$
\cdots \rightarrow \pi_{n+k} \Sigma^k X \xrightarrow{E} \pi_{n+k+1} \Sigma(\Sigma^k X) = \pi_{n+k+1} \Sigma^{k+1} X.
$$
Note that $\pi^s_n$ is defined even if $n$ is negative: in this case the sequence starts with $\pi_0 \Sigma^{-n} X \to \pi_1 \Sigma^{-n+1} X \to \cdots$, and since $-n > 0$ we see that $\pi^s_n X \approx 0$ if $n < 0$. The groups $\pi^s_n(X)$ are called the **stable homotopy groups** of $X$.

We observe the following.

1. $\pi^s_n$ is a homotopy invariant: if $f_0, f_1 : X \to Y$ are based homotopic maps, the induced maps $\pi^s_n X \to \pi^s_n Y$ are the same.
2. There is a suspension isomorphism
   \[\sigma : \pi^s_n X \to \pi^s_{n+1} \Sigma X.\]
   This is induced by the commutative diagram
   \[
   \begin{array}{cccccc}
   \pi_n X & \xrightarrow{E_X} & \pi_{n+1} \Sigma X & \xrightarrow{E_{\Sigma X}} & \pi_{n+2} \Sigma^2 X & \xrightarrow{E_{\Sigma^2 X}} & \cdots \\
   E_X & & E_{\Sigma X} & & E_{\Sigma^2 X} & \\
   \pi_{n+1} \Sigma X & \xrightarrow{E_{\Sigma X}} & \pi_{n+2} \Sigma(\Sigma X) & \xrightarrow{E_{\Sigma(\Sigma X)}} & \pi_{n+3} \Sigma^2 (\Sigma X) & \xrightarrow{E_{\Sigma^2 (\Sigma X)}} & \cdots \\
   \end{array}
   \]
   That this is an isomorphism is because there is an inverse, given by the “shift”.
3. Associated to a HEP map $f : X \to Y$ is an isomorphism.
4. The tautological map
   \[\bigoplus \pi^s_n X_\alpha \to \pi^s_n \bigvee X_\alpha\]
   is an isomorphism.

So $\pi^s_n$ is an example of a (generalized) homology theory.

Describe homology theories arising from spectra.

### 31. **Model categories**

Let $\mathcal{M}$ be a category. A class of maps $\mathcal{W} \subseteq \mathcal{M}$ satisfies **2-out-of-3** if (i) it contains all identity maps, and (ii) if $gf$ is defined, and any two elements of $(f, g, gf)$ are in $\mathcal{W}$, so is the third.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{M}$ is a **weak factorization system (WFS)** if (i) $\mathcal{A} = \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^#$, and (ii) every morphism $f$ can be factored $f = ba$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. (Note that each class determines the other by (i), and we have that $\mathcal{A} \cong \mathcal{B}$.)

A **model category** is a complete and cocomplete category $\mathcal{M}$ equipped with classes $\mathcal{W}, \mathcal{C}, \mathcal{F}$ of morphisms (“weak equivalences”, “cofibrations”, “fibrations”) such that

1. $\mathcal{W}$ satisfies 2-out-of-3, and
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are WFSs.

Easy facts:
- If $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ is a model structure for $\mathcal{M}$, then $(\mathcal{W}^{op}, \mathcal{F}^{op}, \mathcal{C}^{op})$ is a model structure for $\mathcal{M}^{op}$.
- If $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ is a model structure for $\mathcal{M}$, and $A$ is an object of $\mathcal{M}$, then we get a model structure $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ on $\mathcal{M}/A$. This is defined via the forgetful functor $U : \mathcal{M}/A \to \mathcal{M}$, so that a map $f$ in $\mathcal{M}/A$ is in one of the classes iff $U(f)$ is in the corresponding class for $\mathcal{M}$.

We say that an object $X$ is **cofibrant** if $0 \to X$ is a cofibration, **fibrant** if $X \to 1$ is a fibration.

### 32. **Model structure on chain complexes**

Let $\mathcal{M} = \text{Ch}(\text{Mod}_R)$, the category of unbounded chain complexes of $R$-modules. We will produce a model structure on this category.

A chain complex $P$ is called **cellular** if there exists a filtration
\[0 = P^{(-1)} \subseteq P^{(0)} \subseteq P^{(1)} \subseteq \cdots \subseteq P\]
by subcomplexes such that
32.1. Example. Let \( R = k[e]/(e^2) \), and let \( P \) be the complex with \( P_n = R \) for all \( n \), and \( d: P_n \to P_{n-1} \) be given by \( d(a) = ea \). Then \( P \) is degreewise projective, but not cellular. (The only projective submodules of \( R \) are 0 and \( R \). Therefore, any degreewise projective subcomplex \( Q \subseteq P \) is either 0 or has non-zero differential.)

32.2. Remark. Note that if \( P \) is bounded below at 0 (or at any other index), i.e., if \( P_n = 0 \) for \( n < 0 \), then \( P \) is cellular if and only if it is degreewise projective. In this case we can take \( P^{(n)} \) defined by \( P^{(n)}_k = P_k \) if \( k \leq n \), \( P^{(n)}_k = 0 \) if \( k > n \).

Now we define the model structure on chain complexes. A map \( f: C \to D \) is

1. a weak equivalence if it is a quasi-isomorphism, i.e., if \( H_*(f): H_*C \to H_*D \) is an isomorphism,
2. a cofibration if \( f \) is injective and \( D/f(C) \) is cellular,
3. a fibration if \( f \) is surjective.

It takes some work to show that this is a model category, which I will do.

We define two special sets of morphisms.

- Let \( D^n \) be the complex with
  \[(D^n)_n = (D^n)_{n-1} = R, \quad (D^n)_k = 0 \quad \text{if} \quad k \neq n-1, n,
\]
  with \( d \) such that \( d: (D^n)_n \to (D^n)_{n-1} \) is an isomorphism. Let \( i_n: S^{n-1} \to D^n \) be the inclusion of a subcomplex with \( (S^{n-1})_{n-1} = R, (S^{n-1})_n = 0 \). (We can also say \( S^{n-1} = R[n-1] \).

  Then \( i_n: S^{n-1} \to D^n \) is a cofibration, since \( D^n/S^{n-1} \ltimes R[n] \). Let \( \mathcal{I} = \{i_n\} \).

- Let \( j_n: 0 \to D^n \) be the evident inclusion.

  Then \( j_n \) is a cofibration and a weak equivalence. (It is a cofibration because of the filtration \( 0 \subseteq S^{n-1} \subseteq D^n \).) Let \( \mathcal{J} = \{j_n\} \).

32.3. Lemma. Let \( f: C \to D \).

1. \( f \in \mathcal{F} \cap \mathcal{W} \) if and only if \( \mathcal{I} \vartriangle f \).
2. \( f \in \mathcal{F} \) if and only if \( \mathcal{J} \vartriangle f \).

Proof. This is an elementary calculation, using \( \text{Hom}(D^n, C) \approx C_n \) and \( \text{Hom}(S^{n-1}, C) \approx \text{Ker}[C_{n-1} \to C_{n-2}] \), and that \( i_n \) induces the boundary map. \( \square \)

Now note that \( \vartriangle \mathcal{F} \) and \( \vartriangle (\mathcal{F} \cap \mathcal{W}) \) are weakly saturated classes which contain \( \mathcal{J} \) and \( \mathcal{I} \) respectively.

We write \( \overline{\mathcal{K}} \) for the weak saturation of \( \mathcal{K} \), i.e., the class which contains all isomorphisms and elements of \( \mathcal{K} \), and is closed under arbitrary coproducts, pushouts, countable composition, and retracts.

We use the small object argument to construct factorizations \( \mathcal{F} \cdot \mathcal{J} \) and \( (\mathcal{F} \cap \mathcal{W}) \cdot \mathcal{I} \).

Let \( \mathcal{K} = \{A_i \to B_i\} \) be a set of maps such that each \( \text{Hom}(A_i, -) \) commutes with countable sequential colimits.

32.4. Lemma (Small object argument). The pair \( (\text{Cell}(\mathcal{K}), \text{Inj}(\mathcal{K})) \) is a WFS.

Proof. Use the small object argument to produce a factorization. It is clear that \( \mathcal{K} \vartriangle = \overline{\mathcal{K}} \), and that \( \overline{\mathcal{K}} \subseteq \vartriangle (\overline{\mathcal{K}}) \). Suppose \( f \vartriangle (\overline{\mathcal{K}}) \), factor as \( f = pi \) with \( i \in \overline{\mathcal{K}} \). Then you can show that \( f \) is a retract of \( i \), so is in \( \overline{\mathcal{K}} \). \( \square \)
We apply this to chain complexes, giving WFSs \((\text{Cell}(\mathcal{I}), \text{Inj}(\mathcal{I}))\) and \((\text{Cell}(\mathcal{J}), \text{Inj}(\mathcal{J}))\). It remains to show that \(\text{Cell}(\mathcal{I}) = \mathcal{C}\) and \(\text{Cell}(\mathcal{J}) = \mathcal{C} \cap \mathcal{F}\).

### 32.5. Lemma. \(\text{Cell}(\mathcal{I}) \subseteq \mathcal{C}\).

**Proof.** We note that the proof of the small object argument applied to \(\mathcal{I}\) actually shows that any \(f\) can be factored \(f = ip\) with \(p \in \text{Inj}(\mathcal{I})\), and that \(i \in \mathcal{C}\), because \(i\) is explicitly built as a countable direct limit of inclusions \(\mathcal{C} = \mathcal{C}(0) \to \mathcal{C}(1) \to \cdots \to D\), so that
\[
\mathcal{C}(n)/\mathcal{C}(n-1) \approx \bigoplus_i D^{n_i}/S^{n_i-1},
\]
which is a degreewise projective complex with trivial differential. Thus \(i \in \mathcal{C}\).

The "retract argument" then applies to show that if \(f \in \text{Cell}(\mathcal{I})\), then \(f\) is a retract of \(i\).

Thus, it suffices to show that \(\mathcal{C}\) is closed under retracts. Since retracts of monomorphisms are retracts, and the cokernel of the retract is the retract of the cokernel, it suffices to show that cellular complexes \(P\) are closed under retracts. But \(P\) cellular means that \(d_P \equiv 0\), and therefore the same is true for any subcomplex. Furthermore, retracts of projective modules are projective. \(\square\)

### 32.6. Lemma. \(\mathcal{C} \subseteq \text{Cell}(\mathcal{I})\).

**Proof.** Let \(C \to D\) be a cofibration, with filtration \(D^{(n)}\) such that \(D^{(0)} = C\) and \(P = D^{(n)}/D^{(n-1)}\) degreewise projective with trivial boundary map. Then there is a pushout
\[
\begin{array}{ccc}
S^{-1} \otimes_R P & \longrightarrow & C^{(n-1)} \\
\downarrow & & \downarrow \\
D^0 \otimes_R P & \longrightarrow & C^{(n)}
\end{array}
\]
where the map \(\text{Re}_0 \otimes_R P \to D^0 \otimes_R P \to C^{(n)}\) is a lift of \(P \to C^{(n)}/C^{(n-1)}\).

**Proof.** Show that \(\mathcal{C} \subseteq (\mathcal{F} \cap \mathcal{W})\), and therefore \(\mathcal{C} \subseteq (\mathcal{F} \cap \mathcal{W}) = \mathcal{I}\). Conversely, we need to show that \(\mathcal{I} \subseteq \mathcal{C}\). Since \(\mathcal{I} \subseteq \mathcal{C}\), it suffices to show that \(\mathcal{C}\) is closed under the various operations, which is relatively straightforward. \(\square\)

### 32.7. Lemma. \(\mathcal{J} = \mathcal{C} \cap \mathcal{W}\).

**Proof.** We have \(\mathcal{J} \subseteq \mathcal{C} \cap \mathcal{W}\). Show that \(\mathcal{C} \cap \mathcal{W}\) is closed under the weak saturation operations, so \(\mathcal{J} \subseteq \mathcal{C} \cap \mathcal{W}\).

Finally, let \(f \in \mathcal{C} \cap \mathcal{W}\). Factor as \(f = qj\) with \(q \in \mathcal{F}\) and \(j \in \mathcal{J} \subseteq \mathcal{C} \cap \mathcal{W}\). Because weak equivalences satisfy 2-out-of-3 we have \(q \in \mathcal{F} \cap \mathcal{W}\). Since \(f \in \mathcal{C}\) we can show that \(f\) is a retract of \(j\), and therefore \(f \in \mathcal{J}\). \(\square\)

We have actually carried out an example of the “recognition theorem”, applied to \(\mathcal{C} = \mathcal{I}, \mathcal{CW} = \mathcal{J}, \mathcal{F} = \mathcal{J}^\perp, \mathcal{FW} = \mathcal{I}^\perp\).

### 32.8. Theorem. Given \(\mathcal{M}\) together with a class \(\mathcal{W}\) satisfying 2-out-of-3, and WFSs \((\mathcal{C}, \mathcal{FW})\) and \((\mathcal{CW}, \mathcal{F})\) such that
\[
\begin{align*}
(1) \quad & \mathcal{CW} \subseteq \mathcal{C} \cap \mathcal{W}, \\
(2) \quad & \mathcal{FW} \subseteq \mathcal{F} \cap \mathcal{W}, \text{ and} \\
(3) \quad & \text{one of } \mathcal{C} \cap \mathcal{W} \subseteq \mathcal{CW} \text{ or } \mathcal{F} \cap \mathcal{W} \subseteq \mathcal{FW} \text{ holds},
\end{align*}
\]
then the remaining statement of (3) holds and \((\mathcal{W}, \mathcal{C}, \mathcal{F})\) is a model structure.

**Proof.** Suppose we have (1), (2), and \(\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{FW}\). Then given \(f \in \mathcal{C} \cap \mathcal{W}\) factor \(f = qj\) with \(q \in \mathcal{F}\) and \(j \in \mathcal{CW}\). Then 2-out-of-3 gives \(q \in \mathcal{F} \cap \mathcal{W} = \mathcal{FW}\) and \(j \in \mathcal{CW} \subseteq \mathcal{C} \cap \mathcal{W}\) whence we can show that \(f\) is a retract of \(j\) and so \(f \in \mathcal{CW}\). \(\square\)
33. MODEL STRUCTURE ON TOPOLOGICAL SPACES

The category $\text{Top}$ is a model category where a map $f: X \to Y$ is

1. a weak equivalence if (duh),
2. a cofibration if it is a retract of a relative cell map, and
3. a fibration if it is a Serre fibration.

We let

$$\mathcal{I} := \{ S^{n-1} \to D^n \mid n \geq 0 \},$$
$$\mathcal{J} := \{ D^n \times \{0\} \to D^n \times [0,1] \mid n \geq 0 \}.$$

We observe that we can carry out the small object argument with these sets, so that we get WFSs $(\text{Cell}(\mathcal{I}), \text{Inj}(\mathcal{I}))$ and $(\text{Cell}(\mathcal{J}), \text{Inj}(\mathcal{J}))$.

We observe that:

1. $\text{Inj}(\mathcal{J})$ is exactly the class $\mathcal{F}$ of Serre fibrations, by definition.
2. $\text{Cell}(\mathcal{I})$ is exactly the class $\mathcal{C}$ of retractions of relative cell complexes. To see this, note that relative cell complexes are clearly in $\text{Cell}(\mathcal{I})$, and hence so are retractions. Conversely, if $f \in \text{Cell}(\mathcal{I})$, then the small object argument gives a factorization $f = pi$ where $i$ is actually a relative cell complex. Then the retract trick tells us that $f$ is a retract of $i$.
3. We have $\mathcal{J} \subseteq \text{Cell}(\mathcal{I})$, which automatically implies $\text{Cell}(\mathcal{J}) \subseteq \text{Cell}(\mathcal{I})$ and $\text{Inj}(\mathcal{J}) \subseteq \text{Inj}(\mathcal{I})$.
4. We have that $\text{Inj}(\mathcal{I}) \subseteq \mathcal{W}$. This is because if $f \in \text{Inj}(\mathcal{I})$ then using $I \Box f$ we see immediately that (i) $f_\alpha: \pi_0X \to \pi_0Y$ is surjective, and (ii) $\pi_n(f) \approx 0$ for all $n \geq 1$.
5. We have that $\text{Cell}(\mathcal{J}) \subseteq \mathcal{W}$. This is because each element of $\mathcal{J}$ is a deformation retraction, and this property is inherited under pushout and transfinite composition.

To summarize, we have two WFSs $(\text{Cell}(\mathcal{I}), \text{Inj}(\mathcal{I}))$ and $(\text{Cell}(\mathcal{J}), \text{Inj}(\mathcal{J}))$, and we know

- $\text{Cell}(\mathcal{J}) \subseteq \text{Cell}(\mathcal{I}) \cap \mathcal{W}$, and
- $\text{Inj}(\mathcal{I}) \subseteq \text{Inj}(\mathcal{J}) \cap \mathcal{W}$.

By the recognition theorem it suffices to show either $\text{Cell}(\mathcal{I}) \cap \mathcal{W} \subseteq \text{Cell}(\mathcal{J})$ or $\text{Inj}(\mathcal{J}) \cap \mathcal{W} \subseteq \text{Inj}(\mathcal{I})$. We will prove the second one, which says: if a map $p$ is both a Serre fibration and a weak equivalence, then $I \Box p$.

**Claim.** If $p$ is a Serre fibration and $i \in \mathcal{I}$, then $i \Box p$ if and only if $i \Box (F \to \{b_0\})$ for any $b_0 \in B$ and $F = p^{-1}(b_0)$. Then we win, since if $p$ is a weak equivalence, the LES of a Serre fibration tells us that $\pi_* F \approx 0$, and so lifts clearly exist.

In fact, earlier we proved the “covering homotopy extension property”, which says that for any Serre fibration $f: E \to B$ and relative cell complex $j: K \to L$, and any diagram

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & \text{Map}(L,E) \\
\downarrow & & \downarrow \\
I & \longrightarrow & \text{Map}(j,p)
\end{array}
$$

a lift exists. That is, for any 1-parameter family of lifting problems relating $j$ and $p$ such that a lift exists when $t = 0$, a lift exists for all $t \in I$.

Thus, given $p$ which is a Serre fibration and a weak equivalence, we consider a lifting problem $j: S^{n-1} \to D^n$ vs $f$. We choose a deformation retraction $K$ of $D^n$ to a basepoint, and consider

$$
(S^{n-1} \times \{0\}) \cup (\{s_0\} \times I) \xrightarrow{(\alpha, \pi_0)} E
$$

$$
S^{n-1} \times I \xrightarrow{K} D^n \times I \xrightarrow{p} B
$$
and form $H$ since $p$ is a Serre fibration. This data fits together to give a homotopy $(H,K)$ of based maps $(S^{n-1} \to D^n) \Rightarrow (E \to B)$, which at the 0 end is the original diagram, and at at the 1 end factors through $(F \to \{b_0\})$. Thus it suffices to consider the case of $F \to *$ which is a Serre fibration and a weak equivalence, where the claim is clear.

### 34. Homotopy Category

Given a pair $(\mathcal{M}, \mathcal{W})$ of category and subcategory, a **homotopy category** is a functor $\gamma : \mathcal{M} \to h\mathcal{M}$ such that every functor $F : \mathcal{M} \to \mathcal{A}$ which takes elements of $\mathcal{W}$ to isomorphisms factors uniquely through a functor $F' : h\mathcal{M} \to \mathcal{A}$ so that $F'\gamma = F$.

**34.1. Remark.** Let $\mathcal{A}$ be any category such that there is exactly one map $A \to A'$ for every pair of objects. Then every $F : \mathcal{M} \to \mathcal{A}$ sends elements of $\mathcal{W}$ to isomorphisms, so extends uniquely to $h\mathcal{M}$. As a consequence, we can see that $\gamma$ must be bijective on objects: every object of $h\mathcal{M}$ is canonically $\gamma(X)$ for a unique $X$ in $\mathcal{M}$.

If $\mathcal{M}$ is a small category, then $h\mathcal{M}$ exists formally. If it isn’t small, it still “exists”, but lives in a potentially bigger universe of sets.

The basic idea is that if $\mathcal{M}$ is a model category, then one can show that

$$\text{Hom}_{h\mathcal{M}}(\gamma X, \gamma Y) \approx \text{Hom}_{\mathcal{M}}(X_c, Y_f)/\sim,$$

where (i) $X_c$ is any cofibrant object with a chosen weak equivalence $X_c \to X$, (ii) $Y_f$ is any fibrant object with a chosen weak equivalence $Y \to Y_f$, and (iii) $\sim$ is a suitable “homotopy relation” on the set of maps.

**34.2. Example.** In $\mathcal{M} = \text{Ch}(\text{Mod}_R)$ as constructed above, all objects are fibrant, so we can take $D \to D_f$ to be the identity map. Given $C$ we choose a quasi-isomorphism $C_c \to C$ where $C_c$ is cofibrant (i.e., cellular). It turns out that in this case the homotopy relation on $\text{Hom}(C_c, D)$ is exactly chain homotopy.

Given a map $g : A \to B$ in a model category $\mathcal{M}$, let $\mathcal{M}_{/g/}$ denote the category of factorizations of $g$. This is a model category in its own right. Note the initial and terminal objects in $\mathcal{M}_{/g/}$ are

$$A \xrightarrow{\text{id}_A} A \xrightarrow{g} B, \quad A \xrightarrow{g} B \xrightarrow{\text{id}_B} B.$$

Consider the fold map $\nabla_X = (\text{id}, \text{id}) : X \amalg X \to X$. A **cylinder object** on $X$ is on object of $\mathcal{M}_{/\nabla X/}$ which is weakly equivalent to the terminal object. Explicitly, a cylinder object is a pair of maps

$$X \amalg X \xrightarrow{i=(i_0, i_1)} C \xrightarrow{p} X$$

such that $pi = \nabla_X$ and $p$ is a weak equivalence.

Note that the cylinder object is cofibrant in $\mathcal{M}_{/\nabla X/}$ if and only if $i$ is a cofibration; this is also called a “good” cylinder object. Furthermore, the cylinder object is fibrant in $\mathcal{M}_{/\nabla X/}$ if and only if $p$ is a fibration. Cylinder object which are bifibrant (i.e., both cofibrant and fibrant in $\mathcal{M}_{/\nabla X/}$) are also called “very good” cylinder objects.

**Observation.** For a cylinder object as above, both maps $i_0, i_1 : X \to C$ are weak equivalences, by 2-out-of-3.

**34.3. Example.** If $C$ is a chain complex, define $E$ so that

$$E_n = C_n \times C_{n-1} \times C_n, \quad d(x,a,y) = (dx + (-1)^n a, da, dy - (-1)^n a).$$

We have $i : C \oplus C \to E$ by $i(x,y) = (x,0, y)$ and $p : E \to C$ by $p(x,a,y) = x + y$. Verify that $p$ is a quasi-isomorphism, using the exact sequence

$$0 \to K \to E \xrightarrow{p} C \to 0$$
where $K$ is the kernel, isomorphic to $K_0 = C_n \times C_{n-1}$ with $d(x, a) = (dx + (-1)^n a, da)$.

This is not usually “good”, but it is if $C$ is cofibrant=cellular, in which case it is “very good” since $p$ is a fibration.

Fix a cylinder object for $X$. Given any $Y$ we can define a relation $\sim_\ell$ on $\text{Hom}(X,Y)$ using the cylinder object: $f_0 \sim_\ell f_1$ iff there exists $f: X^1 \to Y$ such that $fi_0 = f_0$, $fi_1 = f_1$.

34.4. Remark. If $F: \mathcal{M} \to \mathcal{A}$ is a functor which takes $\mathcal{W}$ to isos (e.g., the universal example $\gamma: \mathcal{M} \to h\mathcal{M}$), then $F(i_0) = F(i_1)$. This is because $\text{id}_X = pi_0 = pi_1$ so $\text{id}_F(X) = F(p)F(i_0) = F(p)F(i_1)$, but $F(p)$ is iso so we can cancel.

Thus, if $f_0 \sim_\ell f_1$, then any $F: \mathcal{M} \to \mathcal{A}$ which takes $\mathcal{W}$ to isos must satisfy $F(f_0) = F(f_1)$, since $F(f_k) = F(f_{ik}) = F(f)f_{ik}$.

We are interested in the converse: we want to know that if $\gamma(f_0) = \gamma(f_1)$ then there exists a left homotopy. This happens under additional hypotheses.

34.5. Lemma. If $f_0 \sim_\ell f_1$ for $f_0,f_1: X \to Y$, then there exists a “good” left homotopy relating them; i.e., one defined on a cofibrant cylinder object.

The left homotopy relation is easily seen to be reflexive and symmetric. Symmetry follows by switching 0 and 1. Reflexivity follows because $X \amalg X \xrightarrow{\gamma X} X \xrightarrow{\text{id}_X} X$ is a cylinder object. However, the left homotopy relation is not generally transitive.

Let $\text{Hom}(X,Y)_\ell$ denote the classes under the equivalence relation generated by left homotopy. In general left homotopy is not an equivalence relation, but we have the following.

34.6. Proposition. If $X$ is cofibrant, then the left homotopy relation on $\text{Hom}(X,Y)$ is transitive, and hence is an equivalence relation.

Proof. We define $\amalg \mathcal{M}/\mathcal{W}_X/ \times \mathcal{M}/\mathcal{W}_X/ \to \mathcal{M}/\mathcal{W}_X$ as follows: given factorizations

$$X \amalg X \xrightarrow{j=(j_0,j_1)} C \xrightarrow{q} X, \quad X \amalg X \xrightarrow{j'=(j'_0,j'_1)} C' \xrightarrow{q'} X,$$

form an object $C \sqcup C'$ by

$$X \amalg X \xrightarrow{j''=(j''_0,j''_1)} C \sqcup_{j_1,X, j'_0} C' \xrightarrow{q''=(q,q')} X,$$

where $j''_0: X \xrightarrow{j_0} C \to C \sqcup_{j_1,X,j'_0} C'$ and $j''_1: X \xrightarrow{j'_1} C' \to C \sqcup_{j_1,X,j'_0} C''$. This defines a monoidal structure on $\mathcal{M}/\mathcal{W}_X/ \amalg$ whose unit object is the terminal object.

To prove the claim, it suffices to show that if $C$ and $C'$ are cofibrant cylinder objects, and $X$ is a cofibrant object of $\mathcal{M}$, then $C \sqcup C'$ is a cylinder object. In fact, we have the diagram

$$\begin{array}{ccc}
X & \xrightarrow{j_0} & C' \\
\downarrow{j_1} & & \downarrow{}
\end{array}$$

$$\begin{array}{ccc}
X & \xrightarrow{j_0} & C \\
\downarrow{j_0} & & \downarrow{}
\end{array}$$

$$\begin{array}{ccc}
C & \xrightarrow{j_1} & C' \\
\downarrow{j'_0} & & \downarrow{}
\end{array}$$

$$\begin{array}{ccc}
C & \xrightarrow{j'_1} & C \\
\downarrow{j'_0} & & \downarrow{}
\end{array}$$

I claim that $j'_0$ is a weak equivalence. Since it is also a cofibration, its cofbase change $C \to C \sqcup_{j_1,X,j'_0} C'$ is a cofibration and weak equivalence. Since $j_0$ is also a weak equivalence, we can use 2-out-of-3 to show $q''$ is a weak equivalence.

If $X$ is cofibrant, then the inclusion $X \to X \amalg X$ of either factor is a cofibration, since it is a cofbase change of $0 \to X$. The claim follows. \hfill \square

We want left homotopy to be compatible with composition. The following is obvious: if $h: Y \to Y'$, then $f_0 \sim_\ell f_1$ implies $hf_0 \sim_\ell hf_1$. 

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34.7. **Proposition.** Suppose $Y$ fibrant, $k: X' \to X$ a map, and $f_0, f_1: X \to Y$. If $f_0 \sim^\ell f_1$ then $f_0k \sim^\ell f_1k$.

We need a lemma.

34.8. **Lemma.** If $f_0, f_1: X \to Y$ with $f_0 \sim^\ell f_1$ and $Y$ fibrant, then there exists a left homotopy which lives on a fibrant cylinder object.

**Proof.** Given the hypotheses and a choice of cylinder $X \amalg X \to C \xrightarrow{q} X$ witnessing the left-homotopy, choose a factorization $q = q'i$ where $i$ is a cofibration and $q'$ is a fibration and weak equivalence. By 2-out-of-3 $i$ is also a weak equivalence. In

$$
\begin{array}{c}
\xymatrix{ X \amalg X \ar[r]^{j_0} \ar[r]_{j_1} & C \ar[r]^{q'} & X \\
& \ar@{.>}[u]^{F} & \ar@{.>}[u]^{F'} }
\end{array}
$$

a dotted arrow exists because $Y$ is fibrant and $i$ is a cofibration and weak equivalence. □

**Proof of proposition.** Suppose $f_0, f_1: X \to Y$ such that $f_0 \sim^\ell f_1$ with $Y$ fibrant. By the lemma we can assume this lives on a fibrant cylinder object $X \amalg X \to C \xrightarrow{q} X$. Choose any cofibrant cylinder object $X' \amalg X' \xrightarrow{j'} C' \xrightarrow{q'} X$. We have a commutative diagram

$$
\begin{array}{c}
\xymatrix{ X' \amalg X' \ar[r]^{j'} & C' \ar[r]^{q'} & X' \\
& \ar@{.>}[u]^{|k|} & \ar@{.>}[u]^{K} & \ar@{.>}[u]^{k} \ar@{.>}[u]^{q} & \ar@{.>}[u]^{X'} }
\end{array}
$$

and since $j'$ is cofibration and $q$ fibration and we, a lift $K$ exists. Then $FK: C' \to Y$ is the desired left homotopy. □

Let $\mathcal{M}_f \subseteq \mathcal{M}$ denote the full subcategory of fibrant objects. Then by the above composition

$$
\text{Hom}(X', X'')_{\ell} \times \text{Hom}(X, X')_{\ell} \to \text{Hom}(X, X'')_{\ell}
$$

is well-defined, so we obtain a “homotopy category” $h\mathcal{M}_f$.

34.9. **Proposition.** If $X$ cofibrant and $g: Y' \to Y$ a trivial fibration, then $\text{Hom}(X, Y')_{\ell} \to \text{Hom}(X, Y)_{\ell}$ is a bijection.

**Proof.** For surjectivity, note that $\text{Hom}(X, Y') \to \text{Hom}(X, Y)$ is already surjective using lifting:

$$
\begin{array}{c}
\xymatrix{ Y' \ar[rd]^{g} & \\
X \ar[r]_{\sim} & Y 
}\end{array}
$$

For injectivity, recall that since $X$ is cofibrant left homotopy is an equivalence relation. Suppose we are given a cylinder object $X \amalg X \xrightarrow{j=(j_0, j_1)} C \xrightarrow{q} X$ and a map $K: C \to Y$ such that $Bj_0 = gf_0$ and $Bj_1 = gf_1$. WLOG we can assume $j$ is a cofibration. Now lift in

$$
\begin{array}{c}
\xymatrix{ X \amalg X \ar[r]^{j} & C \ar[r]^{j} & Y' \\
& \ar@{.>}[u]^{j_0} & \ar@{.>}[u]^{j_1} & \ar@{.>}[u]^{K} & \ar@{.>}[u]^{g} \ar@{.>}[u]^{Y'} }
\end{array}
$$

to get a left homotopy $f_0 \sim^\ell f_1$. □
Next, we note that we can dualize everything. There is a notion of path object for $X$, which is a factorization $X \xrightarrow{\iota} P \xrightarrow{p} X \times X$ such that $\iota$ is a weak equivalence. There is a corresponding notion of the right homotopy relation $\sim_r$ on $\text{Hom}(X,Y)$.

- If two maps are right homotopic, we can choose a right homotopy to a fibrant path object.
- We have $\text{Hom}(X,Y)_r$, the set of maps up to the relation generated by right homotopy.
- Right homotopy on $\text{Hom}(X,Y)$ is an equivalence relation if $Y$ fibrant.
- If $Y,Y',Y''$ are cofibrant, then composition

$$\text{Hom}(Y',Y''), \times \text{Hom}(Y,Y')_r \rightarrow \text{Hom}(Y,Y'')_r$$

is well-defined. We obtain a quotient category $h\mathcal{M}_c$ of the full subcategory $\mathcal{M}_c \subseteq \mathcal{M}$ of cofibrant objects.

- If $X' \rightarrow X$ is a cofibration and weak equivalence, and $Y$ is fibrant, then $\text{Hom}(X,Y)_r \rightarrow \text{Hom}(X',Y)_r$ is a bijection.

34.10. Proposition. Let $f_0,f_1: X \rightarrow Y$. If $X$ cofibrant, then $f_0 \sim_{\ell} f_1$ implies $f_0 \sim_r f_1$. If $Y$ fibrant, then $f_0 \sim_r f_1$ implies $f_0 \sim_{\ell} f_1$.

Proof. Suppose $X$ is cofibrant, and choose a cofibrant cylinder $X \sqcup X \xrightarrow{j_{01}} C \xrightarrow{q} X$ and map $H: C \rightarrow Y$ witnessing the left homotopy. Choose any fibrant path object $Y \xrightarrow{i_0} P \xrightarrow{p} Y \times Y$. Now consider

$$
\begin{array}{ccc}
X & \xrightarrow{i_0} & P \\
\downarrow{j_0} & \sim & \downarrow{p} \\
C & \xrightarrow{(f_0,H)} & Y \times Y
\end{array}
$$

This commutes, since $pi_0 = \Delta_Y f_0 = (f_0, f_0)$ and $(f_0q, H)j_0 = (f_0qj_0, Hj_0) = (f_0, f_0)$. Now choose a lift $K$. Then $Kj_1: X \rightarrow P$ is the desired right homotopy, since $pKj_1 = (f_0q,H)j_1 = (f_0qj_1, Hj_1) = (f_0, f_1)$.

Thus, on the full subcategory $\mathcal{M}_{cf}$ of bifibrant objects, left homotopy and right homotopy coincide, are both equivalence relations, and are compatible with composition. We call this relation "homotopy", and we get a quotient category $h\mathcal{M}_{cf}$.

34.11. Proposition. Let $f: X \rightarrow Y$ be a map between bifibrant objects. Then $f$ is a weak equivalence if and only if it projects to an isomorphism in $h\mathcal{M}_{cf}$.

Proof. Suppose $f$ is a weak equivalence. We want to show $[f]$ has an inverse in $h\mathcal{M}_{cf}$. Since any $f$ can be factored into a trivial cofibration followed by a trivial fibration, it suffices to prove the claim for these two classes.

So suppose $f$ is a trivial cofibration. Since $X$ is fibrant, lifting gives $r: Y \rightarrow X$ such that $rf = \text{id}_X$. Also, $f$ induces a bijection

$$f^*: \text{Hom}(Y,Y) \rightarrow \text{Hom}(X,Y)$$

since it is a trivial cofibration and $Y$ is fibrant. Then since $frf = f$ we get $fr \sim \text{id}_Y$, and the claim is proved. The proof for trivial fibrations is dual.

Next suppose $f: X \rightarrow Y$ is a homotopy equivalence. Factor $f = pi$ with $i$ a trivial cofibration and $p$ a fibration. Since $i$ is a homotopy equivalence so is $p$, and we only need to show that $p$ is a weak equivalence. Thus WLOG assume $f$ is a fibration and a homotopy equivalence.
Let \( g : Y \to X \) be a homotopy inverse for the fibration \( f, \) \( Y \rightrightarrows Y \) \( j = (j_0, j_1) \) \( C \xrightarrow{q} Y \) a cofibrant cylinder object, and \( H : C \to X \) witnessing \( fg \sim \text{id}_Y \). Lift in
\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow^{j_0} & \searrow^{H'} & \downarrow^{f} \\
C & \xrightarrow{q} & Y 
\end{array}
\]
to get \( H' : C \to X \). Define \( s := H' j_1 \), whence \( fs = f H' j_1 = q j_1 = \text{id}_Y \). Note also that \( H' j_0 = g \), so \( H' \) is a homotopy \( g \sim s \). We have
\[
sf \sim gf \sim \text{id}_X.
\]
We use a fact: if \( a_0 \) is a weak equivalence and \( a_0 \sim \ell a_1 \), then \( b \) is also a weak equivalence. (Proof: let \( H : C \to B \) be a left homotopy such that \( H j_0 = a \) and \( H j_1 = b \), where \( a_0, a_1 : A \to B \). If \( a_0 \) is a weak equiv so is \( H \) by 2-of-3, whence so is \( b \) by 2-of-3.)

Thus \( sf : X \to X \) is a weak equivalence. Now note:
\[
\begin{array}{ccc}
X & \xrightarrow{\text{id}_X} & X \\
\downarrow^{f} & \downarrow^{sf} & \downarrow^{f} \\
Y & \xrightarrow{s} X & \xrightarrow{f} Y 
\end{array}
\]
commutes, since \( fsf = f \). The result follows once we prove the following. \( \square \)

34.12. Proposition. In a model category, weak equivalences are closed under retracts.

Now we can construct the homotopy category. For each object \( X \) of \( \mathcal{M} \), make a choice of
- trivial cofibration \( i_X : X \to RX \) to a fibrant object, and
- trivial fibration \( p_Y : QY \to Y \) from a cofibrant object.

Furthermore, let us suppose \( i_X = \text{id}_X \) if \( X \) is already fibrant, and \( p_Y = \text{id}_Y \) if \( Y \) is already fibrant. Note that if \( X \) is already cofibrant, then so is \( RX \), and if \( Y \) is already fibrant so is \( QY \).

We define a category \( \mathcal{M}_{W} \) and functor \( \gamma : \mathcal{M} \to \mathcal{M}_{W} \) as follows.
- The objects of \( \mathcal{M}_{W} \) are in bijective correspondence with the objects of \( \mathcal{M} \).
- Set
\[
\text{Hom}_{\mathcal{M}_{W}}(\gamma X, \gamma Y) := \text{Hom}_{\mathcal{M}_{j}}(RX, RQY)_{h}.
\]
- Given \( f : X \to Y \) in \( \mathcal{M} \), choose any maps \( f' \) and \( f'' \) making the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{p_X} & QX & \xrightarrow{i_QX} & RX \\
\downarrow^{f} & \downarrow^{f'} & \downarrow^{f''} & \downarrow & \downarrow \\
Y & \xleftarrow{p_Y} & QY & \xrightarrow{i_QY} & RQY 
\end{array}
\]
commute. Then set \( \gamma(f) := [f''] \).

Note that since composition is compatible with homotopy for bifibrant objects, \( \mathcal{M}_{W} \) really is a category. We need to make sure that \( \gamma \) is well-defined.

34.13. Lemma. Given \( f : X \to Y \), maps \( f' \) and \( f'' \) exist, and their choice is unique up to homotopy.

Proof. We have bijections \( \text{Hom}(QX, QY)_{h} \xrightarrow{\sim} \text{Hom}(QX, Y)_{h} \) and \( \text{Hom}(RQX, RQY)_{h} \xrightarrow{\sim} \text{Hom}(QX, RQY)_{h} \) since \( p_Y \) is a trivial fibration, \( i_QX \) is a trivial cofibration, and all objects are bifibrant. \( \square \)
Since the construction of $\gamma(f)$ doesn’t depend on the choices, it is easy to show that $\gamma(gf) = \gamma(g)\gamma(f)$.

34.14. **Theorem.** $\gamma: \mathcal{M} \to \mathcal{M}_W$ is a category of fractions with respect to $W$.

**Proof.** First we need to show that $\gamma$ takes weak equivalences to isomorphisms.

If $f$ is a weak equivalence, then so are $f'$ and $f''$. Thus it suffices to show that $f''$ is a homotopy equivalence, which was done above.

Next, suppose $F: \mathcal{M} \to \mathcal{A}$ is a functor taking weak equivalences to isomorphisms. Define $\overline{F}: \mathcal{M}_W$ by

$$\overline{F}(\gamma X) := F(X),$$

and for a map $\gamma X \to \gamma Y$ represented by some $f'': RQX \to RQY$, set

$$\overline{F}([f'']) := F(p_Y)F(i_{QY})^{-1}F(f'')F(i_{QX})F(p_X)^{-1}.$$

We note that this does not depend on the homotopy class of $f''$: we already noted that if $f_0, f_1$ are left homotopic, and $F$ such a functor, then $F(f_0) = F(f_1)$.

It is then straightforward to check that $\overline{F}$ is a functor.

We check that $\overline{F}(\gamma(f)) = F(f)$, using the commutativity of the diagram.

To prove uniqueness, we note that since $\gamma$ is a functor taking weak equivalences to isomorphisms, and that $\text{Hom}(RQX, RQY) \to \text{Hom}(RQX, RQY)_h$ is surjective, then if $[f'']: \gamma X \to \gamma Y$ is a map represented by some $f'': RQX \to RQY$, we have

$$[f''] = (p_Y)\gamma(i_{QY})^{-1}\gamma(f'')\gamma(i_{QX})\gamma(p_X)^{-1}.$$

To see this, note that $QQX = QX$ and $p_{QQX} = id$ and by the diagram

$$\begin{array}{ccc}
QQX & \xrightarrow{i_{QQX}} & RQQX \\
p_{QQX} & & id \\
X & \xleftarrow{p_X} & QX & \xrightarrow{i_{QX}} & RX
\end{array}$$

we have $\gamma(p_X) = [id]$. Likewise

$$\begin{array}{ccc}
QQX & \xrightarrow{i_{QQX}} & RQQX \\
i_{QQX} & & id \\
RQQX & \xleftarrow{id} & QRX & \xrightarrow{id} & RQRX
\end{array}$$

so $\gamma(i_{QX}) = [id].$ \hfill $\square$

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