We need the following notation: $\alpha^k: I^{d-1} \to I^d$ denotes the inclusion of a face of the $d$-cube, defined by

$$\alpha^k(x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{k-1}, \epsilon, x_k, \ldots, x_{d-1}), \quad k = 1, \ldots, d-1, \; \epsilon = 0, 1.$$ 

Last time I defined for each $d$ a singular chain $F^d \in C_d(I^d)$, by the formula

$$F^d = \sum_{\sigma \in \Sigma_d} (-1)^{\text{sgn} \sigma} F_{\sigma},$$

where the sum is over elements of the symmetric group on $d = \{1, \ldots, d\}$, and $F_{\sigma}: \Delta^d \to I^d$ is defined by

$$F_{\sigma} = [v_\{\}, v_{\sigma(1)}, v_{\sigma(1), \sigma(2)}], \ldots, v_d],$$

where for $S \subseteq d$ we write $v_S = (\chi_S(1), \ldots, \chi_S(d)) \in I^d$, where $\chi_S(i) \in \{0, 1\}$ according to whether $i \in S$.

**Lemma 0.1.** In the singular chain complex $C_\bullet(I^d)$ we have

$$\partial F^d = \sum_{k=1}^{d} (-1)^{k-1}((\alpha^k_1)_# F^{d-1} - (\alpha^k_0)_# F^{d-1}).$$

The projection of $F^d$ to $C_d(I^d, \partial I^d)$ is a cycle, representing a generator of $H_d(I^d, \partial I^d)$.

**Proof.** The first claim is an explicit calculation which I will not write out in full. (I suggest writing it down in low dimensions.) To understand it, note that $\partial F_{\sigma}$ will be an alternating sum of codimension 1 simplices of the form

$$[v_{S_0}, \ldots, v_{S_{d-1}}], \quad S_0 \subset \cdots \subset S_{d-1} \subseteq d.$$ 

There are three kinds of codimension 1 simplices of this type, satisfying one of the following possibilities:

1. $S_0 = \{k\} \neq \{\}$ for some $k$.
2. $S_0 = d \setminus \{k\} \neq \{d\}$ for some $k$.
3. The rest, with $S_0 = \{\}$ and $S_{d-1} = d$.

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The first two types lie on the boundary: type (1) is contained in $\alpha^k(I^{d-1})$ while type (2) is contained in $\alpha^k(I^{d-1})$. For type (3), there will be a unique $k = 1, \ldots, d - 2$ such that $S_k \setminus S_{k-1}$ has two elements.

The point is that each simplex of type (3) will appear in the boundary of exactly two $F_\sigma$s, which will correspond to permutations which differ by a single transposition, and so will cancel in $\partial F^d$. Each simplex of type (1) or (2) appears in the boundary of a unique $F_\sigma$, and so will appear in $\partial F^d$.

For the second claim, note that there is an evident $\Delta$-complex structure for $I_d$, with simplices of the form $[v_{S_0}, \ldots, v_{S_k}]$ with $S_0 \subset \cdots \subset S_k \subset d$, and that $F^d \in \Delta_d(I^d) \subseteq C_d(I^d)$. Furthermore, since $\Delta_{d+1}(I^d) = 0$ and $H_d(I^d, I^{d-1}) \approx \mathbb{Z}$, the group of cycles in $\Delta_d(I^d, I^{d-1})$ must be isomorphic to $\mathbb{Z}$. As $F^d$ clearly projects to a cycle in this group and is not an integer multiple of any element, it must be a generator.

Given this, we can prove the claimed description of the cellular chain complex $C_{\bullet}^{CW}(I^d)$.

In $I^m \times I^n$, if we set $u^m = u_1 \otimes \cdots \otimes u_m$ and $u^n = u_{m+1} \otimes \cdots \otimes u_{m+n}$, we see that

$$\partial(u^m \otimes u^n) = \partial(u^m) \otimes u^n + (-1)^m u^m \otimes \partial(u^n).$$

Now consider the homeomorphism $f: D^m \to I^m$. If we give $D^m$ the CW-structure so that the $(m-1)$-skeleton is the boundary sphere $S^{m-1}$, then $f$ is cellular. We know that $f_\#(e^m) = u_1 \otimes \cdots \otimes u_m$. (Up to a sign; choose the homeomorphism so that the sign is positive.) Since

$$f_\#(\partial(e^m)) = \partial(f_\#(e^m)) = \sum (-1)^k u_1 \otimes \cdots \otimes (1_k - 0_k) \otimes \cdots u_m.$$

Thus, considering $f \times g: D^m \times D^n \to I^m \times I^n$, we get $\partial(e^m \otimes e^n) = \partial(e^m) \otimes e^n + (-1)^m e^m \otimes \partial(e^n)$.

To calculate the boundary map of a cell $e_{\alpha}$ in $C_{m+n}(X \times Y)$, note that the case when one of the cells in 0-dimensional is easy. So we can assume $m, n > 0$.

Consider a characteristic map $\Phi_{\alpha}: D^m \to X$. Unfortunately, this map may not be a cellular map, because there is no condition on the image of the basepoint of $S^{m-1}$. However, we may homotopy $\Phi_{\alpha}$ to a cellular map $\tilde{\Phi}_{\alpha}: D^m \to X$ using the homotopy extension property (first choose a homotopy at $* \in S^{m-1}$ which goes to a vertex in $X_0$, then extend to a homotopy of maps $S^{m-1} \to X_{m-1}$, then to a homotopy $D^m \to X_m$). We will still have $(\tilde{\Phi}_{\alpha})_\#(e^m) = e_\alpha \in C_{m}^{CW}(X)$. In other words, every basis element in $C_{m}^{CW}(X)$ is the image of the generator of $C_{m}^{CW}(D^m)$ under a cellular map.

Now we have

$$\partial(e_\alpha \otimes e_\beta) = \partial((\tilde{\Phi}_{\alpha} \times \tilde{\Phi}_{\beta})_\#(e^m \otimes e^n))$$

$$= (\tilde{\Phi}_{\alpha} \times \tilde{\Phi}_{\beta})_\#(\partial(e^m) \otimes e^n)$$

$$= (\tilde{\Phi}_{\alpha})_\#(\partial(e^m)) \otimes (\tilde{\Phi}_{\beta})_\#(e^n) + (\tilde{\Phi}_{\alpha})_\#(e^m) \otimes (\tilde{\Phi}_{\beta})_\#(\partial(e^n))$$

$$= \partial(\tilde{\Phi}_{\alpha})_\#(e^m) \otimes (\tilde{\Phi}_{\beta})_\#(e^n) + (\tilde{\Phi}_{\alpha})_\#(e^m) \otimes \partial(\tilde{\Phi}_{\beta})_\#(e^n)$$

$$= \partial(e_\alpha) \otimes e_\beta + (-1)^m e_\alpha \otimes \partial(e_\beta).$$

**Homology of tensor products of chain complexes.** Let $C_\bullet$ and $D_\bullet$ be chain complexes of modules over a commutative ring $R$, and assume that $C_n = D_n = 0$ for $n < 0.$
There is a map
\[ \mu : H_p(C) \otimes_R H_q(D) \to H_{p+q}(C \otimes_R D), \]
defined by sending classes \([x] \otimes [y] \mapsto [x \otimes y] \). We’ll call this the K"unneth map.

**Exercise.** Check that \( \mu \) is associative, in the sense that the two ways of using \( \mu \) to construct a map
\[ H_p(C) \otimes_R H_q(D) \otimes_R H_r(E) \to H_{p+q+r}(C \otimes_R D \otimes_R E) \]
are the same.

**Exercise.** Check that \( \mu \) is graded commutative. Note that to get an isomorphism \( C \otimes R D \to D \otimes R C \), we need to use the map \( x \otimes y \mapsto (-1)^{|x||y|} y \otimes x \).

**Exercise.** Check that the K"unneth map is natural with respect to maps of chain complexes in either variable.

**Tensor product of modules. Example.** If \( R \) is a commutative ring, the tensor product \( M \otimes_R N \) of \( R \)-modules \( M \) and \( N \) is an \( R \)-module, generated by symbols \( m \otimes n \) for \( m \in M \) and \( n \in N \), subject to the relations:
\[
(m + m') \otimes n = m \otimes n + m' \otimes n, \quad m \otimes (n + n') = m \otimes n + m \otimes n', \quad mr \otimes n = m \otimes rn.
\]
The \( R \)-module structure is given by \( r(m \otimes n) = rm \otimes n \). If \( R = \mathbb{Z} \), this is just the tensor product of abelian groups we already defined.

**K"unneth theorem.** The following theorem is the algebraic component we need to (1) describe homology with arbitrary coefficients, and (2) compute the homology of a product.

**Proposition 0.2.** Let \( R \) be a PID, and let \( C \) and \( D \) be chain complexes of \( R \)-modules. If \( C \) is a complex of free \( R \)-modules, then there is a short exact sequence of chain complexes
\[
0 \to \bigoplus_i H_i(C) \otimes_R H_{n-i}(D) \xrightarrow{\mu} H_n(C \otimes_R D) \to \bigoplus_i \text{Tor}_i^R(H_iC, H_{n-i-1}D) \to 0
\]
which is natural with respect to chain maps in the variables \( C \) and \( D \). Furthermore, the sequence admits a non-natural splitting.

To prove this, we need to define \( \text{Tor} \).

**Brief introduction to homological algebra.** We need some definitions. Much what I say here applies in the general context of abelian categories, but I don’t want to work in that generality. (It makes things tricky and we don’t need it.) I’ll work in the context of modules over a commutative ring \( R \); this includes the case of abelian groups, which are modules over \( \mathbb{Z} \), as well as vector spaces over a field. (It’s not necessary to require \( R \) to be commutative, but facts about tensor products will be easier in this case.)

Given a functor \( F : \text{Mod}_R \to \text{Mod}_S \) we say it is **additive** if \( F : \text{Hom}_R(M, N) \to \text{Hom}_S(F(M), F(N)) \) is a homomorphism of abelian groups. (There are many non-additive functors, such as \( F(M) = M \otimes M \).)

**Example.** Given an \( R \)-module \( N \), the functor \( F(M) \overset{\text{def}}{=} M \otimes_R N \) is an additive functor \( \text{Mod}_R \to \text{Mod}_R \).
Fact. Every additive functor preserves finite direct sums, and takes the 0 group to the 0 group. (Idea of proof: a direct sum splitting of $M$ is given by an idempotent $e: M \to M$.)

Projective resolutions. An $R$-module $P$ is projective if for every surjection $f: M \to N$ and every map $g: P \to N$, there exists map $\tilde{g}: P \to M$ such that $f \tilde{g} = g$.

Exercise. Over an arbitrary ring, projectives are precisely the summands of free modules.

In particular, free modules are always projective.

The idea of defining derived functors is the following. If $F$ is an additive functor on $R$-modules, and $P_\bullet$ is a chain complex of projective $R$-modules with $P_q = 0$ for $q < 0$, and with $H_q(P_\bullet) = 0$ for $q \neq 0$, then $H_q(F(M))$ turns out to depend only on the group $M = H_0P_\bullet$.

The resulting groups will be called $L_qF(M)$, and the construction $M \mapsto L_qF(M)$ is called the $q$th left derived functor of $F$. We will be interested in the left derived functors of $- \otimes_R Q$ (which are $\text{Tor}_q^R(\cdot, Q)$) and the left derived functors of $\text{Hom}_R(Q, \cdot)$ (which are $\text{Ext}^q_R(Q, \cdot)$).

Convention. Any module $M$ will be regarded tautologically as a chain complex of modules $M = M_\bullet$, with $M_0 = M$ and $M_q = 0$ if $q \neq 0$.

A projective resolution of an $R$-module $M$ is a pair $(P_\bullet, \epsilon)$, consisting of a chain complex $P_\bullet$ of $R$-modules such that $P_0 = 0$ if $q < 0$, and $P_q$ is projective for all $q$, and a chain map $\epsilon: P_\bullet \to M$ of complexes of $R$-modules such that $H_q(\epsilon): H_q(P_\bullet) \to H_q(M_\bullet)$ is an isomorphism for all $q$.

That is, $H_q(P_\bullet) = 0$ for $q \neq 0$, and $\epsilon$ induces an isomorphism $H_0(P_\bullet) \approx M$. We can summarize this by saying that the sequence

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. (The map $\epsilon$ is sometimes called an augmentation.)

Given a map $f: M \to N$ of modules, and projective resolutions $P_\bullet \to M$ and $Q_\bullet \to N$, a map of projective resolutions covering $f$ is a map $g: P_\bullet \to Q_\bullet$ of chain complexes such that $H_0(g) \approx f$.

The following omnibus proposition explains why homological algebra works.

Proposition 0.3. Let $R$ be a commutative ring.

1. Every $R$-module admits a projective resolution.
2. Given projective resolutions $P_\bullet \to M$ and $Q_\bullet \to N$ of $R$-modules, every homomorphism $M \to N$ is covered by a map $P_\bullet \to Q_\bullet$ of resolutions.
3. Any two maps of resolutions covering a given homomorphism $M \to N$ are chain homotopic.
4. Given a short exact sequence $0 \to M' \to M \to M'' \to 0$ of $R$-modules, there exists a short exact sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ of projective resolutions covering it.

Proof. This is really an exercise. Claims (1), (2), (3) are proved in a straightforward way, using repeatedly the definition of projective module. The proof of (1) requires the fact that for any module $M$, there exists a free module $F$ and a surjection $F \to M$.

To prove (3), suppose that $g_1, g_2: P_\bullet \to Q_\bullet$ are two chain maps which cover the same homomorphism $M \to N$. Then $g = g_1 - g_2$ covers the 0 homomorphism. Thus, to prove (3) it suffices to produce a chain homotopy $H$ between $g$ and $0: P_\bullet \to Q_\bullet$, since then $\partial H + H\partial = g = g_1 - g_2$.

Thus, for each $q \geq 0$, we must find $H_q: P_q \to Q_{q+1}$ such that $\partial Q H_q + H_{q-1} \partial P = g_q$; when $q = 0$, this condition reduces to $\partial Q H_0 = g_0$. In degree 0, we have that $g_0(P_0) \subseteq
Ker(ε: Q₀ → N) = Im(∂: Q₁ → Q₀), and thus projectivity of P₀ provides a lift H₀: P₀ → Q₁ of P₀ → Im(∂: Q₁ → Q₀).

When q ≥ 1, assume we have constructed Hᵢ₋₁ such that ∂Hᵢ₋₁ + Hᵢ₋₂∂ = gᵢ₋₁. We need to produce Hᵢ: Pᵢ → Qᵢ₊₁ such that ∂Hᵢ = gᵢ − Hᵢ₋₁∂; thus, it suffices to verify that

\[ \text{Im}(gᵢ − Hᵢ₋₁∂) ⊆ \text{Im}(\partial: Qᵢ₊₁ → Qᵦ) = \text{Ker}(\partial: Qᵦ → Qᵦ₋₁), \]

which is clear since

\[ \partial(gᵢ − Hᵢ₋₁∂) = ∂gᵢ − ∂Hᵢ₋₁∂ = ∂gᵢ − (gᵢ₋₁ − Hᵢ₋₂∂)∂ = ∂gᵢ − gᵢ₋₁∂ = 0 \]

since g is a chain map.

Part (4) is the “horseshoe lemma”. First choose any two resolutions P'ᵣ → M' and P''ᵣ → M''. Then we define a resolution Pᵣ → M as follows: let P₀ = P₀' ⊕ P₀'', and let Pᵣ' → Pᵣ and Pᵣ → Pᵣ'' be the evident inclusion and projection maps. It remains to produce a boundary map for Pᵣ and an augmentation P₀ → M, so that it is a resolution of M and so that the maps of complexes are well-defined. Let ε: P₀ → M be a map with ε(P₀') coinciding with P₀' → M' → M, and choose εP₀'' to be a lift of P₀'' → M''; that this is surjective follows from the 5-lemma. Then inductively construct the boundary map in Pᵣ, making sure that Pᵣ' → Pᵣ and Pᵣ → Pᵣ'' are maps of complexes. □

Exercise. If Pᵣ → M and Qᵣ → M are projective resolutions, then there exists a chain homotopy equivalence Pᵣ → Qᵣ covering the identity map of M.

Left derived functors. Homological algebra gives a way of associating to an additive functor F: Modᵣ → Modₛ a collection of functors LᵦF: Modᵣ → Modₛ for q ≥ 0, called left derived functors.

We define

\[ (LᵦF)(M) \overset{\text{def}}{=} Hᵦ(F(Pᵣ)), \]

where Pᵣ → M is any chosen projective resolution. We see that it is a functor, by covering maps by maps of resolutions.

Key observation. If Pᵣ' → M is any other projective resolution, any chain homotopy equivalence covering the identity of M gives an chosen isomorphism Hᵦ(FPᵣ) ≃ Hᵦ(FPᵣ'), and any two such chain homotopy equivalences give the same isomorphism. If Pᵣ'' → M is another projective resolution, then the triangle of isomorphisms relating Hᵦ(FPᵣ), Hᵦ(FPᵣ'), Hᵦ(FPᵣ'') commutes.

Thus, even though there is an “indeterminacy” in the construction of the derived functor (the choice of projective resolution), the resulting LᵦF(M) is defined up to unique isomorphism.

There is one case where the derived functors are trivial. Recall that an additive functor is exact if it takes short exact sequences to short exact sequences.

Lemma 0.4. If F is exact and Cᵣ is a chain complex, then HᵦF(Cᵣ) ≃ F(HᵦCᵣ).

Proof. Instructive exercise: break up chain complex into a sequence of short exact sequences 0 → Zᵦ → Cᵦ → Bᵦ₋₁ → 0, and express homology in terms of short exact sequences 0 → Bᵦ → Zᵦ → HᵦCᵦ → 0. □
Corollary 0.5. If $F$ is exact, then $L_0F = F$, and $L_qF \equiv 0$ if $q \geq 1$.

Given a short exact sequence $0 \to M' \to M \to M'' \to 0$, we obtain a map $\delta : (L_qF)(M'') \to (L_{q-1}F)(M')$ fitting into a long exact sequence, by applying the horshoe lemma. In particular, the terminal edge of the sequence looks like

$$(L_1F)(M) \to (L_1F)(M'') \to (L_0F)(M') \to (L_0F)(M) \to (L_0F)(M'') \to 0.$$  

In particular, the functor $L_0F$ is right exact.

There is a natural map $\alpha_M : L_0F(M) \to F(M)$, which comes from the sequence

$$F(P_1) \to F(P_0) \xrightarrow{F(\delta)} F(M)$$

obtained by applying $F$ to a projective resolution $P_\bullet \to M$.

Proposition 0.6.

1. If $M$ is projective, then $\alpha_M : (L_0F)(M) \to F(M)$ is an isomorphism, and $(L_qF)(M) \approx 0$ for $q > 0$.
2. If $F$ is right exact, then $\alpha$ is an isomorphism for all $M$.

Proof. For (1), note that $M$ (viewed as a chain complex) is a projective resolution of itself. For (2), use the 5-lemma. \qed

Example. For a module $N$, $F(M) = M \otimes_R N$ is right exact; thus $L_0F \approx F$.

Example. For a module $N$, $F(M) = \text{Hom}_R(N, M)$ is not right exact if $N$ is not projective. In particular, $L_0F \to F$ will not be an isomorphism for all $M$. Exercise. Describe $L_0F$ in the case $R = N = \mathbb{Z}/2$.

Homological algebra for PIDs. We need the following fact.

Proposition 0.7. If $R$ is a PID, then every submodule of a free module is free.

Proof. This is proved by a Zorn’s lemma argument. Here is a sketch.

Let $F = RX$ be a free module with basis $X$, and let $M \subseteq F$ be a submodule. Given $S \subseteq X$, let $M_S = M \cap RS$, the submodule of $M$ contained in the free submodule generated by $S$.

Let $\mathcal{F}$ be the set of pairs $(S, f)$, where $S \subseteq X$ and $f : S \to M_S \subseteq M$ is a function which maps $S$ injectively to a free basis of $M_S$. (I.e., the induced homomorphism $RS \to M_S$ is an isomorphism.) The set $\mathcal{F}$ is clearly partially ordered under inclusion, and is non-empty since we can take $(\emptyset, \emptyset) \to M_S = 0$.

Show that for any chain $\{(S_i, f_i)\}$ in $\mathcal{F}$, the pair $(S, f)$ defined by $S = \bigcup S_i$ and $f|_{S_i} = f_i$ is in $\mathcal{F}$. (Checking that $f(S)$ is a basis involves arguments which only require finitely many elements at a time.) Thus, Zorn’s lemma provides at least one maximal element $(S, f)$.

If $M_S \neq M$, derive a contradiction by picking $t \in X \setminus S$ such that $M_S \neq M_T$ (where $T = S \cup \{t\}$). Then show that $M_T/M_S \to RT/RS \approx R$ is injective. Since $R$ is a PID, the image is a free module of rank 1. (This is where you use the PID hypothesis.) Use this to construct $(T, g) \in \mathcal{F}$ with $(S, f) < (T, g)$, a contradiction of maximality. \qed

Recall a special case of this: if $R$ is a field, then every $R$-module is free.

Proposition 0.8. If $R$ is a field, then $L_qF = 0$ for all $q > 0$. If $R$ is a PID, then $L_qF = 0$ for all $q > 1$. 

Proof. If $R$ is a field, then every module is projective, so is its own projective resolution.

If $R$ is a PID, then every submodule of a free module is free (a Zorn’s lemma argument, and the fact that it is true in the finitely generated case). Thus, any choice of surjection $P_0 \to M$ from a free module gives a projective resolution $0 \to P_1 \to P_0 \to M \to 0$. □

Remark 0.9. In general, there may be no bound on a projective resolution. For instance, consider $R = \mathbb{Z}/p^2$, where $p$ is a prime, and $M = \mathbb{Z}/p$. Then every projective $R$-resolution of $M$ is infinite!