LECTURE NOTES (WEEK 2), MATH 526 (FALL 2012)

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W 5 Sep

Cellular chains on products of CW complexes. Review the cellular chain complex:

\[ C^q_{\text{CW}} X = H_q(X_q, X_{q-1}) \]

with boundary map given by

\[ H_q(X_q, X_{q-1}) \rightarrow H_{q-1}(X_{q-1}, X_{q-2}) \]

Write \( e_\alpha \in C^q_{\text{CW}} X \) for the generator corresponding to the cell \( \Phi_\alpha : D^q \rightarrow X \) with \( q \geq 2 \), and recall the degree formula

\[ \partial(e_\alpha) = \sum d_{\alpha \beta} e_\beta, \]

where for \( \Phi_\beta : D^{q-1} \rightarrow X \), we have that \( d_{\alpha \beta} \) is the degree of the composite

\[ S^{q-1} \xrightarrow{\partial_\alpha} X_{q-1} \rightarrow X_{q-1}/X_{q-2} \rightarrow D^{q-1}/S^{q-2} = S^{q-1}. \]

(A different but easy formula works when \( q = 1 \).)

Review cellular chain complexes for \( S^n \) and \( \mathbb{RP}^n \).

Note that \( D^m \times D^n \approx D^{m+n} \). This is more apparent if we choose homeomorphisms \( I^k \approx D^k \) for all \( k \), say by the evident contraction map.

Let \( X \) and \( Y \) be CW-complexes, with characteristic maps \( \{ \Phi_\alpha^X : D^m \rightarrow X \} \) and \( \{ \Phi_\beta^Y : D^n \rightarrow Y \} \). Then we would like to say that \( X \times Y \) is a CW-complex, with characteristic maps \( \{ \Phi_{\alpha \beta} = \Phi_\alpha^X \times \Phi_\beta^Y : D^{m+n} \rightarrow X \times Y \} \). The \( k \)-skeleton of \( X \times Y \) is \( (X \times Y)_k = \bigcup_{i+j=k} X_i \times Y_j \).

This assumes that we have chosen some fixed identification \( D^{m+n} \approx D^m \times D^n \).

If either \( X \) or \( Y \) is a finite dimensional CW complex, this is correct: \( X \times Y \), given the product topology, will be a CW complex with this cell structure. In general, this fails; the product topology may not coincide with the "weak topology" coming from the cell structure. Let’s assume that \( X \times Y \) is given the weak topology, so it is a CW complex.

Let \( C_* \) and \( D_* \) be chain complexes of abelian groups. Their tensor product \( (C \otimes D)_* \) is the chain complex with groups

\[ (C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j, \]

and with boundary map given by

\[ \partial(x \otimes y) = \partial(x) \otimes y + (-1)^{|x|} x \otimes \partial(y). \]

**Proposition 0.1.** If \( X \) and \( Y \) are CW complexes, then there is an isomorphism \( C^\text{CW}_*(X) \otimes C^\text{CW}_*(Y) \approx C^\text{CW}_*(X \times Y) \). With respect to the usual bases on CW-chains given by cells, this isomorphism sends \( e^p_\alpha \otimes e^q_\beta \mapsto e^{p+q}_{\alpha \beta} \).

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Part of the proof is easy. There is an evident isomorphism of abelian groups
\[ \bigoplus_{i+j=n} C^\text{CW}_i X \otimes C^\text{CW}_j Y \to C^\text{CW}_n (X \times Y) \]
which sends \( e_\alpha \otimes e_\beta \mapsto e_{\alpha,\beta} \). We also need to check that the boundary maps coincide. I defer the proof.

**Example.** \( X = \mathbb{R}P^2 \times \mathbb{R}P^2 \). \( C^\bullet_\text{CW}(X) \) is given by
\[
\begin{align*}
e^2 \otimes e^2 &\mapsto 2(e^1 \otimes e^2) + 2(e^2 \otimes e^1), \quad e^2 \otimes e^0 \mapsto 2(e^1 \otimes e^0), \\
e^2 \otimes e^1 &\mapsto 2(e^1 \otimes e^1), \quad e^1 \otimes e^1 \mapsto 0, \\
e^1 \otimes e^2 &\mapsto -2(e^1 \otimes e^1), \quad e^0 \otimes e^2 \mapsto 2(e^0 \otimes e^1), \\
e^1 \otimes e^0 &\mapsto 0, \quad e^0 \otimes e^1 \mapsto 0.
\end{align*}
\]
Thus \( H_0 X \approx \mathbb{Z}, \ H_1 X \approx \mathbb{Z}/2[e^0] \otimes \mathbb{Z}/2[e^1], \ \ H_2 X \approx \mathbb{Z}/2[e^1 \otimes e^1], \ H_3 X \approx \mathbb{Z}/2[e^2 \otimes e^1 + e^1 \otimes e^2], \ H_4 X \approx 0 \). Note that the non-zero class in dimension 3 is not in any sense a “product” of homology classes of \( \mathbb{R}P^2 \).

Calculate \( C^\text{CW}_\bullet(X; \mathbb{F}_2) \).

The above tells us that \( H_\ast(X \times Y) \) is isomorphic to \( H_\ast(C^\bullet_\text{CW}(X) \otimes C^\bullet_\text{CW}(Y)) \). To compute the latter, we want the “algebraic Künneth theorem”.

**Cellular chains of a product.** All we have left to do is compute the formula \( \partial(e_\alpha \otimes e_\beta) = \partial(e_\alpha) \otimes e_\beta + (-1)^|\alpha|e_\alpha \otimes \partial(e_\beta) \) in \( C^\bullet_\text{CW}(X \times Y) \).

If \( (X, x_0) \) and \( (Y, y_0) \) are pointed spaces, we define their **smash product** by \( X \wedge Y \overset{def}{=} X \times Y/(X \times \{y_0\}) \cup (\{x_0\} \times Y) \). It is useful to note that
\[
(X/A) \wedge (Y/B) \approx (X \times Y)/(X \times B \cup A \times Y).
\]
In particular, \( S^m \wedge S^n \approx S^{m+n} \).

The space \( S^1 \wedge X \) is called the **reduced suspension** of \( X \). Compare with the unreduced suspension \( S(X) = C(X)/X \). There is a map \( S(X) \to S^1 \wedge X \), which collapses \( S(\{x_0\}) \) to a point. This map is an isomorphism in homology. We have shown previously that there is an isomorphism \( H_{q+1} S(X) \to H_q X \) in homology which is natural in \( X \), so therefore there is a natural isomorphism \( \tilde{H}_{q+1} S^1 \wedge X \to \tilde{H}_q X \) for good pairs \( (X, x_0) \).

We immediately conclude that if \( f: S^n \to S^n \) is a basepoint preserving map with degree \( d \), then so does \( S^1 \wedge f: S^{n+1} \to S^{n+1} \). Using this, we get

**Proposition 0.2.** If \( f: S^m \to S^m \) and \( g: S^n \to S^n \) are basepoint preserving maps, then \( \deg(f \wedge g) = \deg(f) \deg(g) \).

Recall that a cellular map \( f: X \to Y \) between CW-complexes is a map such that \( f(X_n) \subseteq Y_n \) for all \( n \), and that cellular maps induce \( f_\#: C^\bullet_\text{CW}(X) \to C^\bullet_\text{CW}(Y) \). Furthermore,
\[
f_\#(e_\alpha) = \sum f_{\alpha \beta} e_\beta
\]
where
\[
f_{\alpha \beta} = \deg(S^n = D^n/S^{n-1} \overset{\Phi_0}{\longrightarrow} X_n/X_{n-1} \overset{\psi_\beta}{\longrightarrow} S^n).
\]
Now suppose we have two cellular maps \( f: X \to X' \) and \( g: Y \to Y' \), giving a cellular map \( f \times g: X \times X' \to Y \times Y' \). It is straightforward to show that
\[
(f \times g)_{\#}(e_{\alpha,\beta}) = \sum f_{\alpha \alpha'} g_{\beta \beta'} e_{\alpha',\beta'}.
\]
That is, our isomorphism of groups \( C^*_{\text{CW}}(X \times Y) \approx C^*_{\text{CW}}(X) \otimes C^*_{\text{CW}}(Y) \) is natural with respect to cellular maps in \( X \) and \( Y \).

We can give \( I \) the CW-structure with 0-cells \((0)\) and \((1)\), and 1-cell \( u \). The boundary map in the cellular chain complex is given by \( \partial(u) = (1) - (0) \).

This gives \( I^d \) the product CW-structure. Thus, we can describe the cellular chain complex \( C^*_{\text{CW}}(I^d) \) in terms of: one \( d \)-cell \( u_1 \otimes \cdots \otimes u_d \), \( d-1 \)-cells \( u_1 \otimes \cdots \otimes 0_k \otimes \cdots \otimes u_d \), \( u_1 \otimes \cdots \otimes 1_k \otimes \cdots \otimes u_d \), etc.

**Claim.** \( \partial(u_1 \otimes \cdots \otimes u_d) = \sum (-1)^k u_1 \otimes \cdots \otimes 1_k \otimes \cdots \otimes u_d - \sum (-1)^k u_1 \otimes \cdots \otimes 0_k \otimes \cdots \otimes u_d \).

Hatcher proves the claim using the local degree formula, and “symmetry”: the reflection on a \( I \) factor induces a degree \(-1\) map of \((I^m, \partial I^m)\), as does switching two consecutive \( I \) factors.

It is also possible (and perhaps easier) to prove this by writing down an explicit singular chain in \( C_d(I^d, \partial I^d) \) which represents \( u_1 \otimes \cdots \otimes u_m \in H_d(I^d, \partial I^d) = C_d^{\text{CW}}(I^d) \), and computing its boundary explicitly.

To be continued.