Due Friday, October 19.

1. Hatcher, §3.1, #5 (p. 205).

2. For any space $X$, show that $H^0 X$, as a ring under cup product, is isomorphic to the ring of set functions $\{\pi_0 X \to \mathbb{Z}\}$, where addition and multiplication are defined point-wise.

3. Let $X = A \vee B$, the one point union of pointed spaces $(A, \{a_0\})$ and $(B, \{b_0\})$. Assume that $(A, \{a_0\})$ is a good pair (in the sense of Hatcher p. 114). Prove that the map $H^n X \to H^n A \times H^n B$ induced by inclusions is an isomorphism for $n \geq 1$. Furthermore, prove that with respect to this isomorphism, the cup product on $H^* X$ is given in positive dimensions by

$$\langle \alpha_1, \beta_1 \rangle \smile \langle \alpha_2, \beta_2 \rangle = \langle \alpha_1 \smile \alpha_2, \beta_1 \smile \beta_2 \rangle.$$

Use this to describe the cup product structure on $H^*(S^m \vee S^n)$, where $m, n \geq 1$.

4. Hatcher §3.3, #1 (p. 228).

5. Consider the explicit triangulization of a 3-dimensional lens space built from $n$ tetrahedra, given in exercise #8 in §2.1 of Hatcher (Chapter 2, p. 131).

   a. Use the associated complexes of simplicial chains/cochains to compute $H^*(X; \mathbb{Z}/n)$, describing explicit representatives for generators of these groups.

   b. Explicitly compute the action of the Bockstein operators $\beta: H^i(X; \mathbb{Z}/n) \to H^{i+1}(X; \mathbb{Z}/n)$.

   c. Explicitly compute the cup product in $H^*(X; \mathbb{Z}/n)$.

   d. Now do the same calculation as in (a), (b), and (c) above, but with a modified lens space $X_p$, constructed as follows. For $0 < p < n$ an integer relatively prime to $n$, form a complex as above, except that the bottom face of $T_i$ is identified with the top face of $T_{i+p}$ for all $i$. (The original example was the case $p = 1$.)

   e. For each $p$ relatively prime to $n$, construct an isomorphism of rings $H^*(X_1; \mathbb{Z}/p) \to H^*(X_p; \mathbb{Z}/p)$. Thus, all these spaces have the “same” cohomology with cup product.

In each of these examples, the class $T = \sum T_k \in \Delta_3(X_p; \mathbb{Z})$ gives a generator of $H_3(X_p; \mathbb{Z})$, as does $-T$. We will refer to these two generators $[T], -[T] \in H_3(X_p; \mathbb{Z})$ as fundamental classes of $X_p$. We will also refer to the images of $[T]$ and $-[T]$ under the map $H_3(X_p; \mathbb{Z}) \to H_3(X_p; \mathbb{Z}/n)$ induced by reduction mod $n$ as fundamental class(es). (The group $H_3(X_p; \mathbb{Z}/n)$ may have other generators besides the
fundamental classes; the point is that the fundamental classes in $H_3(X_p; \mathbb{Z}/n)$ are special, because they “come from” generators of integral homology.)

(f) Let $x \in H^1(X_p; \mathbb{Z}/n)$ be any element, and let $\beta(x) \in H^2(X_p; \mathbb{Z}/n)$ be the image of $x$ under the Bockstein. Compute

$$\langle x \smile \beta(x), \pm [T]\rangle \in \mathbb{Z}/n$$

in each of the above examples. (Your answer should depend on the choice of element $x$, the choice of fundamental class in $\{\pm [T]\}$, and the integer $p$.) Show that if $x$ is a generator of $H^1$, then the result is relatively prime to $n$.

(g) Let $(\mathbb{Z}/n)^\times$ be the abelian group of units in the ring $\mathbb{Z}/n$; i.e., integers relatively prime to $n$, considered modulo $n$, with group structure given by multiplication. Let $G_n$ be the quotient group of $(\mathbb{Z}/n)^\times$ by the subgroup generated by (i) squares $k^2$ for $k \in (\mathbb{Z}/n)^\times$, and (ii) $\{1\}$. Show that the equivalence class of $\phi(X_p) = \langle x \smile \beta(x), \pm [T]\rangle \in (\mathbb{Z}/n)^\times$ in $G_n$ does not depend on the choice of generator $x$ of $H^1(X_p; \mathbb{Z}/n)$ or fundamental class $\pm [T] \in H_3(X_p; \mathbb{Z})$. Conclude that $\phi(X_p)$ is a homotopy invariant of the spaces $X_p$. Show that every element of $G_n$ can occur as the $\phi$ invariant of some $X_p$.

You can conclude from this that for $n = 5$ (for instance), $X_1$ and $X_2$ are not homotopy equivalent, and therefore not homeomorphic, even though they have isomorphic cohomology rings.