Singular homology. A singular $n$-simplex in a space $X$ is a continuous map $\sigma: \Delta^n \to X$. The singular $n$-chains are

$$C_n(X) \overset{\text{def}}{=} \bigoplus_{\sigma: \Delta^n \to X} \mathbb{Z}.$$  

The boundary map is given by $\partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma d^i$. The singular homology groups are $H_n(X) = H_n(C_n(X))$.

Note that if $X$ has a $\Delta$-structure, then $\Delta_n \to X$ induces an isomorphism $H_\Delta(X) \approx H_\ast(X)$.

Singular homology is a functor. A map $f: X \to Y$ of spaces induces a chain map $f_\#: C_\ast(X) \to C_\ast(Y)$ by $f_\#(\sigma) = f \sigma$, and thus a map $f_\ast = H_\ast(f_\#): H_\ast(X) \to H_\ast(Y)$. Furthermore, $(gf)_\ast = g_\ast f_\ast$, and $id_\ast = id$.

We also have relative homology. Given a pair $(X,A)$ consisting of a space and a subspace, let $C_\ast(X,A) = C_\ast(X)/C_\ast(A)$, and $H_\ast(X,A) = H_\ast(C_\ast(X,A))$. This is functorial with respect to maps $f: (X,A) \to (Y,B)$ of pairs, which are maps $f: X \to Y$ such that $f(A) \subseteq B$. It is convenient to note that $C_\ast(X,A) = C_\ast(X,\emptyset)$, and so $H_\ast(X) = H_\ast(X,\emptyset)$. I’ll silently identify $X$ with the pair $(X,\emptyset)$ when necessary.

Since the sequence

$$0 \to C_\ast(A) \overset{i_\#}{\longrightarrow} C_\ast(X) \overset{j_\#}{\longrightarrow} C_\ast(X,A) \to 0$$

is exact, there is a boundary operator $\partial: H_n(X,A) \to H_{n-1}(A)$ for each pair.

**Exercise.** Show that boundary operator is “natural”, in the sense that if we have a “map” between two short exact sequences of chain complexes, i.e., a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_\ast & \longrightarrow & B_\ast & \longrightarrow & C_\ast & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'_\ast & \longrightarrow & B'_\ast & \longrightarrow & C'_\ast & \longrightarrow & 0
\end{array}
$$

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in which the rows are exact, then the diagram

\[
\begin{array}{ccc}
H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) \\
\downarrow & & \downarrow \\
H_n(C') & \xrightarrow{\partial} & H_{n-1}(A')
\end{array}
\]

commutes. In particular, if \( f: (X, A) \to (Y, B) \) is a map of pairs, the diagram

\[
\begin{array}{ccc}
H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) \\
f_* & & f_* \\
H_n(Y, B) & \xrightarrow{\partial} & H_{n-1}(B)
\end{array}
\]

commutes.

We will show that singular homology (which is the collection \((H_q(\cdot, \cdot), \partial)\) of functors on pairs and natural boundary operators) satisfies a list of properties, which are known as the Eilenberg-Steenrod axioms. It turns out that two “homology theories” which satisfies these axioms must agree on a large class of spaces, which includes all spaces which are homotopy equivalent to CW-complexes (but not all spaces). Also, any such homology theory will agree with \( H^\Delta_* \) for any \( \Delta \)-complex, as we will see. The axioms are:

- Dimension.
- Sum.
- Exactness.
- Homotopy.
- Excision.

**Proposition 0.1** (Dimension Axiom). If \( X \) is a one point space, then \( H_0(X) \approx \mathbb{Z} \), and \( H_q(X) = 0 \) for \( q \neq 0 \).

**Proposition 0.2** (Sum Axiom). Let \( X = \coprod X_\alpha \), and let \( i_\alpha: X_\alpha \to X \) denote the tautological inclusion. Then

\[
\bigoplus H_*(X_\alpha) \xrightarrow{(i_\alpha)_*} H_* X
\]

is an isomorphism.

**Proposition 0.3** (Exactness Axiom). For any pair \((X, A)\), the sequence

\[
H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A)
\]

is exact.