Covering maps and fundamental group. The lifting property implies that covering maps are injective on homotopy classes of paths.

**Proposition 0.1.** Suppose $p: Y \to X$ is a covering map. Then for $y_0, y_1 \in Y$, with $x_i = p(y_i) \in X$, the induced map

$$p_*: \pi_1(Y; y_0, y_1) \to \pi_1(X; x_0, x_1)$$

is injective. The image consists of all homotopy classes of paths $x_0 \rightsquigarrow x_1$ whose lifts starting at $y_0$ actually end at $y_1$.

**Proof.** Injectivity is because path homotopies lift. The statement about the image is immediate. $\square$

We usually use this in the case when $y_0 = y_1$ and $x_0 = x_1$, in which case we discover

**Corollary 0.2.** The homomorphism $p_*: \pi_1(Y; y_0) \to \pi_1(X; x_0)$ is injective if $p: (Y, y_0) \to (X, x_0)$ is a covering map.

**Example 0.3.** $p: S^1 \to S^1$ by $p(z) = z^n$.

Thus, if $p: (Y, y_0) \to (X, x_0)$ is a covering map, then we can use $p_*$ to identify $H = \pi_1(Y; y_0)$ with a subgroup of $G = \pi_1(X; y_0)$, i.e., $H \approx p_* H \subseteq G$.

There is a subtlety about the choice of basepoint; the subgroup depends on which basepoint we use. However, we have

**Proposition 0.4.** Let $p: Y \to X$ be a covering map. If $y_1, y_2 \in p^{-1}(x_0)$ are connected by a path in $Y$, then $H_1 = p_*(\pi_1(Y; y_1))$ and $H_2 = p_*(\pi_1(Y; y_2))$ are conjugate subgroups.

**Proof.** Pick a path $g: y_1 \rightsquigarrow y_2$. Then $H_2 = [g]^{-1}H_1[g]$. $\square$

**Example 0.5.** A suitable 3-fold cover of $Y \to S^1 \lor S^1$.

**Proposition 0.6.** Let $p: (Y, y_0) \to (X, x_0)$ be as above, and suppose that $Y$ is path connected. Then there is a bijection between the fiber $p^{-1}(x_0)$ and the set $H \setminus G$ of right cosets of $H$ in $G$.

**Proof.** Define

$$H \setminus G \to p^{-1}(x_0) \quad \text{by} \quad H[g] \mapsto \tilde{g}(1),$$

where $\tilde{g}$ is the lift of $g$ starting at $y_0$. This is well-defined, because if $[h] \in H$, then $\tilde{h}$ is a loop at $y_0$, so $(\tilde{h} * \tilde{g})(1) = \tilde{g}(1)$.

Define

$$p^{-1}(x_0) \to H \setminus G \quad \text{by} \quad y \mapsto H[p \circ \tilde{g}],$$

Date: February 19, 2012.
where $\tilde{g}: y_0 \rightsquigarrow y$ is a path in $Y$. This is well-defined, for if $\tilde{g}': y_0 \rightsquigarrow y$ is another such path, then $\tilde{g}' \sim \tilde{g} \ast \tilde{h}$ for some loop $\tilde{h}: y_0 \rightsquigarrow y_0$.

It is straightforward to check that the two functions are inverse to each other. \hfill $\Box$

**Examples.** $p: S^1 \to S^1$; $S^n \to \mathbb{R}P^n$ (thus $|\pi_1 \mathbb{R}P^n| = 2$); example of a cover of $S^1 \vee S^1$.

**Proposition 0.7.** Monodromy satisfies the following.

1. If $e_x$ is the constant path at $x \in X$, then $M(e_x): \text{Fib}_p(x) \to \text{Fib}_p(x)$ is the identity map.
2. If $\gamma: x \rightsquigarrow x'$ and $\delta: x' \rightsquigarrow x''$ are paths in $X$, then $M(\delta) \circ M(\gamma) = M(\gamma \ast \delta)$.
3. If $\gamma, \gamma': x \rightsquigarrow x'$ are path homotopic, then $M(\gamma) = M(\gamma')$.
4. For all paths $\gamma$, $M(\gamma)$ is a bijection.

Thus, monodromy is compatible with path composition, and only depends on the homotopy class of the path. To summarize, we have

- for every $x \in X$, a set $\text{Fib}_p(x)$, and
- for every $\alpha \in \pi_1(X; x_0, x_1)$, a function $M_\alpha: \text{Fib}_p(x_0) \to \text{Fib}_p(x_1)$, such that
  - $M(1) = \text{id}$, and
  - $M(\alpha \ast \beta) = M(\beta) \circ M(\alpha)$.

Thus, monodromy amounts to a contravariant functor

$$M: \Pi_1 X \to \text{Set}.$$

**Exercise.** Suppose $p: Y \to X$ is a covering map, and you are given only: the fundamental groupoid $\Pi_1 X$, and the monodromy functor $M: (\pi_1 X)^{\text{op}} \to \text{Set}$. Describe how to reconstruct the fundamental groupoid $\Pi_1 Y$ from $(\Pi_1 X, M)$, without any other knowledge of $Y$. (Hint: the objects of $\Pi_1 Y$, which are just the points of $Y$, are $\coprod_{x \in X} \text{Fib}_p(x)$. So you just have to describe the homotopy classes of paths.)

**Action of fundamental group on fiber.** Let’s specialize the monodromy construction to the case of loops $\gamma: x_0 \rightsquigarrow x_0$. Set $G = \pi_1(X; x_0)$. The monodromy construction defines a right action of $G$ on $\text{Fib}_p(x_0)$. That is, for $g \in G$ and $y \in \text{Fib}_p(x_0)$, we define

$$y \cdot g \overset{\text{def}}{=} M(g)(y),$$

which satisfies $(y \cdot g_1) \cdot g_2 = M(g_2)(M(g_1)(y)) = M(g_1 * g_2)(y) = y \cdot (g_1 * g_2)$.

For simplicity, suppose that $X$ is path connected. Then:

- the set of path components of $Y$ is in one-to-one correspondence with the set of $G$-orbits in $\text{Fib}_p(x_0)$.\hfill $\Box$
• if \( y_0 \in \text{Fib}_p(x) \), then the \textit{isotropy} subgroup \( G_{y_0} = \{ g \in G \mid y_0 \cdot g = y_0 \} \) of this point is identical to the subgroup \( p_*(\pi_1(Y; y_0)) \subseteq G \), which is isomorphic to \( \pi_1(Y; y_0) \).

• the orbit \( y_0G \subseteq \text{Fib}_p(x) \) of \( y_0 \) is in bijective correspondence with the set of right cosets \( G_{y_0} \backslash G \); a coset \( G_{y_0} \gamma \) corresponds to the point \( y_0 \cdot \gamma \).

Question. If \( y_0, y_1 \in \text{Fib}_p(x) \) are two different points, what can we say about the relationship between \( G_{y_0} \) and \( G_{y_1} \).

Note that this immediately implies

\textbf{Proposition 0.8.} Let \( p: (Y, y_0) \to (X, x_0) \) be a covering map with \( Y \) path connected. Then \( p^{-1}(x_0) \approx H \backslash G \), where \( G = \pi_1(X; x_0) \) and \( H = p_*(\pi_1(Y; y_0)) \).

\textbf{Universal covers.} Let \((X, x_0) \) be a path connected space. A \textbf{universal cover} of \( X \) is a covering map \( p: (Y, y_0) \to (X, x_0) \) such that \( Y \) is simply connected; i.e., \( Y \) is path connected and \( \pi_1(Y, y_0) = 1 \).

We’ve already seen that a universal cover has the property that \( p^{-1}(x_0) \approx \pi_1(X; x_0) \). Explicitly, given a point \( y_0 \in p^{-1}(x_0) \), we define the map \( \phi_{y_0}: p^{-1}(x_0) \to \pi_1(X; x_0) \) by sending \( y \in p^{-1}(x_0) \) to \([p \circ \gamma]\), where \( \gamma : y_0 \sim y \) is any path connecting these two points in \( Y \). The inverse to \( \phi_{y_0} \) is defined by the path lifting construction.

More generally, we have that for any \( x \in X \), \( p^{-1}(x) \approx \pi_1(X; x_0, x) \).

\textbf{Examples.} Universal covers of \( S^1 \), \( S^1 \times S^1 \), \( \mathbb{R} \mathbb{P}^2 \).

\textbf{F 17 Feb}

\textbf{Construction of the universal cover.} I want to show that any “nice” path-connected \((X, x_0)\) has a universal cover. The above discussion gives the idea: if \( p: (Y, y_0) \to (X, x_0) \) is a universal cover, there’s a bijection of sets

\[ Y = \coprod_{x \in X} p^{-1}(x) \to \coprod_{x \in X} \pi_1(X, x_0, x). \]

So, starting with \( X \), we’ll \textit{define} the underlying point set of \( Y \) to be \( \coprod_{x \in X} \pi_1(X, x_0, x) \), and we’ll define \( p: Y \to X \) to be the obvious function. It remains to provide a topology for \( Y \).

Recall that a \textbf{basis} for \( Y \) is a collection of subsets \( \mathcal{B} \) of \( Y \), such that (i) \( Y = \bigcup_{U \in \mathcal{B}} U = Y \) and (ii) if \( U_1, U_2 \in \mathcal{B} \) and \( y \in U_1 \cap U_2 \), then there exists \( V \in \mathcal{B} \) such that \( y \in V \subseteq U_1 \cap U_2 \).

If \( \mathcal{B} \) is a basis, then the collection \( \mathcal{T} \) of all unions of basis elements is a topology for \( Y \). Note that if \( f: X \to Y \), to check that it is continuous it is enough to check that \( f^{-1}U \) is open in \( X \) for \( UB \).

If \( p: Y \to X \) is a covering map, then \( X \) has a basis given by the evenly covered open sets \( U_\alpha \). If we write \( p^{-1}U_\alpha = \coprod V_{\alpha\beta} \) where \( V_{\alpha\beta} \) is open in \( Y \) and \( p|_{V_{\alpha\beta}} \to U_\alpha \) is a homeomorphism, then the collection \( \{V_{\alpha\beta}\} \) is a basis of \( Y \).

\textbf{Idea of the construction.} Here’s what we’ll do:

• We’ll define certain a basis \( \mathcal{B}_X \) of the topology on \( X \).

• For each \( U_\alpha \in \mathcal{B} \), we’ll choose a decomposition \( p^{-1}U_\alpha = \coprod V_{\alpha\beta} \), where each \( p|_{V_{\alpha\beta}}: V_{\alpha\beta} \to U_\alpha \) is a bijection.

• Then we’ll let \( \mathcal{B}_Y \) be the collection of all such sets \( V_{\alpha\beta} \) for all \( U_\alpha \in \mathcal{B} \), and we’ll show it’s a basis \( \mathcal{B}_Y \) for \( Y \).

If we can do all this, then we can topologize \( Y \) using the basis \( \mathcal{B}_Y \), and it will turn out that \( p: Y \to X \) is a continuous covering map.
Here’s how we’ll decompose $p^{-1}U$. Given an open set $U$ in $X$, consider $p^{-1}U = \coprod_{x \in U} \pi_1(X, x, 0)$. We’ll define an equivalence relation “$\approx_U$” on $p^{-1}U$ as follows:

- If $[f_1], [f_2] \in p^{-1}U$, where $f_1: x_0 \rightsquigarrow x_1$ and $f_2: x_0 \rightsquigarrow x_2$ are paths in $X$ with $x_1, x_2 \in U$, we’ll say that $[f_1] \approx_U [f_2]$ if there exists a path $g: I \rightarrow U$ of the form $x_1 \rightsquigarrow x_2$, such that $g \ast f_1 \sim f_2$ as paths in $X$.

(Draw picture of a nice, connected evenly covered $U$. Compare with a non-evenly connected $U$. Then consider a non-connected $U$.)

Write $p^{-1}U = \coprod V_{\alpha, \beta}$ for the decomposition into equivalence classes.

Note that if $U' \subseteq U$, the equivalence relations on their preimages are compatible. That is, if $y_1, y_2 \in p^{-1}U'$, then $y_1 \approx_{U'} y_2$ implies $y_1 \approx_U y_2$. Thus, each equivalence class for $p^{-1}U'$ is contained in a single equivalence class for $p^{-1}U$.

We want to show that each map $p|_{V_{\beta}}: V_{\beta} \rightarrow U$ from a $\approx_U$ equivalence class is actually a bijection.

**Necessary condition for surjectivity.** We’ll need $U$ to be path connected; otherwise $p|_{V_{\beta}}$ won’t be surjective.

**Necessary condition for injectivity.** What do we need for $p|_{V_{\beta}}$ to be injective? Suppose $x \in U$, and that $[f_1], [f_2] \in p^{-1}(x) \cap V_{\beta}$, which means that $f_1$ and $f_2$ are both paths $x_0 \rightsquigarrow x$. I’d like to be able to show that $[f_1] = [f_2]$.

Since $[f_1]$ and $[f_2]$ are in the same equivalence class $V_{\beta} \subseteq p^{-1}U$, this means that there exists a path $g: x \rightsquigarrow x$ in $U$ such that $g \ast f_1 \sim f_2$ as paths in $X$. This means that $[g] = [f_2] \ast [f_1]^{-1}$ in $\pi_1(X, x)$. Since we want $[f_2] \ast [f_1]^{-1} = 1$, this means we need $[g] = 1$ in $\pi_1(X, x)$. In other words, loops in $U$ should be contractible in $X$.

We want the inclusion map $i_U: U \rightarrow X$ to be such that for all $x \in U$, $(i_U)_*(\pi_1(U, x)) = \{1\}$ in $\pi_1(X, x)$.

So we will let $\mathcal{B}_X$ be the collection of open sets $U$ such that

1. $U$ is path connected, and
2. $(i_U)_*: \pi_1(U, x) \rightarrow \pi_1(X, x)$ has trivial image for all $x \in U$.

**Locally path-connected spaces.** We say that a space $X$ is **locally path connected** if for every point $x$ and open set $U$ with $x \in U \subseteq X$, there exists a path connected open set $V$ such that $x \in V \subseteq U$. I.e., every point has “arbitrarily small” path-connected neighborhoods. Equivalently, $X$ has a basis of path connected open subsets.

**Example of a space which is not locally path connected.** The space $\{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is not locally path connected at 0; of course, it is not even path connected.

The “comb” $X$, which is the subspace of $\mathbb{R}^2$ defined by

$$X = [0, 1] \times \{0\} \cup \{0\} \times [0, 1] \cup \bigcup_{n \in \mathbb{N}} \{1/n\} \times [0, 1].$$

This is path connected, but not locally path connected at $x_0 = (0, 1)$.

**Semi-locally simply connected spaces.** We say that a space $X$ is **semi-locally simply connected** if for all $x \in X$, there exists an open neighborhood $U$ of $x$ such that $(i_U)_*: \pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$ has trivial image. Equivalently, $X$ has a basis of open subsets $U$ with the property that loops in $U$ are contractible in $X$. (Note that if $U$ has this property, then any smaller neighborhood $x \in V \subseteq U$ also has this property, so this implies that every point has “arbitrarily small neighborhoods, loops in which are contractible in $X$”.)
Example of a space which is not semi-locally simply connected. This is the “Hawaiian earrings” space $X$, defined as $X = \bigcup_{n \in \mathbb{N}} C_n \subset \mathbb{R}^2$, where $C_n \subset \mathbb{R}^2$ is the circle of radius $1/n$ centered at $(0, 1/n)$. Let $x_0 = (0, 0)$, and let $\gamma_n : I \to C_n \subseteq X$ be the loop at $x_0$ which goes around $C_n$ once. Then $[\gamma_n] \neq 1 \in \pi_1(X, x_0)$ (why?) But for any open neighborhood $U$ of $x_0$, there are loops of the form $\gamma_n$ in $U$.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL

E-mail address: rezk@math.uiuc.edu